Mathematics and Its Applications

P.S. Bullen

Handbook of Means and Their Inequalities

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CONTENTS

‘Means and Their Inequalities”-Preface

There seems to be two types of books on inequalities. On the one hand there are treatises that attempt to cover all or most aspects of the subject, and where an attempt is made to give all results in their best possible form, together with either a full proof or a sketch of the proof together with references to where a full proof can be found. Such books, aimed at the professional pure and applied mathematician, are rare. The first such, that brought some order to this untidy field, is the classical *“Inequalities”* of *Hardy, Littlewood & Pólya*, published in 1934. Important as this outstanding work was and still is, it made no attempt at completeness; rather it consisted of the total knowledge of three front rank mathematicians in a field in which each had made fundamental contributions. Extensive as this combined knowledge was there were inevitably certain lacunæ; some important results, such as Steffensen’s inequality, were not mentioned at all; the works of certain schools of mathematicians were omitted, and many important ideas were not developed, appearing as exercises at the ends of chapters. The later book *“Inequalities”* by *Beckenbach & Bellman*, published in 1961, repairs many of these omissions. However this last book is far from a complete coverage of the field, either in depth or scope. A much more definitive work is the recent *“Analytic Inequalities”* by *Mitrinović*, (with the assistance of Vasić), published in 1970, a work that is surprisingly complete considering the vast field covered.

On the other hand there are many works aimed at students, or non-mathematicians. These introduce the reader to some particular section of the subject, giving a feel for inequalities and enabling the student to progress to the more advanced and detailed books mentioned above. Whereas the advanced books seem to exist only in English, there are excellent elementary books in several languages: *“Analytic Inequalities”* by *Kazarinoff*, *“Geometric Inequalities”* by *Bottema, Djordjević, Janić & Mitrinović* in English; *“Nejednakosti”* by *Mitrinović*, *“Sredine”* by *Mitrinović & Vasić* in Serbo-croatian, to mention just a few. Included in this group although slightly different are some books that list all the inequalities of a certain type—a sort of table of inequalities for reference. Several books by Mitrinović are of this type.

Due to the breadth of the field of inequalities, and the variety of applications, none of the above mentioned books are complete on all of the topics that they take up. Most inequalities depend on several parameters, and what is the most natural domain for these parameters is not necessarily obvious, and usually it is not the widest possible range in which the inequality holds. Thus the author, even

the most meticulous, is forced to choose; and what is omitted from the conditions of an inequality is often just what is needed for a particular application. What appears to be needed are works that pick some fairly restricted area from the vast subject of inequalities and treat it in depth. Such coherent parts of this discipline exist. As Hardy, Littlewood & Pólya showed, the subject of inequalities is not just a collection of results. However, no one seems to have written a treatise on some such limited but coherent area. The situation is different in the collection of elementary books; several deal with certain fairly closely defined areas, such as geometric inequalities, number theoretic inequalities, means.

It is the last mentioned area of means that is the topic of this book. Means are basic to the whole subject of inequalities, and to many applications of inequalities to other fields. To take one example: the basic geometric mean-arithmetic mean inequality can be found lurking, often in an almost impenetrable disguise, behind inequalities in every area. The idea of a mean is used extensively in probability and statistics, in the summability of series and integrals, to mention just a few of the many applications of the subject. The object of this book is to provide as complete an account of the properties of means that occur in the theory of inequalities as is within the authors' competences. The origin of this work is to be found in the much more elementary *"Sredine"* mentioned above, which gives an elementary account of this topic.

A full discussion will be given of the various means that occur in the current literature of inequalities, together with a history of the origin of the various inequalities connecting these means[1]. A complete catalogue of all important proofs of the basic results will be given as these indicate the many possible interpretations and applications that can be made. Also, all known inequalities involving means will be discussed. As is the nature of things, some omissions and errors will be made: it is hoped that any reader who notices any such will let the authors know, so that later editions can be more complete and accurate.

An earlier version of this book was published in 1977 in Serbo-croatian under the title *" Sredine i sa Njima Povezane Nejednakosti"*. The present work is a complete revision, and updating of that work.

The authors thank Dr J. E. Pečarić of the University of Belgrade Faculty of Civil Engineering for his many suggestions and contributions.

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[1] Although not mentioned in this preface the book was devoted to discrete mean inequalities and did not discuss in any detail integral mean inequalities, matrix mean inequalities or mean inequalities in general abstract spaces. This bias will be followed in this book except in Chapter VI.

PREFACE TO THE HANDBOOK

Since the appearance of *Means and Their Inequalities* the deaths of two of the authors have occurred. The field of inequalities owes a great debt to Professor Mitrinović and his collaborator for many years, Professor Vasić. Over a lifetime Professor Mitrinović devoted himself to inequalities and to the promotion of the field. His journal, *Univerzitet u Beogradu Publikacije Elektrotehničkog Fakulteta. Serija Matematika i Fizika*, the *"i Fizika'* was dropped in the more recent issues, has in all of its volumes, from the first in the early fifties, devoted most of its space to inequalities. In addition his enthusiasm has resulted in a flowering of the study both by his students, P. M. Vasić, J. E. Pečarić to mention the most notable, and by many others. The uncertain situation in the former Yugoslavia has lead to many of the researchers situated in that country moving to institutions all over the world. There are now more journals devoted to inequalities, such as the *Journal for Inequalities and Applications* and *Mathematical Inequalities and Applications*, as well as many that devote a considerable portion of their pages to inequalities, such as the *Journal of Mathematical Analysis and Applications*; in addition mention must be made of the electronic *Journal of Inequalities in Pure and Applied Mathematics* based on the website `http://rgmia.vu.edu.au` and under the editorship of S. S. Dragomir. This website has in addition many monographs devoted to inequalities as well as a data base of inequalities, and mathematicians working in the field. Another welcome change has been the many contributions from Asian mathematicians. While there have always been results from Japan, in recent years there has been a considerable amount of work from China, Singapore, Malaysia and elsewhere in that region.

It was taken for granted in the earlier Preface that anyone reading this book would not only be interested in inequalities but would be aware of their many applications. However it would not be out of place to emphasize this by quoting from a recent paper; [*Guo & Qi*].

> "It is well known that the concepts of means and their inequalities not only are basic and important concepts in mathematics, (for example, some definitions of norms are often special means[2]), and have explicit geometric

[2] More precisely ". . . certain means are related to norms and metrics.". See III 2.4, 2.5.7 VI 2.2.1.

meanings[3], but also have applications in electrostatics[4], heat conduction and chemistry[5]. Moreover, some applications to medicine[6] have been given recently."

Due to the extensive nature of the revision in the second edition and the large amount of new material it has seemed advisable to alter the title but this handbook could not have been prepared except for the basic work done by my late colleagues and I only hope that it will meet the high standards that they set.

In addition I want to thank my wife Georgina Bullen who has carefully proofread the non-mathematical parts of the manuscript, has suffered from computer deprivation while I monopolized the screen, and without whose support and help the book would have appeared much later and in a poorer form.

P. S. Bullen
Department of Mathematics
University of British Columbia
Vancouver BC
Canada V6T 1Z2
bullen@math.ubc.ca

[3] See II 1.1, 1.3.1; [*Qi & Luo*].

[4] See[*Pólya & Szegö 1951*].

[5] See[*Walker, Lewis & McAdams*], [*Tettamanti, Sárkany, Králik & Stomfai; Tettamanti & Stomfai*]

[6] See[*Ding*].

BASIC REFERENCES

There are some books on inequalities to which frequent reference will be made and which will be given short designations.

[AI] MITRINOVIĆ, D. S., WITH VASIĆ P. M. *Analytic Inequalities*, Springer-Verlag, Berlin,1970.
[BB] BECKENBACH, E. F. & BELLMAN, R. *Inequalities*, Springer-Verlag, Berlin, 1961.
[HLP] HARDY, G. H., LITTLEWOOD, J. E. & PÓLYA, G. *Inequalities*, Cambridge University Press, Cambridge, 1934.
[MI] BULLEN, P.S., MITRINOVIĆ, D. S & VASIĆ P. M. *Means and Their Inequalities*, D.Reidel. Dordrecht, 1988. [The first edition of this handbook.]
[MO] MARSHALL, A. W. & OLKIN, I. *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

Many inequalities can be placed in a more general setting. We do not follow that direction in this book but find the following an invaluable reference. Much of the material is readily translated to our simpler less abstract setting.

[PPT] PEČARIĆ, J. E., PROSCHAN, F. & TONG, Y. L. *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press Inc., 1992.

There are two books that are referred to frequently in certain parts of the book and for which we also introduce short designations.

[B^2] BORWEIN, J. M. & BORWEIN, P. B. *Pi and the AGM. A Study in Analytic Number Theory and Computational Complexity*, John Wiley and Sons, New York,1987.
[RV] ROBERTS, A. W. & VARBERG, D. E. *Convex Functions*, Academic Press, New York-London, 1973.

In addition there are the two following references. The first is a ready source of information on any inequality, and the second is in a sense a continuation of [AI] and [BB] above, being a report on recent developments in various areas of inequalities.

[DI] BULLEN, P. S. *A Dictionary of Inequalities*, Addison-Wesley Longman, London, 1998.[7]
[MPF] MITRINOVIĆ, D. S., PEČARIĆ, J. E. & FINK, A. M. *Classical and New Inequalities in Analysis*, D Reidel, Dordrecht, 1993.

There are many other books on inequalities and many books that contain important and useful sections on inequalities. These are listed in **Bibliography Books**.

From time to time conferences devoted to inequalities have published their proceedings. In particular, there are the proceedings of three symposia held in the United States, and of seven international conferences held at Oberwolfach.

I1(1965), I2 (1967), I3 (1969) *Inequalities, Inequalities II, Inequalities III*, Proceedings of the First, Second and Third Symposia on Inequalities, 1965, 1967, 1969; Shisha, O., editor, Academic Press, New York, 1967, 1970, 1972.
GI1(1976), GI2 (1978), GI3 (1981), GI4 (1984), GI5 (1986), GI6 (1990), GI7 (1995) *General Inequalities Volumes 1–7*, Proceedings of the First–Seventh International Conferences on General Inequalities, Oberwolfach, 1976, 1978, 1981, 1984, 1986, 1990, 1995; Beckenbach, E. F., Walter, W., Bandle, C., Everitt, W. N., Losonczi, L., [Eds.], International Series of Numerical Mathematics, 41, 47, 64, Birkhaüser Verlag, Basel, 1978, 1980, 1983, 1986,1987, 1992, 1997.

Individual papers in these proceedings, referred to in the text, are listed under the various authors with above shortened references.

Finally there are two general references.

[EM1], [EM2], [EM3], [EM4], [EM5], [EM7], [EM8], [EM9],[EM10], [EMSUPPI], [EMSUPPII], [EMSUPPIII]; HAZELWINKEL, M., [ED.] *Encyclopedia of Mathematics, vol.1–10, suppl. I–III*, Kluwer Academic Press, Dordrecht, 1988–2001.
[CE] WEISSTEIN, E. W. *CRC Concise Encyclopedia of Mathematics*, Chapman & Hall/CRC, Boca Raton, 1998.

[7] Additions and corrections can be found at http://rgmia.vu.edu.au/monographs/bullen.html.

NOTATIONS

1 Referencing

Theorems, definitions, lemmas, corollaries are numbered consecutively in each section; the same is true of formulæ. Remarks and Examples are numbered, using Roman numerals, consecutively in each subsection, and in each sub-subsection. In the same chapter references list the section, (subsection, sub-section, in the case of remarks and examples), followed by the detail: thus 3 Theorem 2, 4(6), or 1.2 Remark (6). Footnotes are numbered consecutively in each chapter and so are referred to by number in that chapter: thus 1.3.1 Footnote 1. References to other chapters are as above but just add the chapter number; thus I 3 Theorem 2(a), II 4(6), IV 1.2 Remark (6), I 1.3.1 Footnote 1. Although there are references for all names and all bibliography entries, in the case of a name occurring frequently, for instance Cauchy, only the most important instances will be mentioned; further names in titles of the basic references are usually not referenced, thus Hardy in [*HLP*].

2 Bibliographic References

Some have been given a shortened form; see **Basic References**. Others are standard, the name, followed by a year if there is ambiguity, or the year with an additional letter, such as 1978a, if there is more than one any given year. Joint authorship is referred to by using &, thus Mitrinović & Vasić.

3 Symbols for Some Important Inequalities

Certain inequalities are referred to by a symbol as they occur frequently.

(B) .... *Bernoulli's inequality* I 2.1 Theorem 1;
(C) .... *Cauchy's inequality* III 2.2;
(GA) .... *Geometric-Arithmetic Mean inequality* II 2.1 Theorem 1;
(H) .... *Hölder's inequality* III 2.1 Theorem 1;
(HA) .... *Harmonic- Arithmetic Mean Inequality* II 2.1 Remark(i);
(J) .... *Jensen's inequality* I 4.2 Theorem 12;
(M) .... *Minkowski's inequality* III 2.4 Theorem 9;
(P) .... *Popviciu's inequality* II 3.1 Theorem 1;
(R) .... *Rado's inequality* II 3.1 Theorem 1;
(r;s) .... *Power Mean inequality* III 3.1.1 Theorem 1;
S(r;s) .... *Elementary Symmetric Polynomial Mean inequality* V 2 Theorem 3(b);

(T), (T_N) .... *Triangle inequality* III 2.4 .
Integral analogues of these means, when they exist, will be written (J)-$\int$, etc; see VI 1.2.1, 1.2.2.; and ($\sim$B), etc. will denote the opposite inequalities; see 9 below.

4 Numbers, Sets and Set Functions

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are standard notations for the sets of integers, rational numbers, real numbers, and complex numbers respectively. The set of extended real numbers, $\mathbb{R} \cup \{\pm\infty\}$, is written $\overline{\mathbb{R}}$.
Less standard are the following:
$\mathbb{N} = \{n;\ n \in \mathbb{Z}, \text{and}\, n \geq 0\} = \{0, 1, 2, \ldots\}$, $\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, \ldots\}$,
$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $\mathbb{R}_+ = \{x; x \in \mathbb{R}\, \text{and}\, x \geq 0\}$, $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\} = \{x; x \in \mathbb{R}\, \text{and}\, x > 0\}$,
$\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$, $\mathbb{Q}_+ = \{x; x \in \mathbb{Q}\, \text{and}\, x \geq 0\}$, $\mathbb{Q}_+^* = \mathbb{Q}_+ \setminus \{0\} = \{x; x \in \mathbb{Q}\, \text{and}\, x > 0\}$.
The non-empty subsets of $\mathbb{N}$, or $\mathbb{N}^*$, are called *index sets*, the collection of these is written $\mathcal{I}$.
If $p \in \mathbb{R}^*$, $p \neq 1$, the *conjugate index of p, written p'*, is defined by

$$p' = \frac{p}{p-1}, \quad \text{equivalently} \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Note that

$$(p')' = p, \qquad p > 1 \iff p' > 1, \qquad p > 0 \text{ and } p' > 0 \iff p > 1, \text{ or } p' > 1.$$

A real function ϕ defined on sets, a *set function*, is said to be *super-additive, respectively log-super-additive*, if for any two non-intersecting sets, I, J say, in its domain

$$\phi(I \cup J) \geq \phi(I) + \phi(J), \text{ respectively, } \phi(I \cup J) \geq \phi(I)\phi(J).$$

If these inequalities are reversed the function is said to be *sub-additive, respectively log-sub-additive*. A set function f is said to be *increasing* if $I \subseteq J \implies f(I) \leq f(J)$, and if the reverse inequality holds it is said to be *decreasing*.
The *image of a set A by a function f* will be written $f[A]$; that is $f[A] = \{y;\ y = f(x),\ x \in A\}$.
A set E is called a *neighbourhood of a point x* if for some open interval $]a, b[$ we have $x \in [a, b[\subseteq E$.

5 Intervals

Intervals in $\mathbb{R}$ are written $[a, b]$, closed; $]a, b[$, open etc. In addition we have the unbounded intervals $[a, \infty[$, $]-\infty, a]$, etc; and of course $]-\infty, \infty[= \mathbb{R}$, $[-\infty, \infty] = \overline{\mathbb{R}}$, $[0, \infty[= \mathbb{R}_+$, $]0, \infty[= \mathbb{R}_+^*$. The symbol (a, b) is reserved for 2-tuples, see the next paragraph. If I is any interval then $\mathring{I}$ denotes the open interval with the same end-points as I.

6 n-tuples

If $a_i \in \mathbb{R}$, $\mathbb{C}$, $1 \le i \le n$, then we write $\underline{a}$ for the, ordered, *real, complex, n-tuple* $(a_1, \ldots, a_n)$ with these *elements or entries*; so if $\underline{a}$ is a real n-tuple $\underline{a} \in \mathbb{R}^n$. The usual vector notation is followed.

In addition the following conventions are used.

(i) For suitable functions f, g etc, $f(\underline{a}) = \big(f(a_1), \ldots, f(a_n)\big)$, and $g(\underline{a}, \underline{b}) = \big(g(a_1, b_1), \ldots, g(a_n, b_n)\big)$ etc. This useful convention conflicts with the standard notation for functions of several variables, $f(\underline{a}) = f(a_1, \ldots, a_n)$, but the context will make clear which is being used.

So: $\underline{a}^2 = (a_1^2, \ldots, a_n^2)$, $\underline{a}\,\underline{b} = (a_1b_1, \ldots, a_nb_n)$; but $\max \underline{a} = \max_{1 \le i \le n} a_i$, and $\min \underline{a} = \min_{1 \le i \le n} a_i$.

(ii) $\underline{a} \le \underline{b}$ means that $a_i \le b_i, 1 \le i \le n$. In addition $m \le \underline{a} \le M$ means that $m \le a_i < M, 1 \le i \le n$. The extensions to the other inequality signs is obvious. In particular if $\underline{a} > 0, \underline{a} \ge 0$ etc, we say that the *n-tuple is positive, non-negative* etc.

The set of non-negative n-tuples is written $\mathbb{R}_+^n$, and the set of positive n-tuples $(\mathbb{R}_+^*)^n$

(iii) An n-tuple all of whose elements are all zero except the i-th that is equal to 1 is written $\underline{e}_i$. The n-tuples $\underline{e}_i$, $1 \le i \le n$, form the standard basis of $\mathbb{R}^n$. As usual if $n = 2, 3$ these basis vectors are written $\underline{e}_1 = \underline{i}$, $\underline{e}_2 = \underline{j}$, $\underline{e}_3 = \underline{k}$.

(iv) $\underline{e}$ is the n-tuple each of whose entries is equal to 1 ; and $\underline{0}$, or just 0, is the n-tuple each of whose entries is equal to 0.

(v) If $1 \le i \le n$ then $\underline{a}_i'$ denotes the $(n-1)$-tuple obtained from $\underline{a}$ by omitting the element a_i.

(vi) $\underline{a} \sim \underline{b}$, *$\underline{a}$ is proportional to $\underline{b}$*, means that for some $\lambda, \mu \in \mathbb{R}$, not both zero, $\lambda\underline{a} + \mu\underline{b} = \underline{0}$; that is the two n-tuples are linearly dependent.

(vii) If $a_i = c, 1 \le i \le n$, we say that *$\underline{a}$ is constant, or is a constant.*

(viii) For $\Delta^k a_n$ and see I 3.1. An n-tuple is called an *arithmetic progression* if $\Delta^1 a_k, 1 \le k \le n-1$, is constant, equivalently if $\Delta^2 a_k = 0, 1 \le k \le n-2$,

(ix) For $\underline{a} \star \underline{b}$ see I 3.2 Definition 4.

(x) For $\underline{a} \prec \underline{b}$ see I 3.3 Definition 11.

(xi) Given an n-tuple $\underline{w}$ we will write $W_k = \sum_{i=1}^k w_i, 1 \le i \le n$; and if necessary $W_0 = 0$. More generally if $\underline{w}$ is a sequence and $I \in \mathcal{I}$, an index set, see 4, we write $W_I = \sum_{i \in I} w_i$.

(xii) If f is a function of k variables, $\underline{a}$ and n-tuple, $1 \le k \le n$, then $\sum_k! f(a_{i_1}, \ldots, a_{i_k})$ means that the summation is taken over all permutations of k elements from the $a_1, \ldots, a_n$. In the case that $k = n$ this will be written as $\sum! f(\underline{a})$.

(xiii) The *inner product of two n-tuples, $\underline{a}, \underline{b}$, written $\langle \underline{a}, \underline{b} \rangle$*, is $\sum_{i=1}^n a_i b_i$ if the n-tuples are real and $\sum_{i=1}^n a_i \bar{b}_i$ if the n-tuples are complex. In both cases $\langle \underline{a}, \underline{a} \rangle = \sum_{i=1}^n a_i^2$.

Where appropriate the above notations will be used for *sequences* $\underline{a} = (a_1, a_2, \ldots)$, provided the relevant series converge.

7 Matrices

An $m \times n$ *matrix* is $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$; the a_{ij} are the *entries* and if they are real, complex then A is said to be *real, complex*. The *transpose of* A is $A^T = (a_{ji})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$; $A^* = (\overline{a}_{ji})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ is the *conjugate transpose of A*.

If $A = A^T$ then A is *symmetric*; if $A = A^*$ then A is *Hermitian*; in these cases of course $m = n$ and the matrix is said to be *square*. The family of all $n \times n$ Hermitian matrices will be written $\mathcal{H}_n$; the subset of positive definite $n \times n$ Hermitian matrices is written $\mathcal{H}_n^+$.

If $A, B \in \mathcal{H}_n$ we write $A \le B$ if $B - A$ is positive semi-definite, and $A < B$ if $B - A$ is positive definite. This defines an order on $\mathcal{H}_n$ called the *Loewner ordering*.

A square matrix D with all non-diagonal elements zero is called a *diagonal matrix*; if the diagonal elements are $a_{ii} = d_i$, $1 \le i \le n$, we write $D = D(\underline{d}) = D(d_1, \ldots, a_d)$. If $\underline{d} = \underline{e}$ the diagonal matrix is called a *unit matrix*, usually written I, or I_n if it is important to show that it is an $n \times n$ unit matrix[8]. The square matrix all of whose entries are equal to 1 is written J, or if the order needs to be noted J_n. A matrix of any order all of whose entries are zero, *a zero matrix*, will be written O, the order being understood from the context.

If $A \in \mathcal{H}_n$ with eigenvalues $\underline{\lambda}$, then all of its eigenvalues real and $A = U^* D(\underline{\lambda}) U$ where U is a *unitary matrix*, that is $UU^* = I$; so for a unitary matrix $U^{-1} = U^*$.

If $A \in \mathcal{H}_n$ and f is a real-values function defined on an interval that contains the eigenvalues of A we define the associated *matrix function of order n*, using the same symbol, $f : \mathcal{H}_n \mapsto \mathcal{H}_n$ by $f(A) = UD\big(f(\underline{\lambda})\big)U^*$; see [*MO p.462*]. Further $\langle f(A)\underline{a}, \underline{a} \rangle = \sum_{i=1}^{n} |b_i|^2 f(\lambda_i)$ where $\underline{b} = U\underline{a}$.

8 Functions

A function of n variables f is said to be *symmetric* if its domain is symmetric and if its value is not altered by a permutation of the variables. Precisely, if $f : D \mapsto \mathbb{R}$ and (i) $D \subseteq \mathbb{R}^n$, (ii) $(a_1, \ldots, a_n) \in D \Longrightarrow (a_{i_1}, \ldots, a_{i_n}) \in D$ for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$, (iii) $f(a_{i_1}, \ldots, a_{i_n}) = f(a_1, \ldots, a_n)$ for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$, then f is said to be symmetric.

A related concept is that of an *almost symmetric function*: this is a function of n variables and n parameters, defined on a symmetric domain, that is invariant under the simultaneous permutation of the variables and the parameters.

EXAMPLES(i) The arithmetic mean with equal weights, $f(\underline{a}) = (a_1 + \cdots + a_n)/n$, is symmetric; the weighted arithmetic mean, $f(\underline{a}) = f(\underline{a}; \underline{w}) = (w_1 a_1 + \cdots + w_n a_n)/(w_1 + \cdots + w_n)$, is almost symmetric; see II 1.1.

The *Gamma* or *factorial function* is denoted by $x!$; that is

$$x! = \Gamma(x + 1).$$

The *identity function* is written ι, and then the *power functions* $\iota^s, s \in \mathbb{R}^*$. That is

$$\iota(x) = x; \qquad \iota^s(x) = x^s;$$

[8] This conflicts with the use of I to denote an interval but in practice there will be no confusion.

the domain being clear from the context.
The *maximum function* is defined as usual by $\max\{f,g\}(x) = \max\{f(x),g(x)\}$: also,

$$x^+ = \max\{x,0\};\ x^- = \max\{-x,0\};\ x = x^+ - x^-,\ |x| = x^+ + x^-.$$

By analogy if f is any real-valued function then $f^+ = \max\{f,0\} = \max \circ f$; when $f = f^+ - f^-$, $|f| = f^+ + f^-$.
The function that is equal to 1 on the set A and to zero off A, the *indicator function*[9] *of* A is written 1_A. That is

$$1_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

The *integral or integer part function* $[\cdot] : \mathbb{R} \mapsto \mathbb{Z}$ is defined by:

$$[x] = n, \text{ where } n \in \mathbb{Z} \text{ is the unique integer such that } n \le x < n+1.$$

9 Various

Proofs begin with a square, □, on the left, and end with one on the right.
If (n) denotes an inequality then (~n) will denote the inequality obtained by changing the inequality sign in (n); the *opposite, inverse or reverse inequality.*
A different concept is the *converse or complementary inequality*. Such inequalities arise as follows: in general an inequality is of the form $F \ge G$ or equivalently $F - G \ge 0$. Usually both F, G are continuous functions and the inequality holds on some non-compact set. If now we restrict the domain to a compact set we will get that $F - G$ attains a maximum on that set, $F - G \le \Lambda$ say. Another form arises using the equivalent $F/G \ge 1$, giving on the compact set $F/G \le \lambda$ say. The inequalities $F - G \le \Lambda$, $F/G \le \lambda$ are converse inequalities for the original inequality $F \ge G$.
If $\lim_{x\to x_0} f(x)/g(x) = 0$ we say that f *is little-oh of* g *as* $x \to x_0$, written $f(x) = o\big(g(x)\big)$, $x \to x_0$. In particular $f(x) = o(1)$, $x \to x_0$ means that $\lim_{x\to x_0} f(x) = 0$.
If there is some positive constant M such that for all x in some neighbourhood of x_0 $|f(x)| \le Mg(x)$ we say that f *is big-oh of* g *for* x *as* $x \to x_0$, written $f(x) = O\big(g(x)\big)$ $x \to x_0$. In particular $f(x) = O(1)$, $x \to x_0$ means that f is bounded at x_0. If $f(x) = O\big(g(x)\big)$ and $g(x) = O\big(f(x)\big)$ for x in a given set then we say that $f(x)$ *and* $g(x)$ *are of the same order of magnitude for* x *in the given set*, written $f(x) \sim g(x)$, $x \to x_0$.
Means are denoted throughout by Gothic letters: $\mathfrak{A}, \mathfrak{G}, \mathfrak{H} \ldots$ etc.
Classes of functions are denoted throughout by calligraphic letters; thus $\mathcal{C}(I)$, the space of functions continuous on I, $\mathcal{C}^n(I)$, the space of functions with continuous nth order derivatives on I, etc.

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[9] This function is also called the *characteristic function of A*.

A LIST OF SYMBOLS

The following is a list of the symbols used in the text.

The Gothic alphabet: $\mathfrak{a}, \mathfrak{A}, \mathfrak{b}, \mathfrak{B}, \mathfrak{c}, \mathfrak{C}, \mathfrak{d}, \mathfrak{D}, \mathfrak{e}, \mathfrak{E}, \mathfrak{f}, \mathfrak{F}, \mathfrak{g}, \mathfrak{G}, \mathfrak{h}, \mathfrak{H}, \mathfrak{i}, \mathfrak{I}, \mathfrak{j}, \mathfrak{J}, \mathfrak{k}, \mathfrak{K}, \mathfrak{l}, \mathfrak{L}, \mathfrak{m}, \mathfrak{M}, \mathfrak{n}, \mathfrak{N}, \mathfrak{o}, \mathfrak{O}, \mathfrak{p}, \mathfrak{P}, \mathfrak{q}, \mathfrak{Q}, \mathfrak{r}, \mathfrak{R}, \mathfrak{s}, \mathfrak{S}, \mathfrak{t}, \mathfrak{T}, \mathfrak{u}, \mathfrak{U}, \mathfrak{v}, \mathfrak{V}, \mathfrak{w}, \mathfrak{W}, \mathfrak{x}, \mathfrak{X}, \mathfrak{y}, \mathfrak{Y}, \mathfrak{z}, \mathfrak{Z}$.

The Greek alphabet: $\alpha, \beta, \gamma, \Gamma, \delta, \Delta, \epsilon, \varepsilon, \zeta, \eta, \theta, \Theta, \iota, \kappa, \lambda, \Lambda, \mu, \nu, \xi, \Xi, \omega, \pi, \Pi, \varpi, \rho, \varrho, \sigma, \Sigma, \tau, \upsilon, \Upsilon, \phi, \Phi, \varphi, \chi, \psi, \Psi. \omega, \Omega$

AN INTRODUCTORY SURVEY

This book is aimed at a wide audience. For those in the field of inequalities the table of contents, and the abstracts at the beginnings of chapters should suffice as a guide to the material. However for others it may be useful to indicate the basic material so as to allow the avoidance of the specialised sections.

The basic material can be found in Chapters I, II, III and VI while the remaining chapters deal mainly with material that is more technical. However even in the basic chapters material of less interest to the general reader occur. Such avoidable sections are: I 3.1, 3.2, 4.5.3, 4.6, 4.8, 4.9; II 3.2–3.6, 3.8, 4, 5.4, 5.6–5.9; III 2.3, 2.5, 3.1.2–3.1.4, 3.2, 4, 5.2.3, 5.4,, 6.2–6.4; VI 1.3, 2.1.4, 2.4, 3.2.2, 4.3–4.6, 5.

The core of the book is the properties of various means. These means are listed in the **Index** both separately, such as *Arithmetic mean*, and collectively, *Means.* The simplest means are two variable means such as the classical *arithmetic mean*, $\frac{1}{2}(x+y)$ and *geometric mean*, $\sqrt{xy}$, and the less well known *logarithmic mean*, $(y-x)/(\log y - \log x)$; see II 1.1, 1.2. 5.5, VI 2, 3.

This leads to the question — what properties should a function $f(x,y)$ have for it to be considered as a mean? Clearly we require f to be continuous, positive, defined for all positive values of both variables; a little less obvious are the properties of *symmetry*, $f(x,y) = f(y,x)$, *reflexivity*, $f(x,x) = x$, *homogeneity*, $f(\lambda x, \lambda y) = \lambda f(x,y)$, and *monotonicity*, $x \le x', y \le y' \implies f(x,y) \le f(x',y')$; finally there is the crucial property of *internality* that justifies the very name of mean, $\min\{x,y\} \le f(x,y) \le \max\{x,y\}$.

Most of the means introduced are easily defined for n-tuples, $n \ge 2$, when these various properties are suitably extended; see II 1.1 Theorem 2, 1.2 Theorem 6, III Theorem 2(e) and VI 6. Of course there are many other properties of means that have been identified as of interest; these are listed in the **Index** under *Mean Properties.*

The inequalities between the various means defined form the core material of the book. Again the two variable cases are the easiest II 2.2.1, 2.2.2, 5.5, VI 2, 3.1, 3.2.1. The fundamental result is the inequality between the arithmetic and geometric means, (GA), discussed in detail in II 2.4 where well over 70 proofs are given or mentioned; most are extremely elementary. The next basic result generalizes this and is the inequality between the power means, (r;s), III 3.1.1. Integral forms of these results are also give; VI 1.2.2.

From Notations 9 we see that every inequality between means of n-tuples can be regarded as saying a certain function of n is non-negative. For instance (GA) im-

plies that $R(n) = n(\mathfrak{A}_n - \mathfrak{G}_n) \geq 0$. A stronger property of this function of n would be to show that it increases; stronger because $R(1) = 0$; a similar discussion occurs for related functions that are not less than 1, $(\mathfrak{A}_n/\mathfrak{G}_n)^n \geq 1$ for example. This leads to the so called *Rado-Popoviciu type extensions* of the original inequality. Such are discussed for (GA) in II 3.1; the analogous discussion for (r;s) in III 3.2 is much more technical as the simplicity of the (GA) case has been lost.

Finally many well-known inequalities arise from the discussion of mean inequalities— in particular the inequalities of *Cauchy, (C), Hölder, (H), Minkowski, (M), Čebišev* and the *triangle inequality*, (T); see II 5.3, III 2.1, 2.2, 2.4.

I INTRODUCTION

In this chapter we will collect some results and concepts used in the main body of the text. There is no intention of being exhaustive in any of the topics discussed and often the reader will be referred to other sources for proofs and full details.

1 Properties of Polynomials

Simple properties of polynomials can be used to deduce some of the basic inequalities to be discussed in this book. In addition certain simple inequalities, needed at various places, are most easily deduced from the properties of some special polynomials. These results are collected together in this section for ease of reference.

CONVENTION In this section, unless otherwise specified, all polynomials will have real coefficients.

1.1 SOME BASIC RESULTS The results given here are standard and proofs are easily available in the literature; see for instance [*CE pp.420,1573; DI p.70; EM3 p.59; EM8 p.175*], [*Uspensky*].

THEOREM 1 *A polynomial of degree n has n complex zeros, and if n is odd at least one zero is real.*

THEOREM 2 [DESCARTES' RULE OF SIGNS] *A polynomial cannot have more positive zeros than there are variations of signs in its sequence of coefficients.*

THEOREM 3 [ROLLE'S THEOREM] *If p is a polynomial then p' has at least one zero between any two distinct real zeros of p.*

THEOREM 4 [INTERMEDIATE VALUE THEOREM] *A polynomial always has a zero between any two numbers at which its values are of opposite sign.*

THEOREM 5 *A zero of a polynomial is a zero of its derivative if and only if it is a multiple zero.*

The following result is basic to a variety of applications; see [*BB p.11; HLP pp.104–105*], [*Milovanović, Mitrinović & Rassias pp.70–71; Newton*], [Dunkel 1908/9; *Kellogg; Maclaurin; Sylvester*]. The present form is that given in [*HLP pp. 104–105*].

THEOREM 6 *If $f(x,y) = \sum_{i=0}^{n} c_i x^i y^{n-i}$ has, as a function of y/x, all of its zeros real then the same is true of all polynomials, not all of whose coefficients are zero, derived from f by partial differentiations with respect to x or y. Further if a zero of one of these derived polynomials has multiplicity $k, k > 1$, then it is a zero of multiplicity $k+1$ of the polynomial from which it was obtained by differentiation.*

□ The proof is immediate by repeated applications of Theorems 1, 3, 4 and 5. □

COROLLARY 7 *If $n \geq 2$, and*

$$p(x) = \sum_{i=0}^{n} c_i x^i = \sum_{i=0}^{n} \binom{n}{i} d_i x^i \tag{1}$$

is a polynomial of degree n with $c_0 \neq 0$ and all zeros real, then if $1 \leq m \leq k+m \leq n$ the polynomial $q(x) = \sum_{i=0}^{m} \binom{m}{i} d_{k+i} x^i$ has all of its zeros real.

□ Let $f(x,y) = \sum_{i=0}^{n} \binom{n}{i} d_i x^i y^{n-i}$. We are given that $d_0 \neq 0$, so $(0,y), y \neq 0$, is not a zero of f. Hence, by Theorem 6, $(0,y), y \neq 0$, is not a multiple zero of any derived polynomial. This implies that no two consecutive coefficients of f can vanish.
Save for a numerical factor $\partial^{n-m} f / \partial^k x \partial^{n-k-m} y$ is equal to $\sum_{i=0}^{m} \binom{m}{i} d_{k+i} x^i y^{m-i}$, which by the previous remark does not have all of its coefficients zero. Hence the result is an immediate consequence of Theorem 6. □

REMARK (i) It follows from the above proof that if p is a polynomial of degree n, as in (1), with $c_0 \neq 0$, and if for some k, $2 \leq k \leq n-1, c_k = c_{k-1} = 0$, then p has a complex, non-real, zero; [*Wagner C*].

COROLLARY 8 *If $n \geq 2$, and p a polynomial of degree n given by (1) with $c_0 \neq 0$ and all of its zeros real, then for $k, 1 \leq k \leq n-1$,*

$$d_k^2 \geq d_{k-1} d_{k+1}, \tag{2}$$

$$c_k^2 > c_{k-1} c_{k+1}. \tag{3}$$

The inequalities (2) are strict unless all the zeros are equal.

□ By Corollary 7 the roots of all the equations

$$d_{k-1} + 2 d_k x + d_{k+1} x^2 = 0, \quad 1 \leq k \leq n-1, \tag{4}$$

are real; from this (2) is immediate.
Now from (2) and the definition of $d_k, 1 \le k \le n$, see (1),

$$c_k^2 \ge \frac{(k+1)(n-k+1)}{k(n-k)} c_{k-1}c_{k+1} > c_{k-1}c_{k+1},$$

unless possibly either $c_{k-1} = 0$, or $c_{k+1} = 0$; but then, by Remark (i), $c_k \ne 0$; and so in all cases (3) is proved.
Finally, if for any k there is equality in (2) the associated quadratic equation (4) has a double root, and so, by Theorem 6, the original polynomial p of (1), has a single zero of multiplicity n. □

REMARK (ii) Inequality (2) is sometimes called *Newton's inequality* and will reappear later, II 2.4 proof *(ix)*, V 2 Theorem 1; [*DI p.192*]. A direct proof can be found in [*Nowicki 2001*]. A converse to this inequality has been given; [*Whiteley 1969*].

REMARK (iii) The above implies that if for some $k, 1 \le k \le n-1$, $c_k^2 \le c_{k-1}c_{k+1}$, then the above polynomial p has a non-real zero.

REMARK (iv) By writing (2) in the form $d_k^{2k} \ge d_{k-1}^k d_{k+1}^k$, $1 \le r \le k \le n-1$, multiplying, and assuming that $c_n = 1$, we get the important inequalities, see V 2(6);

$$d_{n-r}^{r+1} \ge d_{n-r-1}^{r}; \tag{5}$$

similarly,

$$c_{n-r}^{r+1} > c_{n-r-1}^{r}. \tag{6}$$

EXAMPLE (i) A very important case of (1) occurs when $-a_1, \ldots, -a_n$ are the real distinct zeros of p, when $p(x) = \prod_{k=1}^n (x + a_k)$; then $c_n = 1$ and $c_{n-k} = \frac{1}{k!}\sum_i ! a_{i_1} \ldots a_{i_k}$, $1 \le k \le n$; in particular $c_{n-1} = \sum_{i=1}^n a_i, c_0 = \prod_{i=1}^n a_i$; see V 1(7).

1.2 SOME SPECIAL POLYNOMIALS (a) [*DI p.207*], [*Bullen 1996a; Čakalov 1963*]. Define the polynomial $p_n, n \ge 1$, by

$$p_n(x) = x^{n+1} - (n+1)x + n.$$

In particular then $p_1(x) = (x-1)^2 > 0$, $x \ne 1$.
Clearly $x = 1$ is a zero of both p_n and of p_n'. So, by 1.1 Theorem 5, $x = 1$ is a double zero of p_n. [This can also be seen from the identity $p_n(x) = (x-1)^2 \sum_{i=0}^n (n-i)x^i$.]
This, by 1.1 Theorem 2, implies that $x = 1$ is the only positive zero of p_n.

Since $p_n(0) = n > 0$, and $p_n(n) = n^{n+1} - n^2 > 0$ if $n > 1$, we have by 1.1 Theorem 4, and the special case above that:

$$\text{if} \quad x \geq 0,\ x \neq 1, \quad \text{then} \quad x^{n+1} > (n+1)x - n. \tag{7}$$

(b) [*DI p.207*]. Define the polynomial $q_n, n \geq 1$, by

$$q_n(x) = (x+n)^{n+1} - (n+1)^{n+1}x.$$

Then using an argument similar to that in (a), $x = 1$ is the only positive zero of q_n. Hence

$$\text{if} \quad x \geq 0,\ x \neq 1, \quad \text{then} \quad (x+n)^{n+1} > (n+1)^{n+1}x. \tag{8}$$

2 Elementary Inequalities

In this section we collect some important elementary inequalities.

2.1 BERNOULLI'S INEQUALITY Inequality 1.2 (8) leads to one of the basic elementary inequalities.

Take the $(n+1)$-th root of both sides in 1.2 (8) and put $n+1 = \dfrac{1}{r}$, and $y = x - 1$ to get:

$$\text{if} \quad y \geq -1,\ y \neq 0, \quad \text{then} \quad (1+y)^r < 1 + ry; \tag{1}$$

This inequality (1), or ($\sim$1), can be shown to hold for arbitrary exponents r; [*AI p.34; HLP p.40*], [*Herman, Kučera & Šimša p.109*].

THEOREM 1 [BERNOULLI'S INEQUALITY] *If $x \geq -1, x \neq 0$, and if $0 < \alpha < 1$ then*

$$(1+x)^\alpha < 1 + \alpha x; \tag{B}$$

if either $\alpha > 1$ or $\alpha < 0$, when of course $x \neq -1$, then ($\sim$ B) holds.

□ (*i*) $0 < \alpha < 1$

Let $f(x) = 1 + \alpha x - (1+x)^\alpha$, $x \geq -1$. Then $f'(x) = \alpha\big(1 - (1+x)^{\alpha-1}\big)$. Hence $f(0) = f'(0) = 0$ and it is easy to see that this point is the minimum of f. Alternatively we can look at $f''(x) = \alpha(1-\alpha)(1+x)^{\alpha-2}$, which shows that f is strictly convex, see 4.1 below for a definition.

Either argument shows that $f(x) > 0, x \neq 0$, which is just (B).

(*ii*) $\alpha < 0$ or $\alpha > 1$

The argument in (*i*) shows that ($\sim$B) holds.

(iii) Alternatively we can show that the result for $\alpha < 0$ follows from that for $\alpha > 0$

If $1 + \alpha x < 0$ there is nothing to prove. So suppose that $\alpha < 0$ and $1 + \alpha x > 0$ and choose β so that $\beta > 1$ and $0 < -\alpha/\beta < 1$. Then from case (i) $(1+x)^{-\alpha/\beta} < 1 - \dfrac{\alpha}{\beta}x$, or

$$(1+x)^{\alpha/\beta} > \frac{1}{1 - \dfrac{\alpha}{\beta}x} = \frac{1 + \dfrac{\alpha}{\beta}x}{1 - \dfrac{\alpha^2}{\beta^2}x^2} > 1 + \frac{\alpha}{\beta}x.$$

Hence $(1+x)^\alpha > (1 + \dfrac{\alpha}{\beta}x)^\beta > 1 + \alpha x$, by case (ii). □

Remark (i) This important result is called *Bernoulli's inequality*, although the original inequality of this name is ($\sim$B) in the case $\alpha = 2, 3, \ldots$; [*CE p.111; DI pp. 28–29*]. It is still the subject of study; see [*MPF pp.65–81*], [*Alzer 1990a,1991c; Mitrinović & Pečarić 1990a*].

Remark (ii) (B) is equivalent to many of the fundamental inequalities to be discussed; see for instance II 2.2.2 Remark (i), III 3.1.2 Theorem 6.

Remark (iii) The history of Bernoulli's inequality is discussed in [*Mitrinović & Pečarić 1993*].

Remark (iv) Proofs of (B) that differ from the one above are given later: see below 4.1 Example(iii), and II 2.2.2 Remark(i).

Remark (v) (B) can be extended to $\prod_{i=}^{n}(1 + x_i) > 1 + \sum_{i=1}^{n} x_i$, $-1 < \underline{x} < 1$, $\underline{x} \neq 0$; see [*AI p.35; DI p.29; HLP p.60*] and 3.3 Corollary 19(b). For further generalizations see II 2.5.3 Theorem 22 and [*Pečarić 1983a*].

(B) is given a symmetric form in the following corollary.

Corollary 2 *If $a, b > 0$, $a \neq b$ and $0 < \alpha < 1$ then*

$$\alpha a^{\alpha-1}(b-a) > b^\alpha - a^\alpha > \alpha b^{\alpha-1}(b-a), \tag{2}$$

or

$$\frac{b^\alpha}{a^{\alpha-1}} < \alpha b + (1-\alpha)a; \tag{3}$$

if $\alpha > 1$ or $\alpha < 0$ then ($\sim$2) and ($\sim$3) hold.

□ Substitute $x = \dfrac{b}{a} - 1$ in (B) to give the left inequality in (2); interchange a and b to get the right inequality. □

REMARK (vi) A direct proof of (2) can be given using the mean-value theorem of differentiation[1]; [*Rüthing 1984*]

REMARK (vii) For a refinement of an integral form of (B) see [*Alzer1991c*].

It is possible to obtain more precise inequalities; [*HLP p.41*].

THEOREM 3 *If $x > 1$, and $0 < r < 1$, or if $x < 1$ and $r > 1$ then*

$$\frac{1}{2}(1-r)\frac{(x-1)^2}{x^2} < (x-1) - \frac{x^r - 1}{r} < \frac{1}{2}(1-r)x(x-1)^2. \tag{4}$$

If $x < 1$ and $0 < r < 1$ or if $x > 1$ and $r > 1$ then (~4) holds .

□ These inequalities follow using a calculus argument similar to that used to prove (B). □

COROLLARY 4

$$(1+x)^r = 1 + rx + O(x^2), \text{ as } x \to 0; \tag{5}$$

$$\left(1 + O(r^2)\right)^{1/r} = 1 + O(r), \text{ as } r \to 0. \tag{6}$$

□ These follow easily from (4); [note that if $x > 0$, then $x^r = 1 + O(r)$ as $r \to 0$.]

□

REMARK (viii) An alternative sharpening of (B) can be found in [*Haber*].

2.2 INEQUALITIES INVOLVING SOME ELEMENTARY FUNCTIONS (a) THE EXPONENTIAL FUNCTION [*DI pp.81–83*] If $x \neq e$ then

$$e^x > x^e. \tag{7}$$

To see this note that $y = x/e$ is a tangent to the curve $y = \log x$ at the point $(e, 1)$. Hence by the strict concavity of the log function, see 4.1 Corollary 3 below, $\dfrac{x}{e} > \log x, \quad x \neq e$, which is equivalent to (7).

REMARK (i) The graphs of e^x and x^e touch at (e, e^e). In general the graphs of a^x and x^a, $a > 0$, cross at the two solutions of the equation $a^x = x^a$, $x = a$, and

[1] The *mean-value theorem of differentiation*, sometimes named after Lagrange, states: *If f is continuous on $[a,b]$, differentiable on $]a,b[$ then there is a c, $a<c<b$, such that $f'(c) = \frac{f(b)-f(a)}{b-a}$*; [*CE p.1153*]. We call such a point c a *mean-value point of f*. Further note that the mean-value theorem of differentiation can be used to prove that: if $f' \geq 0$ and every sub-interval contains a point at which $f' > 0$, in particular if $f' \geq 0$ with $f'(x) = 0$ at only a finite number of points, then f is strictly increasing.

$x = \alpha$, say. If $a = e, \alpha = a$, if $1 < a < e$ then $\alpha > e$, while if $a > e$, then $1 < \alpha < e$; for $a \le 1$ take $\alpha = \infty$. If $I = [\alpha, a]$, or $[a, \alpha]$, then $a^x < x^a$ if $x \in \overset{\circ}{I}$ and $a^x > x^a$ if $x \notin I$; [*Bullen 2000; Qi & Debnath 2000b; Smirnov*].
If $x \neq 0$ then

$$e^x > 1 + x. \tag{8}$$

This follows in several ways: (i) the strict convexity of the function e^x implies by 4.1 Corollary 3 that at each point its graph lies above the tangent, and $y = x + 1$ is the tangent to the graph at $x = 0$;
(ii) by Taylor's theorem[2] since $e^x = 1 + x + \dfrac{x^2}{2}e^y$, for some y between 0 and x;
(iii) note that $f(x) = e^x - 1 - x$ has a unique maximum at $x = 0$.

Remark (ii) Of course the Taylor's theorem argument can be used to prove that $e^x > 1 + x + \cdots + \dfrac{x^n}{n!}$, $x > 0$. In particular, taking $n = 2$ and replacing x by $x - 1$ we get $e^{x-1}/x > \dfrac{1}{2}(x + 1/x)$, $x > 1$.
Another well-known fact is that

$$\Big(1 + \frac{1}{n}\Big)^n < e < \Big(1 + \frac{1}{n}\Big)^{n+1}, \quad n \in \mathbb{N}^*.$$

It can be shown that the left-hand side increases to e, and the right-hand side decreases to e, as $n \to \infty$; see II 2.5.3(δ).
A simple deduction from these inequalities is a crude estimate for the factorial function $n!$, $n \in \mathbb{N}^*$; [*Klambauer p.410*].

$$\frac{n^n}{e^{n-1}} < n! < \frac{n^{n+1}}{e^{n-1}}.$$

More generally, [*DI p.81*], [*Wang C L 1989a*], if $x \neq 0$,

$$e^x\Big(1 + \frac{1}{n}\Big)^{-x} < \Big(1 + \frac{x}{n}\Big)^n < e^x, \quad n \in \mathbb{N}^*,$$

which shows that $\lim_{n\to\infty} \big(1 + \dfrac{x}{n}\big)^n = e^x$.
(b) The Logarithmic Function [*DI pp.159–160*] If $x > 0, x \neq 1$ then

$$\frac{x - 1}{x} < \log x < x - 1. \tag{9}$$

[2] Taylor's theorem states: *if f has $n, n \ge 0$, continuous derivatives on $[a,b]$, and if $f^{(n+1)}$ exists on $]a,b[$ then $f(x) = T_{n+1}(f;a;x) + R_{n+1}(f;a;x)$ where $T_{n+1}(f;a;x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the* Taylor polynomial of order $n+1$ centred at a, *and* $R_{n+1}(f;a;x) = \frac{1}{(n-1)!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$ *is the* $(n+1)$-st Taylor remainder centred at a; also for some $c, a<c<b$, $R_{n+1}(f;a;x) = f^{(n+1)}(c)\frac{(b-a)^{n+1}}{(n+1)!}$. The expression $T_{n+1} + R_{n+1}$ is called the *Taylor expansion of f*. [*EM9 pp.1355–136*].

The right-hand inequality is immediate because $\log x = \int_1^x \frac{1}{t}\,dt < x-1$; or because if $f(x) = 1-x+\log x$ then f has a unique maximum at $x = 1$. In addition a simple change of variable changes (8) into the right-hand inequality in (9). The left-hand inequality follows by a simple change of variable in the right-hand inequality.

REMARK (iii) For another amusing way of proving (9) see 4.5.3 Example (iv).

A useful inequality is the following, [*Wang C L 1989a*]; if $x \neq 1$ then

$$\log x < n(x^{1/n} - 1) < x^{1/n} \log x, \quad n \in \mathbb{N}^*,$$

which shows that $\lim_{n\to\infty} n(x^{1/n} - 1) = \log x$.

If $x > -1$ then

$$\frac{2|x|}{2+x} < |\log(1+x)| < \frac{|x|}{\sqrt{1+x}}. \tag{10}$$

Consider the functions $f(x) = \log(1+x) - \dfrac{2x}{2+x}$, and $g(x) = \log(1+x) - \dfrac{x}{\sqrt{1+x}}$. Then simple calculations show that $f(0) = g(0) = 0$ and $g' \leq 0 \leq f'$, which implies (10).

REMARK (iv) For other inequalities involving the logarithmic function see II 5.5 Corollary 13 and Remark (ii), and [*Mitrinović 1968b*].

(c) THE BINOMIAL FUNCTION [*DI pp.35–37*] Another useful inequality is the following.

THEOREM 5 *If either (a) $x > 0$ and $0 < q < p$, or (b) $0 < q < p$ and $-q < x < 0$, or (c) $q < p < 0$ and $-p > x > 0$ then*

$$\left(1 + \frac{x}{q}\right)^q \leq \left(1 + \frac{x}{p}\right)^p. \tag{11}$$

Inequality (~11) holds if either (d) $q < 0 < p$ and $q > x > 0$; or (e) $q < 0 < p$ and $-p < x < 0$.

□ For a simple proof by Bush see [*AI p.365*], [*Bush*]. □

This inequality is a generalization of (B); for a special case see 3.3 (20), and II 2.5.3(δ).

In addition there is:

$$(1+x)^n > 1 + nx + \cdots + \binom{n}{k} x^k, \tag{12}$$

provided $x > 0, 0 \leq k \leq n, n \neq 1$. Inequality (12) and the inequality in Remark (ii) are special cases of *Gerber's inequality*, [*DI pp.101–102*], [*Alić, Bullen, Pečarić & Volenec 1998; Alzer 1990a; Gerber 1968*].

(d) THE CIRCULAR AND HYPERBOLIC FUNCTIONS [*DI pp.131–132, 250–252*]

$$\cos x < \frac{\sin x}{x} < 1, \quad x \neq 0; \quad \sin x < x < \tan x, \quad 0 < x < \frac{\pi}{2}; \tag{13}$$

$$\cosh x > \frac{\sinh x}{x} > 1, \; x \neq 0; \quad \tanh x < x < \sinh x, \quad x > 0. \tag{14}$$

REMARK (v) For an extension of the right-hand set of inequalities see II 5.5 Remark (vi). Further the left-hand side of the first set of inequalities in (14) can be improved to $(\cosh x/3)^3$, see VI 2.1.1 Remark (x).

(e) MAXIMUM AND MINIMUM FUNCTION INEQUALITIES [*DI pp.134*] If $\underline{a}$ and $\underline{b}$ are real n-tuples then

$$\max \underline{a} + \min \underline{b} \leq \max(\underline{a} + \underline{b}) \leq \max \underline{a} + \max \underline{b}, \tag{15}$$

$$\max \underline{a} \min \underline{b} \leq \max \underline{a}\,\underline{b} \leq \max \underline{a} \max \underline{b}. \tag{16}$$

The right-hand inequalities are strict unless for some i, $a_i = \max \underline{a}$ and $b_i = \max \underline{b}$. Similar inequalities hold for sequences or infinite sets of real numbers but in that case we usually write inf and sup for min and max respectively.

(f) A SIMPLE RATIONAL FUNCTION Let $a, b > 0$ and put

$$f(x) = \frac{x}{a} + \frac{b}{x}, \quad x > 0.$$

Noting that $f'(x) = \dfrac{1}{ax^2}(x^2 - ab)$ we see that f has a unique minimum at $x = x_0 = \sqrt{ab}$ and that $f(x_0) = y_0 = 2\sqrt{b/a}$. In particular f is strictly convex, strictly decreasing on the interval $]0, x_0]$, and strictly increasing on the interval $[x_0, \infty[$. Alternatively the equation $f(x) = y$ has two roots, x_1, x_2 say, and $x_1 x_2 = ab$, $x_1 + x_2 = ay$. So

$$\frac{x}{a} + \frac{b}{x} \geq 2\sqrt{\frac{b}{a}}, \tag{17}$$

with equality if and only if $x = \sqrt{ab}$;

further if $x \geq x_3 \geq \sqrt{ab}$ then

$$\frac{x}{a} + \frac{b}{x} \geq \frac{x_3}{a} + \frac{b}{x_3}, \tag{18}$$

with equality if and only if $x = x_3$;

and finally if $x_1 \leq x \leq x_2$ then

$$2\sqrt{\frac{b}{a}} \leq \frac{x}{a} + \frac{b}{x} \leq \frac{x_1}{a} + \frac{b}{x_1} = \frac{x_2}{a} + \frac{b}{x_2}, \tag{19}$$

with equality on the left if and only if $x = x_0$, and on the right if and only if $x = x_1$, or $x = x_2$.

Some special cases are of some interest.

(i) If $a = b = 1$ inequality (17) gives the familiar

$$x + \frac{1}{x} > 2, \; x \neq 1. \tag{20}$$

A direct proof follows immediately on noting that for $x \neq 1$ (20) is equivalent to $(x-1)^2 > 0$, or by noting that $(x-1)(1-1/x) > 0$.

(ii) If $a = \dfrac{1}{\alpha} \leq 1$, $b = 1$ then $x_0 = \dfrac{1}{\sqrt{\alpha}} < 1$ So taking $x_3 = 1$ in (18) we get that if $x \geq 1$ then

$$\alpha x + \frac{1}{x} > \alpha + 1, \tag{21}$$

with equality if and only if $x = 1$. This is an inequality due to Chong, [*Chong K M 1983*], A direct proof can be given as follows:

$$\begin{aligned} \alpha x + \frac{1}{x} =& (\alpha - 1)x + (x + \frac{1}{x}) \\ \geq& (\alpha - 1)x + 2, \quad \text{by (20)}, \\ \geq& \alpha + 1. \end{aligned}$$

For equality at the first step we need $x = 1$, and then there is equality at the second step.

Inequality (21) can be written in an equivalent form: put $\alpha = u/v$, $u \geq v > 0$ then

$$ux + \frac{v}{x} \geq u + v, \tag{22}$$

with equality if and only if $x = 1$.

(iii) Given m, M, $0 < m < M$ choose $x_1 = m, x_2 = M$ and $a = b = \sqrt{mM}$ then from (19)

$$2 \leq \frac{x}{\sqrt{mM}} + \frac{\sqrt{mM}}{x} \leq \sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}} = \frac{m+M}{\sqrt{mM}}, \tag{23}$$

with equality in the left inequality if and only if $x = \sqrt{mM}$, and in the right inequality if and only if $x = m$ or $x = M$.

Inequality (20) has been generalized by Korovkin; [*Korovkin p.8*].

LEMMA 6 *If $\underline{x}$ is a positive n-tuple, $n \geq 2$, then*

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq n, \tag{24}$$

with equality if and only if $\underline{x}$ is constant.

□ The proof is by induction, noting that the case $n = 2$ is just (20) with $a = b = x_2$ and $x = x_1$. So assume (24) holds for integers less than n, and that $x_n = \min \underline{x}$. Then the left-hand side of (24) is equal to

$$\begin{aligned}
&\frac{x_1}{x_2}+\cdots+\frac{x_{n-1}}{x_1}+\left(\frac{x_{n-1}}{x_n}+\frac{x_n}{x_1}-\frac{x_{n-1}}{x_1}\right)\\
&\quad\geq(n-1)+\left(\frac{x_{n-1}}{x_n}+\frac{x_n}{x_1}-\frac{x_{n-1}}{x_1}\right), \quad \text{by the induction hypothesis,}\\
&\quad= n+\frac{(x_n-x_{n-1})(x_n-x_1)}{x_1x_n}\geq n.
\end{aligned}$$

The case of equality is immediate. □

Remark (vi) For further generalizations see II 2.4.5 Lemma 19 and II 2.5.3 (ϕ) Theorem 25.

3 Properties of Sequences

3.1 Convexity and Bounded Variation of Sequences Let $\underline{a} = (a_1, a_2, \ldots)$ be a real sequence and define the sequences $\Delta^k \underline{a}$, $k \in \mathbb{N}^*$, by recursion as follows:[3]

$$\Delta^1 a_n = \Delta a_n = a_n - a_{n+1}, \quad n \in \mathbb{N}^*; \qquad \Delta^k a_n = \Delta\big(\Delta^{k-1} a_n\big), \quad n \in \mathbb{N}^*, k \geq 2.$$

Then it easily follows that

$$\Delta^k a_n = \sum_{i=0}^{n} (-1)^i \binom{k}{i} a_{n+i}, \; n \in \mathbb{N}^*, \tag{1}$$

and that if $1 \leq j \leq k$,

$$\Delta^j a_n = \sum_{i=1}^{\infty} \binom{i+k-j-2}{k-j-1} \Delta^k a_{i+n-1}. \tag{2}$$

Convention Throughout this section we assume that:

$$\binom{-1}{-1} = 1, \text{ and, if } n \geq 2, \text{ that } \binom{n-2}{-1} = 0.$$

---

[3] We will also use the notation $\tilde{\Delta}^k \underline{a}$, where $\tilde{\Delta} a_n = -\Delta a_n = a_{n+1} - a_n$, $\tilde{\Delta}^k a_n = \tilde{\Delta}\big(\tilde{\Delta}^{k-1} a_n\big)$, $n \in \mathbb{N}^*$, $k \geq 2$. So $\tilde{\Delta}^k a_n = (-1)^k \Delta a_n$, $k, n \in \mathbb{N}^*$.

DEFINITION 1 *(a) A sequence $\underline{a}$ is said to be* k-convex, $k \in \mathbb{N}^*$, *if the sequence $\Delta^k \underline{a}$ is non-negative.*

(b) A sequence $\underline{a}$ is said to be of bounded k-variation, $k \in \mathbb{N}^*$, *if*

$$\sum_{i=1}^{\infty} \binom{i+k-2}{k-1} |\Delta^k a_i| < \infty.$$

EXAMPLE (i) A sequence that is 1-convex is exactly a decreasing sequence; a 2-convex sequence, usually called *a convex sequence*, has $a_n - 2a_{n+1} + a_{n+2} \geq 0$, $n \geq 1$.

EXAMPLE (ii) A sequence $\underline{a}$ is of 1-bounded variation, or just *bounded variation*, if $\sum_{i=1}^{\infty} |a_i - a_{i+1}| < \infty$. Clearly the sequence of partial sums of a series is of bounded variation if and only if the series is absolutely convergent.

REMARK (i) If f is a convex function, see 4.1 Definition 1, then $a_n = f(n), n = 1, \ldots$ is a convex sequence; more generally if f is k-convex, see 4.7, then $a_n = (-1)^n f(n), n = 1, \ldots$ is k-convex. [*DI pp.64–65, 191, EM2 p.419*].

REMARK (ii) If the sequence $-\underline{a}$ is convex, k-convex we say that $\underline{a}$ is *concave*, *k-concave*.

REMARK (iii) The definition of a *k-convex n-tuple*, $n > k$, is immediate.

The partial sums of a series can be written as the difference of the partial sums of two positive termed series, $B_n = \sum_{i=1}^{n} b_n = \sum_{i=1}^{n} b_n^+ - \sum_{i=1}^{n} b_n^- = P_n - N_n$; and the series $\sum b_n$ is absolutely convergent if and only if both of the series $\sum_{i=1}^{n} b_n^+$ and $\sum_{i=1}^{n} b_n^-$ converge. This can be expressed in terms of the present concepts as: if a sequence is bounded then it is of bounded variation if and only if it can be written as the difference of two bounded decreasing sequences. The main result of this section, Theorem 3 below, is a generalization of this result due to Dawson, [*Dawson*].

First we prove a basic lemma

LEMMA 2 *If $\underline{a}$ is a sequence that is bounded and k-convex, respectively of bounded k-variation, $k \geq 2$, then:*

(a) $\underline{a}$ *is p-convex, respectively bounded p-variation, $1 \leq p \leq k-1$;*

(b)
$$\lim_{n \to \infty} \binom{n+j-1}{j} \Delta^j a_n = 0, \ 1 \leq j \leq k-1;$$

(c)
$$\sum_{i=1}^{\infty} \binom{i+j-2}{j-1} \Delta^j a_i = a_1 - \lim_{n \to \infty} a_n, \ 1 \leq j \leq k.$$

□ (i): $k = 2$ and $\underline{a}$ is bounded and convex.

This implies that $\Delta\underline{a}$ is decreasing and so $\lim_{n\to\infty} \Delta a_n$ exists. Further the assumption that $\underline{a}$ is bounded implies that this limit is zero; hence $\Delta\underline{a} \geq 0$.

This is (a) in this case.

Now

$$a_1 - a_{n+1} = \sum_{i=1}^{n} \Delta a_i = \sum_{i=1}^{n-1} i\Delta^2 a_i + n\Delta a_n, \tag{3}$$

and noting that $\Delta\underline{a} \geq 0, \Delta^2\underline{a} \geq 0$, and that $\underline{a}$ is bounded, the series in (3) converges, which implies (c).

Hence $\lim_{n\to\infty} a_n$ exists and is finite, which from (3) and the convergence of the series shows that $\lim_{n\to\infty} n\Delta a_n$ exists and is non-negative, and so is zero.

This gives (b) and completes the proof of this case.

(ii): $k = 2$ and $\underline{a}$ is bounded and of bounded 2-variation.

From (3) the sequence $n\Delta a_n, n \in \mathbb{N}^*$, is bounded and hence $\sum_{i=1}^{\infty} |\Delta a_i|$ converges, which is (a) in this case.

Further $\lim_{n\to\infty} a_n$ exists, and so from (3) $\lim_{n\to\infty} n\Delta a_n$ exists, A say, and assume that $A \neq 0$. Then for some $n_0 \in \mathbb{N}^*$,

$$\begin{aligned}\frac{|A|}{2}\sum_{i=n_0}^{\infty}\left|1 - \frac{\Delta a_{i+1}}{\Delta a_i}\right| &\leq \sum_{i=n_0}^{\infty} i|\Delta a_i|\left|1 - \frac{\Delta a_{i+1}}{\Delta a_i}\right| \\ &= \sum_{i=n_0}^{\infty} i|\Delta^2 a_i| < \infty.\end{aligned}$$

Hence $\prod_{i=n_0}^{\infty}\left(\frac{\Delta a_{i+1}}{\Delta a_i}\right)$ converges absolutely to some non-zero limit. This contradicts the convergence of $\sum_{i=1}^{\infty} |\Delta a_i|$, and so $A = 0$.

The lemma in this case now follows from (3).

(iii): $k > 2$ and $\underline{a}$ is bounded and k-convex.

Since $\Delta^k\underline{a} = \Delta^2(\Delta^{k-2}\underline{a})$ the sequence $\Delta^{k-2}\underline{a}$ is convex and bounded. Hence (a), in this case, follows by induction.

In particular $\underline{a}$ is convex and so (b) and (c) hold with $j = 1$, and $j = 1, 2$ respectively.

Suppose that $1 < j_0 < k$ and that (c) holds for $j = j_0$. Then since

$$\sum_{i=1}^{n}\binom{i+j_0-2}{j_0-1}\Delta^{j_0} a_i = \sum_{i=1}^{n-1}\binom{i+j_0-1}{j_0}\Delta^{j_0+1} a_i + \binom{n+j_0-1}{j_0}\Delta^{j_0} a_n, \tag{4}$$

this implies $\sum_{i=1}^{\infty}\binom{i+j_0-1}{j_0}\Delta^{j_0+1} a_i < \infty$, and $\lim_{n\to\infty}\binom{n+j_0-1}{j_0}\Delta^{j_0+1} a_n$ exists, A say; assume that $A \neq 0$.

Since

$$\binom{n+j_0-1}{j_0}\Delta^{j_0}a_n = \left(\sum_{i=1}^{n}\binom{i+j_0-2}{j_0-1}\right)\Delta^{j_0}a_n \le n\binom{n+j_0-2}{j_0-1}\Delta^{j_0}a_n,$$

it follows that there is an $n_0 \in \mathbb{N}^*$ such that if $n > n_0$ then

$$\frac{A}{2n} \le \frac{1}{n}\binom{n+j_0-1}{j_0}\Delta^{j_0+1}a_n \le \binom{n+j_0-2}{j_0-1}\Delta^{j_0}a_n.$$

This contradicts the convergence of $\sum_{i=1}^{\infty}\binom{i+j_0-2}{j_0-1}\Delta^{j_0}a_i$; so we have (c) in the case $k = j_0$.

Hence $A = 0$; that is (b) holds with $j = j_0$ and so, from (4) and (c) with $j = j_0$,

$$\sum_{i=1}^{\infty}\binom{i+j_0-1}{j_0}\Delta^{j_0+1}a_i = \sum_{i=1}^{\infty}\binom{i+j_0-2}{j_0-1}\Delta^{j_0+1}a_i = a_1 + \lim_{n\to\infty} a_n;$$

that is to say we have (c) with $j = j_0 + 1$.

This completes the proof of this case.

(*iv*): $k > 2$ and $\underline{a}$ is bounded and of bounded k-variation.

Suppose that no subsequence of $\binom{n+k-2}{k-1}\Delta^{k-1}a_n, n \in \mathbb{N}^*$ converges to zero. Then for some $n_0 \in \mathbb{N}^*$ and $A > 0$ we have that

$$\begin{aligned} A\sum_{i=n_0}^{\infty}\left|1 - \frac{\Delta^{k-1}a_{i+1}}{\Delta^{k-1}a_i}\right| &\le \sum_{i=n_0}^{\infty}\binom{i+k-2}{k-1}|\Delta^{k-1}a_i|\left|1 - \frac{\Delta^{k-1}a_{i+1}}{\Delta^{k-1}a_i}\right| \\ &= \sum_{i=n_0}^{\infty}\binom{i+k-2}{k-1}|\Delta^k a_i|. \end{aligned}$$

Hence by arguing as in (*ii*),

$$\lim_{n\to\infty}\Delta^{k-1}a_n = A \ne 0. \tag{5}$$

Since $\underline{a}$ is of bounded k-variation, and

$$\sum_{i=1}^{\infty} i\Delta^2|\Delta^{k-2}a_i| \le \sum_{i=1}^{\infty}\binom{i+k-2}{k-1}|\Delta^k a_i| < \infty,$$

we have that $\Delta^{k-2}\underline{a}$ is of bounded 2-variation.

So, by the case $k = 2$, $\Delta^{k-2}\underline{a}$ is of bounded variation; that is $\sum_{i=1}^{\infty}|\Delta(\Delta^{k-2}a_i)| = \sum_{i=1}^{\infty}|\Delta^{k-1}a_i| < \infty$, which contradicts (5). Hence there is a strictly increasing

sequence $n_i \in \mathbb{N}^*, i \in \mathbb{N}^*$, such that $\lim_{i\to\infty} \binom{n_i+k-2}{k-1}\Delta^{k-1}a_{n_i} = 0$, which implies that for some M, $\binom{n_i+k-2}{k-1}|\Delta^{k-1}a_{n_i}| < M$. Now,

$$\begin{aligned}\sum_{i=1}^{n_j}\binom{i+k-3}{k-2}|\Delta^{k-1}a_i| &\le \sum_{i=1}^{n_j-1}\binom{i+k-2}{k-1}|\Delta^k a_i| + \binom{n_j+k-2}{k-1}|\Delta^{k-1}a_{n_j}| \\ &< \sum_{i=1}^{n_j-1}\binom{i+k-2}{k-1}|\Delta^k a_i| + M.\end{aligned}$$

Since $\underline{a}$ is of bounded k-variation this last inequality implies that $\underline{a}$ is also of bounded $(k-1)$-variation, and a simple induction completes the proof of (a) in this case.
The two series in (4) with $j_0 = k-1$ converge, and $\lim_{i\to\infty} \binom{n_i+k-2}{k-1}\Delta^{k-1}a_{n_i} = 0$; so from (4), $\lim_{n\to\infty} \binom{n+k-2}{k-1}\Delta^{k-1}a_n = 0$, and (b) follows by induction.
Finally, from (4) with $j_0 = k-1$, $\sum_{i=1}^{\infty}\binom{i+k-2}{k-1}\Delta^k a_i = \sum_{i=1}^{\infty}\binom{i+k-3}{k-2}\Delta^{k-1}a_i$, and (c) follows by induction, and (5) is already proved when $k = 2$. □

We can now prove the main result of this section.

THEOREM 3 *If $\underline{a}$ is bounded then it is of bounded k-variation if and only if it is the difference of two bounded k-convex sequences*

□ We may assume $k > 1$ as the case $k = 1$ is well known, see the comments preceding Lemma 2.
(i) Assume that $\underline{a} = \underline{b} - \underline{c}$, where $\underline{b}$ and $\underline{c}$ are bounded and k-convex.
Then,

$$\sum_{i=1}^{\infty}\binom{i+k-2}{k-1}|\Delta^k a_i| \le \sum_{i=1}^{\infty}\binom{i+k-2}{k-1}|\Delta^k b_i| + \sum_{i=1}^{\infty}\binom{i+k-2}{k-1}|\Delta^k c_i| < \infty,$$

by Lemma 2(c) with $j = k$.

(ii) Assume that $\underline{a}$ is bounded and of bounded k-variation.
Then by Lemma 2(c) $\lim_{n\to\infty} a_n$ exists, A say.
From a simple extension of Lemma 2(c), $\sum_{i=1}^{\infty}\binom{i+k-2}{k-1}\Delta^k a_{i+n} = a_{n+1} - A$.
Now define

$$b_{n+1} = \sum_{i=1}^{\infty}\binom{i+k-2}{k-1}|\Delta^k a_{i+n}|,\ n \in \mathbb{N}.$$

Then $\underline{b}$ is bounded and

$$\begin{aligned}\Delta b_n =&\sum_{i=1}^{\infty}\binom{i+k-2}{k-1}|\Delta^k a_{i+n-1}| - \sum_{i=1}^{\infty}\binom{i+k-2}{k-1}|\Delta^k a_{i+n}|\\
=&|\Delta^k a_{i+n-1}| + \sum_{i=2}^{\infty}\left(\binom{i+k-1}{k-1} - \binom{i+k-3}{k-1}\right)|\Delta^k a_{i+n-1}|\\
=&|\Delta^k a_{i+n-1}| + \sum_{i=2}^{\infty}\binom{i+k-3}{k-2}))|\Delta^k a_{i+n-1}|\\
=&\sum_{i=1}^{\infty}\binom{i+k-3}{k-2}|\Delta^k a_{i+n-1}| \geq 0,\end{aligned}$$

so, by an obvious induction,

$$\Delta^j b_n = \sum_{i=1}^{\infty}\binom{i+k-j-2}{k-j-1}|\Delta^k a_{i+n-1}| \geq 0,\ 1 \leq j \leq k;$$

showing that $\underline{b}$ is k-convex.
Now define $\underline{c} = \underline{b} - \underline{a}$ when, using (2),we have for $1 \leq j \leq k$ that,

$$\begin{aligned}\Delta^j c_n =&\Delta^j b_n - \Delta^j a_n\\
=&\sum_{i=1}^{\infty}\binom{i+k-j-2}{k-j-1}|\Delta^k a_{i+n-1}| - \sum_{i=1}^{\infty}\binom{i+k-j-2}{k-j-1}\Delta^k a_{i+n-1} \geq 0.\end{aligned}$$

This completes the proof. □

Remark (iv) More results on convex sequences can be found in the following reference, where further references are given; [*Pečarić, Mesihović, Milovanović & Stojanović*].

3.2 Log-convexity of Sequences

Definition 4 *Given two real sequences* $\underline{a} = (a_0, a_1, \dots)$ *and* $\underline{b} = (b_0, b_1, \dots)$ *the sequence* $\underline{c} = \underline{a} \star \underline{b}$ *defined by*

$$c_n = \sum_{r=0}^{n} a_r b_{n-r},\ n \in \mathbb{N}, \tag{6}$$

or by the product of the formal power series

$$\sum_{r\in\mathbb{N}} c_r x^r = \left(\sum_{r\in\mathbb{N}} a_r x^r\right)\left(\sum_{r\in\mathbb{N}} b_r x^r\right), \tag{7}$$

is called the convolution of $\underline{a}$ and $\underline{b}$.

DEFINITION 5 *(a) If* $0 \le \alpha \le \infty$ *then the positive sequence* $\underline{c} = (c_0, c_1, \ldots)$ *is said to be* α-logarithmically convex, *or just* α-log-convex, *if for* $n \in \mathbb{N}^*$,

$$c_n^2 \le \begin{cases} \left(\dfrac{\alpha+n-1}{\alpha+n}\right)\left(\dfrac{n+1}{n}\right) c_{n+1}c_{n-1}, & \text{if } \alpha \ne \infty, \\ \left(\dfrac{n+1}{n}\right) c_{n+1}c_{n-1}, & \text{if } \alpha = \infty. \end{cases} \tag{8}$$

When $\alpha = 1$ *we will just say* log-convex, *and when* $\alpha = \infty$ *we will say* weakly log-convex.

(b) If inequality (~8) holds then we say that the sequence is α-log-concave; *or just* log-concave *if* $\alpha = 1$; *if* $\alpha = \infty$ *we say the sequence is* strongly log-concave. *[DI p.163].*

REMARK (i) In the convex case the smaller α the stronger the condition to be satisfied, while in the concave case the larger α the stronger the condition.

REMARK (ii) The sequence $\underline{c}$ is log-convex if and only if

$$c_n^2 \le c_{n+1}c_{n-1}, \quad n \in \mathbb{N}^*; \tag{9}$$

equivalently if and only if

$$\frac{c_1}{c_0} \le \frac{c_2}{c_1} \le \cdots \le \frac{c_m}{c_{m+1}} \le \ldots \le \frac{c_{m+n}}{c_{m+n+1}} \le \cdots, \quad n, m \in \mathbb{N}^*; \tag{10}$$

in the case of log-concavity the inequalities (~9) and (~10) hold.

EXAMPLE (i) The sequence $\underline{\alpha}$ defined by

$$\alpha_n = (-1)^n \binom{-\alpha}{n} = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!}, \ n \in \mathbb{N}^* \tag{11}$$

is α'-log-convex, respectively concave, if $\alpha' \ge \alpha$, respectively $\alpha' \le \alpha$.

EXAMPLE (ii) The sequence $b_n = \dfrac{1}{n}, n \in \mathbb{N}^*$, is 0-log-concave and so α-log-convex for all $\alpha \ge 0$.

EXAMPLE (iii) The sequence $c_n = \dfrac{1}{n!}, n \in \mathbb{N}^*$, is weakly log-convex and so α-log-convex for all $\alpha \le \infty$.

EXAMPLE (iv) From 1 Corollary 8 we see that if a polynomial has all of its roots real then its coefficients are log-concave; in particular $\{\binom{n}{k}, \ 0 \le k \le n\}$ is log-concave.

REMARK (iii) It is useful to notice that $\underline{c}$ is α-log-convex, respectively concave, if and only if $\tilde{\underline{c}}$ is log-convex, respectively concave, where for $n \in \mathbb{N}^*$, $\tilde{c}_n = c_/\alpha_n$ if $0 < \alpha < \infty$, $\tilde{c}_n = nc_n$ if $\alpha = 0$, and $\tilde{c}_n = n!c_n$ if $\alpha = \infty$; where α_n is defined in (11).

The results of this section show that the property of log-concavity is preserved under the operation of convolution. Not all the results are needed in the sequel but are included for the sake of completeness.

THEOREM 6 *If $\underline{a}$ and $\underline{b}$ are log-convex, respectively log-concave, so is $\underline{c}$ where*

$$c_n = \sum_{r=0}^{n} \binom{n}{r} a_r b_{n-r}, n \in \mathbb{N}. \tag{12}$$

□ (*i*): $\underline{a}$ and $\underline{b}$ are log-convex; [*Davenport & Pólya*].
The proof of inequalities (9) is by induction on n.
The case $n = 1$, $c_0c_2 - c_1^2 = a_0^2(b_0b_2 - b_1^2) + b_0^2(a_0a_2 - a_1^2) \geq 0$, follows from the log-convexity of $\underline{a}$ and $\underline{b}$.
Now assume that (9) holds for all n, $1 \leq n \leq k-1$.
Then, noting that $\binom{k}{r} = \binom{k-1}{r} + \binom{k-1}{r-1}$, we can write $c_k = c'_{k-1} + c''_{k-1}$ where the sequences $\underline{c}', \underline{c}''$ are defined using (12) with the sequences $\underline{a}', \underline{b}$, and $\underline{a}, \underline{b}''$ respectively; where $\underline{a}' = \underline{a}'_0 = (a_1, a_2, \ldots)$, $\underline{b}'' = \underline{b}'_0 = (b_1, b_2, \ldots)$.
Hence,

$$\begin{aligned}
c_k^2 =& (c'_{k-1} + c''_{k-1})^2 = c'^2_{k-1} + 2c'_{k-1}c''_{k-1} + c''^2_{k-1} \\
\leq& c'_{k-2}c'_k + 2\sqrt{c'_{k-2}c'_kc''_{k-2}c''_k} + c''_{k-2}c'_k, \text{ by the induction hypothesis },\\
\leq& c'_{k-2}c'_k + c'_{k-2}c''_k + c''_{k-2}c'_k + c''_{k-2}c'_k, \text{ by (GA), see II 2.2.1 (2)},\\
=& c_{k-1}c_{k+1},
\end{aligned}$$

as had to be proved.

(*ii*): $\underline{a}$ and $\underline{b}$ are log-concave; [*Whiteley 1962b,1965*].

Rewrite definition (12) as: $c_n = \sum_{r=0}^{n-1} \binom{n-1}{r} (a_{r+1}b_{n-r-1} + a_rb_{n-r})$.

So:

$$\begin{aligned}
c_n^2 -& c_{n-1}c_{n+1} = \\
&\left[\sum_{r=0}^{n} \binom{n}{r} a_r b_{n-r}\right] \left[\sum_{r=0}^{n-1} \binom{n-1}{r} (a_{r+1}b_{n-r-1} + a_rb_{n-r})\right] \\
&\qquad - \left[\sum_{r=0}^{n-1} \binom{n-1}{r} a_r b_{n-r-1}\right] \left[\sum_{r=0}^{n} \binom{n}{r} (a_{r+1}b_{n-r} + a_rb_{n-r+1})\right] \\
&= A + B.
\end{aligned}$$

where

$$A = \sum_{\substack{r,s\ge 1\\ r+s\le n+1}} (b_s b_{r+s+1} - b_{r+s} b_{s-1}) \left(\binom{n}{r+s-1}\binom{n}{r+s-1} a_{n-r-s+1} a_{n-s} \right.$$
$$\left. - \binom{n}{s-1}\binom{n-1}{r+s-1} a_{n-s+1} a_{n-r-s} \right),$$

and B is obtained from A by interchanging $\underline{a}$ and $\underline{b}$.

The log-concavity of $\underline{b}$ shows that the first bracket in A is non-negative, (10); further, since $\binom{n}{r+s-1}\binom{n-1}{s-1} > \binom{n}{s-1}\binom{n-1}{r+s-1}$, the second bracket in A exceeds $\binom{n}{s-1}\binom{n-1}{r+s-1}(a_{n-r-s+1}a_{n-s} - a_{n-s+1}a_{n-r+s})$, which is non-negative by the log-concavity of $\underline{a}$, (10).

A similar argument shows that B is non-negative and completes the proof. □

REMARK (iv) It is not difficult to see that there is equality in inequalities (9) for $\underline{c}$, defined by (12), if and only if equality occurs in all the corresponding inequalities for $\underline{a}$ and $\underline{b}$.

Noting that if $d_n = c_n x^n$, $n \in \mathbb{N}$, then $\underline{c}$ and $\underline{d}$ are α-log-concave together leads to the following generalization of Theorem 6.

COROLLARY 7 *If If $\underline{a}$ and $\underline{b}$ are log-convex, respectively log-concave, so is $\underline{c}(x,y)$ where*

$$c_n(x,y) = \sum_{r=0}^{n} \binom{n}{r} a_r b_{n-r} x^r y^{n-r}, n \in \mathbb{N}, x > 0, y > 0. \tag{13}$$

THEOREM 8 *(a) If If $\underline{a}$ and $\underline{b}$ are log-concave, so is $\underline{a} \star \underline{b}$.*

(b) If If $\underline{a}$ is α-log-convex, and $\underline{b}$ is $(1-\alpha)$-log-convex, $0 < \alpha < 1$, then $\underline{a} \star \underline{b}$ is log-convex.

□ (a) [*Menon 1969a*] If $\underline{c} = \underline{a} \star \underline{b}$ then,

$$\begin{aligned} c_n^2 - c_{n-1}c_{n+1} =& \Big(\sum_{r=0}^{n} a_r b_{n-r}\Big)^2 - \Big(\sum_{r=0}^{n-1} a_r b_{n-r-1}\Big)\Big(\sum_{r=0}^{n+1} a_r b_{n-r+1}\Big) \\ =& \Big(\sum_{r=0}^{n-1} a_r b_{n-r}\Big)\Big(\sum_{r=0}^{n} a_r b_{n-r}\Big) - \Big(\sum_{r=0}^{n-1} a_r b_{n-r-1}\Big)\Big(\sum_{r=0}^{n} a_r b_{n-r+1}\Big) \\ & + a_n b_0 \Big(\sum_{r=0}^{n} a_r b_{n-r}\Big) - a_{n+1} b_0 \Big(\sum_{r=0}^{n-1} a_r b_{n-r-1}\Big) \\ =& A + B + C, \end{aligned}$$

say, where

$$\begin{aligned}
A =&\sum_{r=0}^{n-1}\sum_{k=1}^{n} a_r a_k\big(b_{n-r}b_{n-k} - b_{n-r-1}b_{n-k+1}\big) = \sum_{r=0}^{n-1}\sum_{k=1}^{n} d_{r,k}, \text{ say,}\\
B =&\sum_{r=0}^{n-1} a_r a_0\big(b_{n-r}b_n - b_{n-r-1}b_{n+1}\big),\\
C =&a_n b_0\sum_{r=0}^{n} a_r b_{n-r} - a_{n+1}b_0\sum_{r=0}^{n-1} a_r b_{n-r-1}\\
=&a_n b_n a_0 b_0 + b_0\sum_{r=1}^{n} b_{n-r}\big(a_n a_r - a_{n+1}a_{r-1}\big).
\end{aligned}$$

B and C are easily seen to be non-negative by the log-concavity of $\underline{a}$ and $\underline{b}$. Since, in A, $d_{r,r+1} = 1$ we get on combining $d_{r-1,k}$ and $d_{k-1,r}$ that,

$$\begin{aligned}
A =& \sum_{\substack{r,k=1\\ r<k+1}}^{n} \big(d_{r-1,k} + d_{k-1,r}\big)\\
=& \sum_{\substack{r,k=1\\ r<k+1}}^{n} \big(a_r a_{k-1} - a_{r-1}a_k\big)\big(b_{n-r}b_{n-k+1} - b_{n-r+1}b_{n-k}\big),
\end{aligned}$$

which is non-negative by (10).

(b) [*Davenport & Pólya*] Using (11), Example (i) and Remark (iii), and letting $\beta = 1-\alpha$,

$$c_n = \sum_{r=0}^{n} a_r b_{n-r} = \sum_{r=0}^{n} \alpha_r\beta_{n-r}\tilde{a}_r\tilde{b}_{n-r}. \tag{14}$$

Simple calculations from (11) lead to:

$$\begin{aligned}
\alpha_r\beta_{n-r} =&\binom{n}{r}\frac{(\alpha+r-1)!(\beta+n-r-1)!}{n!(\alpha-1)!(\beta-1)!}\\
=&\binom{n}{r}\frac{1}{(\alpha-1)!(\beta-1)!}\int_0^1 t^{\alpha+r-1}(1-t)^{\beta+n-r-1}\,\mathrm{d}t.
\end{aligned}$$

Substituting in (14) and using the notation of (13)

$$\begin{aligned}
c_n =&\frac{1}{(\alpha-1)!(\beta-1)!}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\tilde{c}_n(t,1-t)\,\mathrm{d}t\\
\le&\frac{1}{(\alpha-1)!(\beta-1)!}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\sqrt{\tilde{c}_{n-1}(t,1-t)\tilde{c}_{n+1}(t,1-t)}\,\mathrm{d}t,
\end{aligned}$$

by Corollary 7 and (9).

A simple application of (C)-$\int$, see VI 1.2.1 Theorem 3(b), gives (9) and completes the proof. □

REMARK (v) In the proof of part (a) of this theorem $C > 0$, and so the inequalities (9), for $\underline{c}$, are strict. In part (b) however these inequalities can be equalities, and are so if and only if all the inequalities for $\underline{a}$, and for $\underline{b}$ are equalities.

REMARK (vi) The results in (b) is best possible in that the analogue of (a) is in general false; further the result does not hold in general if $\alpha = 1$. The following examples illustrate this.

EXAMPLE (v) If $\underline{\alpha}$, $\underline{\beta}$ be two sequences defined as in (11); then $\underline{\alpha} \star \underline{\beta} = \underline{\alpha} + \underline{\beta}$ and so if $\alpha+\beta > 1$ the convolution is not log-convex but only α'-log-convex, $\alpha' \geq \alpha+\beta$.

EXAMPLE (vi) Let $a_n = \dfrac{1}{n}, b_n = 1, n \in \mathbb{N}^*$, then $\underline{a}$ is 0-log-convex, see Example (ii), and $\underline{b}$ is log-convex; but $\underline{a} \star \underline{b} = \underline{c}$ where $c_n = \sum_{r=1}^{n} 1/r$, $n \in \mathbb{N}^*$, which is not log-convex.

3.3 AN ORDER RELATION FOR SEQUENCES

DEFINITION 9 *An $n \times n$ matrix is $S = (s_{ij})_{1 \leq i,j \leq n}$ with non-negative entries is called* doubly stochastic *if*

$$\sum_{j=1}^{n} s_{ij} = 1,\ 1 \leq i \leq n, \quad \text{and} \quad \sum_{i=1}^{n} s_{ij} = 1,\ 1 \leq j \leq n. \tag{15}$$

If in addition in any row or column all elements except one are zero the matrix is called a permutation matrix; *equivalently a doubly stochastic matrix is called a permutation matrix if each row, and column, contains an element equal to* 1.

[CE pp.1349,1743; EM9 p.9; MO p.19]

EXAMPLE (i) If $q_i, q_i' \in \mathbb{R}_+^*$ and $q_i + q_i' = 1$, $1 \leq i \leq n$, then the following matrices, all $n \times n$ except the Q_i, are doubly stochastic:

$$I;\ \frac{1}{n}J;\ Q_i = \begin{pmatrix} q_i & q_i' \\ q_i' & q_i \end{pmatrix};\ P_i = \begin{pmatrix} I_{i-1} & O & O \\ O & Q_i & O \\ O & O & I_{n-i-1} \end{pmatrix},\ 1 \leq i \leq n-1;$$

$$P_n = \begin{pmatrix} q_n & O & q_n' \\ O & I_{n-2} & O \\ q_n' & O & q_n \end{pmatrix}; \qquad P = \prod_{i=1}^{n} P_i$$

DEFINITION 10 *A real n-tuple $\underline{b}$ is called* an average of the real n-tuple $\underline{a}$ *if for some doubly stochastic matrix S, $\underline{b} = \underline{a}S$.*

REMARK (i) To say that $\underline{b}$ is an average of $\underline{a}$ is a statement that does not depend on the order of the elements in either of the n-tuples, and is a *transitive relation*

on the set of n-tuples; that is if $\underline{a}$ is an average of $\underline{b}$, and $\underline{b}$ is an average of $\underline{c}$ then $\underline{a}$ is an average of $\underline{c}$.

REMARK (ii) To say that $\underline{b}$ is an average of $\underline{a}$ is to say that each element of $\underline{b}$ is a weighted arithmetic mean of the elements of $\underline{a}$; see II 1(3).

REMARK (iii) If $\underline{b}$ is an average of $\underline{a}$ and $\underline{a}$ is an average of $\underline{b}$ then $\underline{b}$ is a permutation of $\underline{a}$; conversely $\underline{b}$ is a permutation of $\underline{a}$ then $\underline{b}$ is an average of $\underline{a}$ and $\underline{a}$ is an average of $\underline{b}$.

REMARK (iv) If $1 \le m < n$ and $\underline{b} = (\underline{b}^{(1)}, \underline{b}^{(2)})$ where $\underline{b}^{(1)} = (b_1, \ldots, b_m)$ and $\underline{b}^{(2)} = (b_{m+1}, \ldots, b_n)$, with $\underline{a} = (\underline{a}^{(1)}, \underline{a}^{(2)})$ defined analogously; then $\underline{b}^{(i)}$ an average of $\underline{a}^{(i)}$, $i = 1, 2$, implies that $\underline{b}$ is an average of $\underline{a}$.

DEFINITION 11 *(a) If $\underline{a}, \underline{b}$ are decreasing real n-tuples we say that $\underline{b}$* precedes *$\underline{a}$, $\underline{b} \prec \underline{a}$, if:*

$$\begin{aligned} B_n &= A_n, \\ B_k &\le A_k, \ 1 \le k < n. \end{aligned} \tag{16}$$

(b) Generally if $\underline{a}, \underline{b}$ are real n-tuples we say that $\underline{b} \prec \underline{a}$ if when the two n-tuples are arranged in decreasing order they satisfy (16).

REMARK (v) This relation is an order relation on the set of decreasing real n-tuples, and a pre-order on the set of real n-tuples.

EXAMPLE (ii) If $\underline{a}$ is an n-tuple with$a_1 + \cdots + a_n = 1$, then $(1/n, \ldots, 1/n) \prec \underline{a} \prec (1, 0, \ldots, 0)$; use (16) and $(1/n, \ldots, 1/n) = \underline{a}(\frac{1}{n}J)$, see Example (i).

EXAMPLE (iii) If $\underline{c}$ is a constant real n-tuple and $\underline{b} \prec \underline{a}$, respectively $\underline{b}$ is an average of $\underline{a}$, then $\underline{b} + \underline{c} \prec \underline{a} + \underline{c}$, respectively $\underline{b} + \underline{c}$ is an average of $\underline{a} + \underline{c}$.

REMARK (vi) A full discussion of this concept can be found in [*MO*], and an introductory survey is given in the article [*Marshall & Olkin 1981*]; see also [*DI pp.198–199*].

LEMMA 12 *$\underline{b}$ is an average of $\underline{a}$ if and only if it lies in the convex hull of the $n!$ points obtained by permuting the elements of $\underline{a}$*

□ This follows from a theorem of Birkhoff; see [*MO p.19*], [*Marshall & Olkin 1964*]. [A definition of convex hull is given in 4.6 Definition 35(b) below]. □

LEMMA 13 [MUIRHEAD'S LEMMA] *If $\underline{b} \prec \underline{a}$ then $\underline{b} = \underline{a}T_1T_2 \cdots T_{n-1}$, where T_i is an $n \times n$ matrix of the form $\lambda I + (1 - \lambda)Q_i$, where Q_i is a permutation matrix that differs from I in just two rows, or equivalently just two columns, $1 \le i < n - 1$.*

□ See [*MO pp.21–22*]. □

THEOREM 14 [HARDY, LITTLEWOOD & PÓLYA] *A real n-tuple $\underline{b}$ is an average of $\underline{a}$ if and only if $\underline{b} \prec \underline{a}$.*

□ See [*HLP p. 49; MO p.22*], [Rado]. □

REMARK (vii) If (16) is replaced by

$$B_k \leq A_k, \; 1 \leq k \leq n, \tag{17}$$

we say that $\underline{b}$ *weakly precedes* $\underline{a}$, $\underline{b} \prec^w \underline{a}$; [*MO pp.9–11*].

A simple but useful idea is given in the following definition.

DEFINITION 15 *Two n-tuples $\underline{a}, \underline{b}$ are said to be* similarly ordered *when,*

$$(a_j - a_k)(b_j - b_k) \geq 0, \qquad 1 \leq j, k \leq n; \tag{18}$$

if this inequality is always reversed then the n-tuples are said to be oppositely ordered.

REMARK (viii) Clearly $\underline{a}$ and $\underline{b}$ are similarly ordered if and only if a simultaneous permutation of them both leads to decreasing n-tuples; and they are oppositely ordered if and only if a simultaneous permutation leads to one increasing n-tuple and one decreasing n-tuple; see [*AI p.10; HLP p.43; MO pp.139–140*].
A very simple but useful result is the following; see [*DI pp.222–223; HLP pp.261–262*], [*Herman, Kučera & Šimša pp.145–148*], [*Vince*].

THEOREM 16 *If two n-tuples $\underline{a}$ and $\underline{b}$ are given except in arrangement then the sum $\sum a_i b_i$ is greatest if they are similarly ordered and least if they are oppositely ordered.*

□ It is sufficient to note that $(a_jb_j + a_kb_k) - (a_jb_k + a_kb_j) = (a_j - a_k)(b_j - b_k)$; for full details see the references above. □

REMARK (ix) If any of the inequalities in (18) is strict then the resulting inequalities in Theorem 16 are also strict; so there is equality if and only if either $\underline{a}$ or $\underline{b}$ is constant.

COROLLARY 17 *If $\underline{a}'$ is a rearrangement of the positive n-tuple $\underline{a}$ then*

$$\sum_{i=1}^{n} \frac{a_i'}{a_i} \geq n. \tag{19}$$

Further the inequality is strict unless $\underline{a}' = \underline{a}$.

□ It is sufficient to remark that $\underline{a}$ and $\underline{a}^{-1}$ are oppositely ordered and the right-hand side of (19) is just $\sum_{i=1}^{n}\left(a_i\dfrac{1}{a_i}\right)$. The case of equality is immediate. □

REMARK (x) This result of Chong K M, [*Chong K M 1976c*], has been generalized; see [*Ioanoviciu*].
The following theorem is a simple application of these concepts due to Chong K M; [*Chong K M 1976c*].

THEOREM 18 *If $\underline{b} \prec \underline{a}$ then $\prod_{i=1}^{n} b_i \geq \prod_{i=1}^{n} a_i$.*

□ Suppose that $\underline{b} = \lambda\underline{a} + (1-\lambda)\underline{a}'$, where $\underline{a}'$ is a permutation of $\underline{a}$. Then,

$$\begin{aligned}
\prod_{i=1}^{n}\frac{a_i}{b_i} &= \prod_{i=1}^{n}\left(\lambda + (1-\lambda)\frac{a_i'}{a_i}\right)\\
&= \sum_{k=0}^{n}\lambda^{n-k}(1-\lambda)^k \sum_{1\leq i_1<\cdot<i_k\leq n}\prod_{j=1}^{k}\frac{a_{i_j}'}{a_{i_j}}\\
&\geq \sum_{k=0}^{n}\lambda^{n-k}(1-\lambda)^k\binom{n}{k}, \text{ by Corollary17,}\\
&=1,
\end{aligned}$$

The general case follows using Lemma 13. □

COROLLARY 19 *(a) If $x > -1$ then for $n \geq 2$,*

$$\left(1+\frac{x}{n-1}\right)^{n-1} \leq \left(1+\frac{x}{n}\right)^n. \tag{20}$$

(b) If either $a_i > 0$ or $-1 < a_i < 0$, $0 \leq i \leq n$ then

$$1+\sum_{i=1}^{n} a_i \leq \prod_{i=1}^{n}(1+a_i). \tag{21}$$

□ (a) Use Theorem 18 and the fact that:

$$\begin{aligned}
&(1+x/n, 1+x/n, \cdots, 1+x/n)\\
&\qquad \prec \big(1+x/(n-1), 1+x/(n-1), \cdots, 1+x/(n-1), 1\big).
\end{aligned}$$

(b) Use Theorem 18 and the fact that $\big(1+a_1, \ldots, 1+a_n\big) \prec \big(1+\sum_{i=1}^{n} a_i, 1, \ldots, 1\big)$. □

REMARK (xi) (20) is a special case of 2.2(11).

REMARK (xii) The order relation discussed in this section has been extended to functions, see VI 1.3.4.

4 Convex Functions

The concept of convexity is very important for this book. However this subject receives full treatment in several readily available sources and so this section will merely collect results. For further details and proofs the reader is referred to the following books: [*AI pp.10–22; HLP pp.70–83; RV; PPT*], [*Aumann & Haupt; Popoviciu*], and for general information see [*CE p.326; DI pp.61–64; EM2 pp.415–416*]. The basic reference for all the topics of this section is [*RV*] where full proofs and many references can be found. As the topic is one of active research the more recent [*PPT*] contains much of this newer material as well as an extensive and up-to-date bibliography.

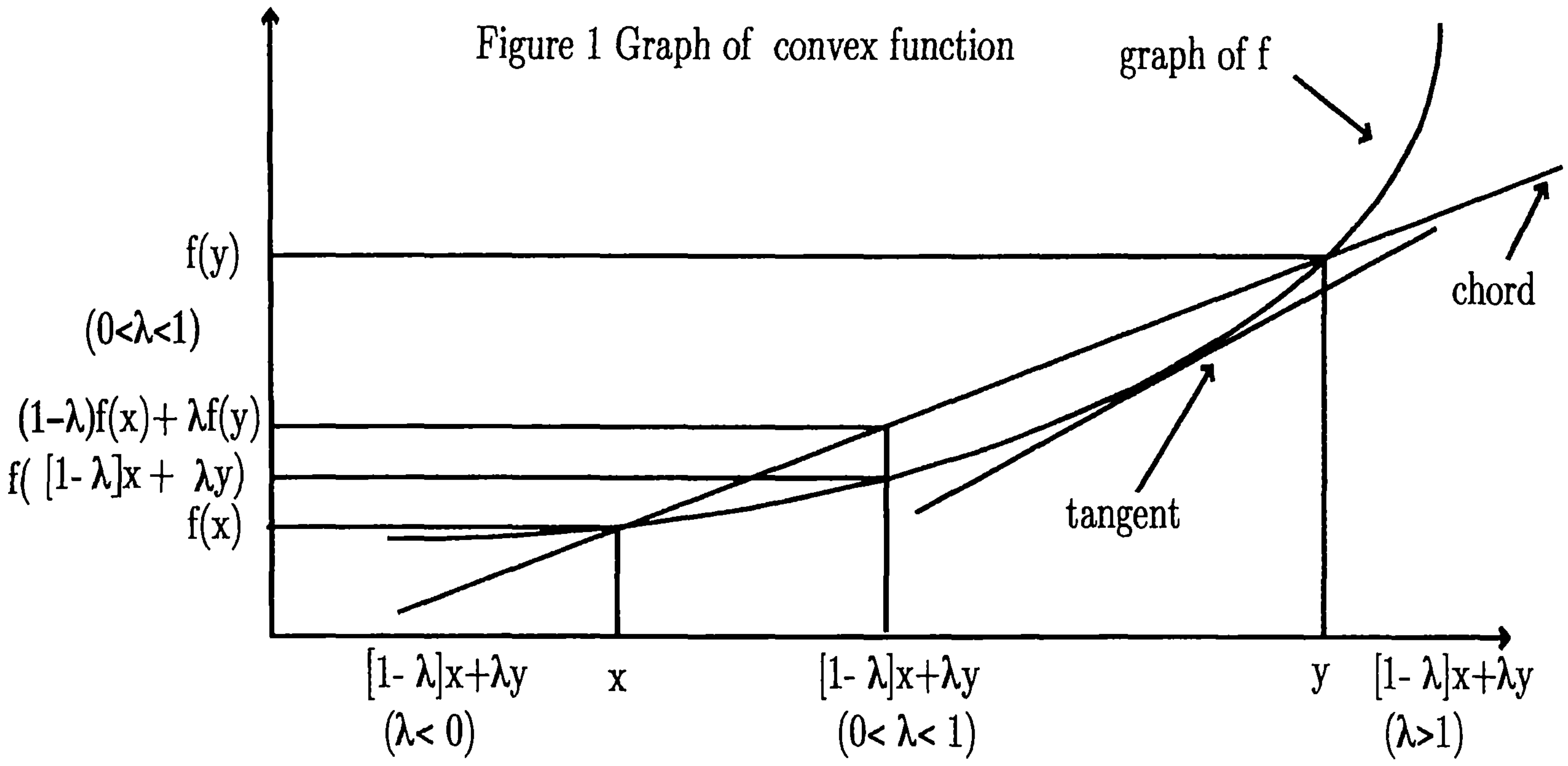

Figure 1 Graph of convex function

4.1 CONVEX FUNCTIONS OF A SINGLE VARIABLE

DEFINITION 1 *(a) Let I be an interval in $\mathbb{R}$, then $f : I \mapsto \mathbb{R}$ is said to be* convex, *on I, if for all $x, y \in I$, and for all $\lambda, 0 \le \lambda \le 1$,*

$$f\big(\lambda x + \overline{1-\lambda} y\big) \le \lambda f(x) + (1-\lambda) f(y). \tag{1}$$

If (1) is strict whenever $x \neq y$ and $\lambda \neq 0, 1$, then f is said to be strictly convex, *on I.*

(b) If in (a) inequality (1) is replaced by ($\sim$1) then f is said to be concave, strictly concave, *on I.*

REMARK (i) If f is both convex and concave then for some m and c, $f(x) = mx+c$; that is to say, f is *affine*.

Remark (ii) The graph of a (strictly) convex, concave, function is also said to be (strictly) convex, concave.

Remark (iii) The simple geometric interpretation of (1) is that the graph of f lies below the chords of that graph; see Figure 1.

Remark (iv) If $x_i, 1 \le i \le 3$, are three points of I with $x_1 < x_2 < x_3$, then (1) is equivalent to:

$$f(x_2) \le \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3), \tag{2}$$

or more symmetrically,

$$\frac{f(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{f(x_2)}{(x_2 - x_3)(x_2 - x_1)} + \frac{f(x_3)}{(x_3 - x_1)(x_3 - x_2)} \ge 0; \tag{2*}$$

or even

$$\begin{vmatrix} 1 & x_1 & f(x_1) \\ 1 & x_2 & f(x_2) \\ 1 & x_3 & f(x_3) \end{vmatrix} \ge 0. \tag{2**}$$

Lemma 2 *If f is convex on I and if*

$$n(x, y) = \frac{f(y) - f(x)}{y - x}, \; x, y \in I, \, x \ne y$$

then n is increasing in both variables and strictly increasing if f is strictly convex.

□ Another way of writing (2) is: $\dfrac{f(x_1) - f(x_2)}{x_1 - x_2} \le \dfrac{f(x_2) - f(x_3)}{x_2 - x_3}$. So, if $x_1, y_1, x_2, y_2 \in I$ with $x_1 \le y_1, x_2 \le y_2$,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(y_2) - f(y_3)}{y_2 - y_1}. \tag{3}$$

The lemma is an immediate consequence of (3). □

Remark (v) In other words the slopes of chords to the graph of a convex function increase to the right; further these slopes increase strictly if f is strictly convex.

This has some interesting implications.

Corollary 3 *(a) At any point on the graph of a convex function where there is a tangent the graph lies above the tangent.*
(b) If f is convex on I but not strictly convex then it is affine on some closed sub-interval of I.

Remark (vi) The property (a) is illustrated in Figure 1.

REMARK (vii) The intervals where a convex, not strictly convex, function is affine can be dense as the integral of the Cantor function[4]shows.

THEOREM 4 *Let f be convex on I then:*
(a) f is Lipschitz on any closed interval in $\mathring{I}$;
(b) $f'_{\pm}$ exist and are increasing on $\mathring{I}$ and $f'_{-} \leq f'_{+}$; further if f is strictly convex these derivatives are strictly increasing;
(c) f' exists except on a countable set, on the complement of which it is continuous.
(d) If f and g are convex then so is $\lambda f + \mu g$ if $\lambda, \mu > 0$.
(e) If f and g are convex, increasing (decreasing) and non-negative then fg is also convex.
(f) If f is convex on I and g is convex in J, $f[I] \subseteq J$, g increasing, then $g \circ f$ is convex on I.

□ See [*RV pp.4–7,15–16*].[5] □

REMARK (viii) If f is an affine function then (f) holds without the assumption that g is increasing.

COROLLARY 5 *(a) f is convex, respectively strictly convex, on $]a, b[$ if and only if for each x_0, $a < x_0 < b$, there is an affine function $S(x) = f(x_0) + m(x - x_0)$ such that $S(x) \leq f(x)$, $a < x < b$, respectively $S(x) < f(x)$, $a < x < b$, $x \neq x_0$.*
(b) If f is differentiable and strictly convex on an interval I and if for $c \in I$ we have $f'(c) = 0$ then $f(c)$ is the minimum value of f on I and this minimum is unique

□ (a) If f is convex, respectively strictly convex, take in S any value of m such that $m \in [f'_{-}(x_0), f'_{+}(x_0)]$.
Conversely suppose that such an S exists at x_0 and let $x_0 = \lambda x + (1 - \lambda)y$ where $x, y \in]a, b[$, $0 \leq \lambda \leq 1$. Then $f(x_0) = S(x_0) = \lambda S(x) + (1 - \lambda)S(y) \leq \lambda f(x) + (1 - \lambda)f(y)$, showing that f is convex. The strictly convex case is similar.
(b) This is an immediate consequence of Theorem 4(b) and Corollary 3(a); see [*Tikhomirov, p.117*]. □

REMARK (ix) The affine function S is called a *support of f at x_0*; its graph is a *line of support for f at x_0*; [*RV p.12*].

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[4] The *Cantor function* is continuous, increases from 0 to 1, and is constant on the dense set of intervals [1/3,2/3],[1/9,2/9],[7/9,8/9],..., where it takes the values 1/2,1/4,3/4,...; see [*CE p.187; EM2 p.13; Pólya & Szegö p.206*]. A set, or a set of intervals, is *dense* if every neighbourhood of every point meets the set, or the set of intervals; [*CE p.416; EM3 pp.46,434*].

[5] A function f is *Lipschitz* if for some $M>0$ and all x,y, $|f(y)-f(x)|<M|y-x|$; then M is called a *Lipschitz constant* of the function; [*CE p.1091; EM5 p.532*].

THEOREM 6 *(a) $f : [a, b] \mapsto \mathbb{R}$ is (strictly) convex if and only if there is a (strictly) increasing function $g :]a, b[\to \mathbb{R}$ and a $c, a < c < b$, such that for all $a, a < x < b$,*

$$f(x) = f(c) + \int_c^x g.$$

(b) If f'' exists on $]a, b[$ then f is convex if and only if $f'' \geq 0$; if every subinterval contains a point where $f'' > 0$ then f is strictly convex.

□ See [*RV pp.9–11*]. □

REMARK (x) For applications to inequalities the conditions of (b) usually suffice; that is we are dealing with functions that are twice differentiable, with second derivative positive on a dense set[6]— usually the complement of a finite set; see [*Bullen 1998*]. It might be noted that, using the mean-value theorem of differentiation[7], we can, deduce from the last property that the chord slope increases strictly; see Lemma 2.

COROLLARY 7 *(a) If $r > 1$ or $r < 0$ and $f(x) = x^r, x > 0$, then f is strictly convex; if $0 < r < 1$, f is strictly concave. The exponential function is strictly convex and the logarithmic function is strictly concave.*
(b) If $f \in C^2(a, b)$ with $m = \min f'', M = \max f''$ then both of the functions $\dfrac{Mx^2}{2} - f(x)$ and $mf(x) - \dfrac{mx^2}{2}$ are convex.

□ Simple applications of Theorem 6(b). □

EXAMPLE (i) Other examples are: $x \log x$, $x > 0$ is strictly convex; if f is (strictly) convex, twice differentiable on I and if $f < 1$ then $1/(1-f)$ is also (strictly) convex on I. These simple examples and those in Corollary 7(a) are used in many places to prove various inequalities; [*Abou-Tair & Sulaiman 1999*].

EXAMPLE (ii) The factorial function $x!$, $x > -1$, is strictly convex but more is true in this case; see 4.5.2 Example (i).

EXAMPLE (iii) From Corollaries 7(a) and 3(a) we have: if $x > 0, x \neq 1$, then $x^r <, [>], 1 + r(x-1)$, if $0 < r < 1, [r > 1,$ or $r < 0]$. This, with a simple change of variable and notation is just 2.1 Theorem 1, that is (B), [($\sim$B)] .

REMARK (xi) Corollary 7(b) has been used to extend many of the inequalities implied by convexity; [*Andrica & Raşa; Raşa*].

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[6] See Footnote 4

[7] See Footnote 1.

A simple property of convexity and concavity, that we will use later, is given in the following lemma.

LEMMA 8 [KUANG] *If $f : [0,1] \mapsto \mathbb{R}$ is strictly increasing and either convex or concave, then for all $n \geq 1$,*

$$\frac{1}{n}\sum_{i=1}^{n} f\Big(\frac{i}{n}\Big) > \frac{1}{n+1}\sum_{i=1}^{n+1} f\Big(\frac{i}{n+1}\Big) > \int_0^1 f$$

□ The last inequality holds since f is strictly increasing.
Suppose then that f is convex and that $1 \leq i < n+1$, then

$$\begin{aligned}
\frac{i-1}{n} f\Big(\frac{i-1}{n}\Big) &+ \Big(1 - \frac{i-1}{n}\Big) f\Big(\frac{i}{n}\Big)\\
&\geq f\Big(\big(\frac{i-1}{n}\big)^2 + \big(1 - \frac{i-1}{n}\big)\frac{i}{n}\Big), \quad \text{by (1)},\\
&= f\Big(\frac{i(n-1)+1}{n^2}\Big)\\
&> f\Big(\frac{i}{n+1}\Big), \quad \text{since } f \text{ is strictly increasing and } i < n+1.
\end{aligned}$$

Summing we get that

$$\frac{1}{n}\sum_{i=1}^{n}\Big((i-1)f\big(\frac{i-1}{n}\big) + (n-i+1)f\big(\frac{i}{n}\big)\Big) > \sum_{i=1}^{n} f\Big(\frac{i}{n+1}\Big).$$

So

$$\frac{1}{n}\sum_{i=1}^{n-1}\Big((i-1)f\big(\frac{i-1}{n}\big) + (n-i+1)f\big(\frac{i}{n}\big)\Big) + \frac{1}{n}f(1) > \sum_{i=1}^{n+1} f\Big(\frac{i}{n+1}\Big) - f(1).$$

This on simplification gives the first inequality.
A similar argument, starting with $\frac{i}{n+1} f\Big(\frac{i+1}{n+1}\Big) + \Big(1 - \frac{i}{n+1}\Big) f\Big(\frac{i}{n+1}\Big)$ leads to the same inequality when f is concave. □

REMARK (xii) This result is a special case of a slightly more general result; [*Kuang; Qi 2000a*].

Another useful result is the following; see [*DI pp.122–123*].

THEOREM 9 [HADAMARD-HERMITE INEQUALITY] *If $f : [a,b] \mapsto \mathbb{R}$ is convex and if $a \leq c < d \leq b$, then*

$$f\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d f \leq \frac{f(c)+f(d)}{2}, \tag{4}$$

and

$$\frac{1}{n}\sum_{i=0}^{n-1} f\left(a+\frac{i}{n-1}(b-a)\right) \le \frac{1}{b-a}\int_a^b f. \tag{5}$$

REMARK (xiii) Inequality (4) attributed Hadamard, and often called *Hadamard's inequality*, is actually due to Hermite[8]; see [*Mitrinović & Lacković*] for an interesting discussion of this. Recently much work has been done on generalizations of this inequality; see[*PPT p.137*], [*Dragomir & Pearce*], [*Lacković, Maksimović & Vasić*].

REMARK (xiv) The left-hand side of (5) increases with n, having the right-hand side as its limit; see [*Nanjundiah 1946*].

Finally mention should be made of the important connection between convexity and the order relations of 3.3.

THEOREM 10 *(a)* [KARAMATA] *If* $\underline{a}, \underline{b} \in I^n$, *$I$ an interval in* $\mathbb{R}$, *then* $\underline{b} \prec \underline{a}$ *if and only if for all functions f convex on I,*

$$\sum_{i=1}^{n} f(b_i) \le \sum_{i=1}^{n} f(a_i).$$

(b) [TOMIĆ] *If* $\underline{a}, \underline{b} \in I^n$, *$I$ an interval in* $\mathbb{R}$, *then* $\underline{b} \prec^w \underline{a}$ *if and only if for all functions f convex and increasing on I,*

$$\sum_{i=1}^{n} f(b_i) \le \sum_{i=1}^{n} f(a_i).$$

□ See [*MO pp.108–109*]. □

COROLLARY 11 *If $\underline{a}$ and $\underline{b}$ are decreasing non-negative n-tuples such that*

$$\prod_{i=1}^{k} a_i \le \prod_{i=1}^{k} b_i, \ 1 \le k \le n,$$

then for all p, $p > 0$, $\underline{a}^p \prec^w \underline{b}^p$.

□ We can without loss in generality assume that the terms in the n-tuples are all greater than 1. Then apply Theorem 10(b)with function $f(x) = e^{px}$ and the n-tuples $\log \underline{a}, \log \underline{b}$. □

4.2 JENSEN'S INEQUALITY One of the most important convex function inequalities is *Jensen's inequality*; in fact it is almost no exaggeration to say that all

[8] It is sometimes called the *Hermite-Hadamard inequality*

known inequalities are particular cases of this famous result. It has been the object of much research full details of which can be found in the various references; [*AI pp.10–14; CE p.953; DI pp.139–141; EM5 pp.234–235; HLP pp.70–75; PPT pp.43–57; RV p.89*].

THEOREM 12 [JENSEN'S INEQUALITY] *If I is an interval in $\mathbb{R}$ on which f is convex, if $n \ge 2$, $\underline{w}$ a positive n-tuple, $\underline{a}$ an n-tuple with elements in I then:*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right) \le \frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i). \tag{J}$$

If f is strictly convex then (J) is strict unless $\underline{a}$ is constant.

□ It should first be remarked that the left-hand side of (J) is well defined since the argument of f is the arithmetic mean of $\underline{a}$ with weight $\underline{w}$ and so its value is between the max $\underline{a}$, and min $\underline{a}$, in particular it is in I; see II 1.1(2). Following later usage we will refer to $\underline{w}$ as the n-tuple of *weights.*

(i) This proof, by Jensen, is by induction on n and the case $n = 2$ is just the definition of convexity, 4.1 (1); see for instance [*Hrimic 2001*].

Suppose then that (J) holds for all k, $2 \le k \le n-1$.

$$\begin{aligned} f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right) &= f\left(\frac{w_n}{W_n}a_n + \frac{W_{n-1}}{W_n}\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i a_i\right) \\ &\le \frac{w_n}{W_n}f(a_n) + \frac{W_{n-1}}{W_n}f\left(\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i a_i\right), \text{ by the case } n = 2, \\ &\le \frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i), \text{by the induction hypothesis.} \end{aligned}$$

The case of equality is easily considered.

(ii) A very simple inductive proof for the equal weight case has been given by Aczél, [*Aczel 1961a,b; Pečarić 1990c*]. Assume that $n \ge 2$ and that the result is known for all k, $2 \le k < n$.

$$\begin{aligned} f\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) &= f\left(\frac{1}{2}\left(\frac{1}{n-1}a_n + \frac{n-2}{n(n-1)}\sum_{i=1}^{n} a_i + \frac{1}{n-1}\sum_{i=1}^{n-1} a_i\right)\right) \\ &\le \frac{1}{2}\left(f\left(\frac{1}{n-1}a_n + \frac{n-2}{(n-1)}\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\right) + f\left(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i\right)\right), \\ &\qquad \text{by the case } n = 2, \\ &\le \frac{1}{2}\left(\left(\frac{1}{n-1}f(a_n) + \frac{n-2}{n-1}f\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\right) + \frac{1}{n-1}\sum_{i=1}^{n-1} f(a_i)\right), \end{aligned}$$

by the case $n = 2$, and the induction hypothesis.

On rearranging this gives

$$f\Big(\frac{1}{n}\sum_{i=1}^{n} a_i\Big) \leq \frac{1}{n}\sum_{i=1}^{n} f(a_i).$$

(*iii*) A simple geometric induction can be given if we take as the definition of strict convexity that $f'' \geq 0$ and in every sub-interval there is a point where f'' is positive.; see 4.1 Theorem 6(b), Remark (x).
First we must prove (J) in the case $n = 2$; that is, show that 4.1 (1) holds strictly if $x \neq y$, $\lambda \neq 0, 1$.
With a slight change of notation we can rewrite (J) in this case as:

$$f\big(\overline{1-s}\,x + sy\big) \leq (1-s)f(x) + sf(y), \quad 0 < s < 1, \tag{J$_2$}$$

with equality only if $x = y$. To prove (J_2) consider the following function obtained from the difference between its two sides

$$D_2(x, y; s) = D_2(s) = (1-s)f(x) + sf(y) - f\big(\overline{1-s}\,x + sy\big)$$

where $0 \leq s \leq 1$ and, without loss in generality, $x < y$.
Then (J_2) is equivalent to $D_2(s) > 0$, $0 < s < 1$.
Since $D_2(0) = D_2(1) = 0$ for some s_0, $0 < s_0 < 1$, $D_2'(s_0) = 0$; then $\overline{1-s_0}\,x + s_0 y$ is a mean-value point for f on $[x, y]$[9]. Further

$$\begin{aligned} D_2'(s) &= f(y) - f(x) - (y-x)f'\big(\overline{1-s}\,x + sy\big) \\ D_2''(s) &= -(y-x)^2 f''\big(\overline{1-s}\,x + sy\big) \end{aligned}$$

By our assumption and Theorem 6(b), D is strictly concave and D_2' is strictly decreasing. Hence s_0 is unique, with D_2' positive to the left of the mean-value point, and negative to the right. Since then D_2 is not constant we have that it is positive except at $s = 0, 1$, which proves (J_2), and that f is convex in the sense of 4,1 Definition 1.
The case $n = 3$ of (J) can be written as,

$$f\big(\overline{1-s-t}\,x + sy + tz\big) \leq (1-s-t)f(x) + sf(y) + tf(z), \tag{J$_3$}$$

where $0 < s+t < 1, 0 < s < 1, 1 < t < 1$ with equality only if $x = y = z$. To prove (J_3) we, by analogy with the above, consider the function,

$$D_3(x, y, z; s, t) = D_3(s, t) = (1-s-t)f(x) + sf(y) + tf(z) - f\big(\overline{1-s-t}\,x + sy + tz\big)$$

[9] See Footnote 1.

where without loss in generality we have $x < y < z$, and

$$0 \le s \le 1,\ 0 \le t \le 1,\ 0 \le s+t \le 1. \tag{T}$$

Since D_3 is continuous it attains both its maximum and its minimum on T and if this occurs in the interior of T then it occurs at a turning point. Now

$$\begin{aligned}\frac{\partial}{\partial s}D_3(s,t) &= f(y) - f(x) - f'\big(\overline{1-s-t}\,x + sy + tz\big)(y-x),\\ \frac{\partial}{\partial t}D_3(s,t) &= f(z) - f(x) - f'\big(\overline{1-s-t}\,x + sy + tz\big)(z-x).\end{aligned}$$

So for a turning point at (s,t) we must have that

$$f'\big(\overline{1-s-t}\,x + sy + tz\big) = \frac{f(y)-f(x)}{y-x} = \frac{f(z)-f(x)}{z-x}.$$

By a 4.1 Lemma 2, $\big(f(y)-f(x)\big)/(y-x) < \big(f(z)-f(x)\big)/(z-x)$. So D_3 has no turning points in T and attains its minimum on the boundary of T.
However on a side of T the problem reduces to the previous case; for instance when $t = 0$, $D_3(x,y,z;s,0) = D_2(x,y;s)$. Hence D_3 attains its minimum value of zero at the corners of T, the points $(0,0), (0,1), (1,0)$, which proves (J_3).
Now it is clear that this argument easily extends to the general case. □

Remark (i) Of course $D_2(s)$ is defined for all $s \in \mathbb{R}$ for which $(1-s)x + sy \in I$ and the argument given above shows that D_2' is strictly decreasing for all such s. This implies that D_2 is negative outside $[0,1]$; that is

$$D_2(s)\begin{cases} = 0 & \text{if } s = 0, 1;\\ > 0 & \text{if } 0 < s < 1;\\ < 0 & \text{if } s < 0 \text{ or } 1 < s.\end{cases}$$

So in the case $n = 2$, (J_2) only holds for positive weights. If $n \ge 3$ the situation is more interesting, see the Jensen-Steffensen inequality, see below 4.3 Theorem 20. More precisely if $n = 2$ then ($\sim J_2$) holds if either $s \le 0$ or $s \ge 1$, see Figure 1; in the case $n = 3$ ($\sim J_3$) will hold if either (i) $s+t \ge 1, t \ge 1$, (ii) $s+t \ge 1, t \le 0$, (iii) $s+t \le 0, t \le 0$, or (iv) $s+t \le 0, t \ge 1$.

Remark (ii) Proof (*i*), as well as other proofs, can be found in [*PPT p.43; RV p.189*], [*McShane; Mitrinović & Mitrović; Pop; Zajta*]. Proof (iii) is in [*Bullen 1994b, 1998*] See also [*CE p.953; DI pp.129–141; EM5 pp.234–235*].

Remark (iii) It is immediate that (J) is equivalent to (1) and so to convexity.

Remark (iv) Another form of ($\sim$J) can be found in III 2.1 Theorem 1 Proof (*iii*).

A completely different proof of the equal weight case of (J) has been given in the interesting paper by Wellstein; [*Wellstein*]. Given an $c > 0$ define $H_n(c)$ as $H_n(c) = \{\underline{a};\ \underline{a} \in \mathbb{R}^n \text{ and } \sum_{i=1}^n a_i = nc\}$; further if $\underline{p}$ is an n-tuple define $\underline{q}$ and $\underline{q}'$ by $q_i = p_i/(p_i + p_{i+1})$, $1 \le i \le n$, $\underline{q}' = 1 - \underline{q}$, where $p_{n+1} = p_1$. Using this notation and that of 3.3 Example(i) we have the following result.

Theorem 13 *If I is an interval in $\mathbb{R}$ and if $f : I^n \cap H_n(c) \mapsto \mathbb{R}$ is continuous at c and for all i, $1 \le i \le n$, $f(\underline{a}) \ge f(\underline{a}P_i)$ with equality if and only if $a_{i+1} = a_i$ then*

$$f(\underline{a}) \ge f(c\underline{e}),$$

with equality if and only if $\underline{a}$ is constant.

□ The proof depends on certain properties of doubly stochastic matrices. In particular, again using the notation of 3.3 Example(i) and that in Notations 7, $\lim_{k\to\infty} P^k = \dfrac{1}{n}J$. Details can be found in the reference. □

Taking $\underline{p} = \underline{e}$ and $f(\underline{a}) = \sum_{i=1}^n g(a_i)$, where g is convex this theorem gives a proof of the equal weight case of (J).

As has been suggested (J) includes many of the classical inequalities as special cases, and many other inequalities are often a much disguised form of (J); see [*He J K*]. For instance we have the following.

Corollary 14 *If $\underline{w}$ is a positive n-tuple then*

$$W_n^{W_n} \ge \prod_{i=1}^n w_i^{w_i}.$$

□ Use the concavity of the logarithmic function. □

The following interesting refinement of (J) has been given in [*Vasić & Mijalković*]. Let $I \in \mathcal{I}$, an index set, we extend the notation introduced earlier, Notations 6(xi), by defining the following function on $\mathcal{I}$:

$$F(\underline{w}; I) = F(I) = \sum_{i\in I} w_i f(a_i) - W_I f\left(\frac{1}{W_I}\sum_{i\in I} w_i a_i\right).$$

Using this notation Theorem 12 can be expressed as follows.

If f is convex then F is non-negative, and if f strictly convex F is positive unless $\underline{a}$ is constant.

In fact more is true, as the following theorem shows.

THEOREM 15 *(a) If f is convex then F is super-additive as a function of I, that is if $I_1, I_2 \in \mathcal{I}$ with $I_1 \cap I_1 = \emptyset$,*

$$F(I_1) + F(I_2) \leq F(I_1 \cup I_2). \tag{6}$$

If f is strictly convex then (6) is strict unless

$$\frac{1}{W_{I_1}} \sum_{i \in I_1} w_i a_i = \frac{1}{W_{I_2}} \sum_{i \in I_2} w_i a_i.$$

(b) If f is convex then F is super-additive as a function of $\underline{w}$, that is if $\underline{u}, \underline{v}$ are n-tuples, then

$$F(\underline{u}; I) + F(\underline{v}; I) \leq F(\underline{u} + \underline{v}; I)$$

□ (a) As remarked above (J) implies that $F(I_1 \cup I_2) \geq 0$. In this inequality replace a_i by b_i where

$$b_i = \begin{cases} \dfrac{1}{W_{I_1}} \displaystyle\sum_{i \in I_1} w_i a_i, & \text{if } i \in I_1, \\ \dfrac{1}{W_{I_2}} \displaystyle\sum_{i \in I_2} w_i a_i, & \text{if } i \in I_2. \end{cases}$$

Then (6) is immediate and the case of equality follows from that for (J).
(b) See [*Dragomir, Pečarić & Persson*]. □

Since (6) gives a lower bound for $F(I_1 \cup I_2)$ that is in general positive it extends (J). This is made more precise in the following corollary. In the case $I_k = \{1, 2, \ldots, k\}, 1 \leq k \leq n$ let us write $F(I_k) = F_k(\underline{a}; \underline{w}; f)$

COROLLARY 16 *If f is convex then*

$$F_n(\underline{a}; \underline{w}; f) \geq F_{n-1}(\underline{a}; \underline{w}; f) \geq \cdots \geq F_2(\underline{a}; \underline{w}; f) \geq 0. \tag{7}$$

In particular $F_n(\underline{a}; \underline{w}; f) \geq F_2(\underline{a}; \underline{w}; f)$, or more generally

$$F_n(\underline{a}; \underline{w}; f) \geq \max_{\substack{1 \leq i,j \leq n \\ i \neq j}} \left\{ w_i f(a_i) + w_j f(a_j) - (w_i + w_j) f\left(\frac{w_i a_i + w_j a_j}{w_i + w_j} \right) \right\}. \tag{8}$$

□ (7) is an immediate consequence of (6) and the rest follows noting that $F(I_n)$ is unchanged if the elements of $\underline{a}$ and $\underline{w}$ are simultaneously permuted. □

REMARK (v) Further extensive study of the function F can be found in [*Dragomir & Goh 1997*a].

Another very simple lower bound for $F_n(\underline{a}; \underline{w}; f)$ in the case of convex functions with continuous second derivatives is given in the following theorem; [*Mercer A 1998*].

THEOREM 17 *If I is an interval in $\mathbb{R}$ on which f is convex, $f \in \mathcal{C}^2(I)$, $n \geq 2$, $\underline{w}$ a positive n-tuple with $W_n = 1$, $\underline{a}$ an n-tuple with elements in I then for some ξ, $\min \underline{a} < \xi < \max \underline{a}$,*

$$F_n(\underline{a}; \underline{w}; f) = \frac{1}{2} F_n(\underline{a}; \underline{w}; \iota^2) f''(\xi) \geq 0,$$

where $\iota^2(x) = x^2$.

□ Let $\overline{a} = \dfrac{1}{W_n} \sum_{i=1}^{n} w_i a_i$ then, by Taylor's theorem, see 2.2 Footnote 2, for some ξ_i between a_i and $\overline{a}$, $f(a_i) = f(\overline{a}) + (a_i - \overline{a}) f'(\overline{a}) + \frac{1}{2}(a_i - \overline{a})^2 f''(\xi_i)$, $1 \leq i \leq n$. Multiplying by w_i and summing over i gives

$$\begin{aligned} F_n(\underline{a}; \underline{w}; f) &= \frac{1}{2} \sum_{i=1}^{n} w_i (a_i - \overline{a})^2 f''(\xi_i) \\ &= \frac{1}{2} \Big(\sum_{i=1}^{n} w_i (a_i - \overline{a})^2 \Big) f''(\xi), \min \underline{a} < \xi < \max \underline{a}, \text{ by the continuity of } f'', \\ &= \frac{1}{2} F_n(\underline{a}; \underline{w}; \iota^2) f''(\xi). \end{aligned}$$

□

REMARK (vi) If in this theorem f is strictly convex then for some $m > 0$ we have that $f'' > m$ so that

$$\sum_{i=1}^{n} w_i f(a_i) - f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i a_i \right) \geq \frac{m}{2} \sum_{i=1}^{n} w_i (a_i - A)^2 > 0,$$

giving an improvement for (J).

Mitrinović & Pečarić proved the following theorem; for a proof of a more general version see [*PPT pp.90–91*].

THEOREM 18 *Let f be convex on the interval I, with A, B and $r \in \mathbb{R}$, r being in an interval J such that if $x \in I$ then both of $\big((1-\lambda)x + \lambda r\big)A + (1-\lambda)(r-x)B$ and $\lambda(r-x)A + (\lambda x + (1-\lambda)r)B$ are in I. If*

$$\begin{aligned} g(x) = \lambda f\big(&(1-\lambda)x + \lambda r)A + (1-\lambda)(r-x)B\big) \\ &+ (1-\lambda) f\big(\lambda(r-x)A + (\lambda x + (1-\lambda)r)B\big), \end{aligned}$$

then g is convex and if $xy > 0$ and $|x| < |y|$ then $g(x) \leq g(y)$.

Finally we give an extension of (J) due to Rooin, [*Rooin 2001b*].

THEOREM 19 *Let I be an interval in $\mathbb{R}$ on which f is convex and $\underline{p}$ be a matrix of affine functions, $p_{ij}(t) = (1-t)\alpha_{ij} + t\beta_{ij}$, $1 \le i,j \le n$, $0 \le t \le 1$, with*

$$\alpha_{ij},\ \beta_{ij} \ge 0,\ 1 \le i,j \le n,\ \textit{and} \sum_{i=1}^{n} \alpha_{ij} = \sum_{j=1}^{n} \alpha_{ij} = \sum_{i=1}^{n} \beta_{ij} = \sum_{j=1}^{n} \beta_{ij} = 1,$$

and for a given n-tuple $\underline{a}$ with elements in I we define

$$F(t) = \frac{1}{n}\sum_{i=1}^{n} f\Big(\sum_{j=1}^{n} p_{ij}(t)a_j\Big),\ 0 \le t \le 1.$$

Then F is convex on $[0,1]$ and if $\underline{w}$ is a positive n-tuple, $\underline{t}$ an n-tuple with elements in $[0,1]$

$$f\Big(\frac{1}{n}\sum_{j=1}^{n} a_j\Big) \le F\Big(\frac{1}{W_n}\sum_{i=1}^{n} w_i t_i\Big) \le \frac{1}{W_n}\Big(\sum_{i=1}^{n} w_i F(t_i)\Big) \le \frac{1}{n}\sum_{i=1}^{n} f(a_i),$$

in particular

$$f\Big(\frac{1}{n}\sum_{j=1}^{n} a_j\Big) \le F(t) \le \frac{1}{n}\sum_{i=1}^{n} f(a_i),\ 0 \le t \le 1,$$

$$f\Big(\frac{1}{n}\sum_{j=1}^{n} a_j\Big) \le \int_0^1 F \le \frac{1}{n}\sum_{i=1}^{n} f(a_i).$$

REMARK (vii) The last inequality is a discrete version of the Hadamard-Hermite inequality, 4.1 Theorem 9.

4.3 THE JENSEN-STEFFENSEN INEQUALITY The inequality (J) is not generalized if we allow the weights $\underline{w}$ to be non-negative, with of course $W_n \ne 0$. For if k elements of $\underline{w}$ are zero, $0 \le k < n$ then (J) becomes the same result but for an $(n-k)$-tuple; the case of equality when f is strictly convex has "*all elements of $\underline{a}$ that have non-zero weights are equal*" instead of " *$\underline{a}$ is constant*"; we then will say that *$\underline{a}$ is essentially constant* as it is convenient to call the elements of $\underline{a}$ that have non-zero weights the *essential elements of $\underline{a}$* for the given weights
Also in the case $n = 2$ if we allow negative weights (J) does not hold, see 4.2 Remark (i). However if $n \ge 3$ real weights are possible as the following extension of (J) due to Steffensen shows; see [*DI p.142*], [*Steffensen, 1919*].

THEOREM 20 [JENSEN-STEFFENSEN INEQUALITY] *If I is an interval in $\mathbb{R}$ on which f is convex and if $\underline{a}$ is a monotonic n-tuple, $n > 2$, with elements in I and if $\underline{w}$ is a real n-tuple satisfying*

$$W_n \ne 0,\ \textit{and}\ 0 \le \frac{W_i}{W_n} \le 1,\ 1 \le i \le n, \tag{9}$$

then (J) holds; further if f is strictly convex (J) is strict unless $\underline{a}$ is essentially constant.

Remark (i) Condition (9) is equivalent to

$$W_n \neq 0, \; and \, 0 \leq \frac{W_n - W_i}{W_n} \leq 1, \; 1 \leq i \leq n.$$

In addition there is no loss in generality is assuming $W_n > 0$ when (9) is equivalent to

$$0 \leq W_i \leq W_n, \quad \text{or to} \quad 0 \leq W_n - W_i \leq W_n, \; 1 \leq i \leq n.$$

□ In this proof we will assume, without loss in generality by Remark (i), that $\underline{a}$ is increasing.

As some of the weights may now be zero or negative it is not as obvious as in 4.2 Theorem 12 that the left-hand side of (J) is well defined. However $\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i = a_n + \sum_{i=1}^{n-1} \frac{W_i}{W_n} \Delta a_i$ and so by (9), and the hypothesis on $\underline{a}$, $a_1 \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i a_i \leq a_n$, in particular then $\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i \in I$.

(*i*) The first proof is by induction on n so let us assume the result for all k, $3 \leq k < n$, and assume without loss in generality that $W_n > 0$; this implies that $w_1, w_n \geq 0$. Further if $\underline{w} \geq 0$ the result reduces to (J) with a value of n the number of non-zero elements in $\underline{w}$; so assume there is a $k, 1 < k < n$, such that $w_i \geq 0, 1 \leq i < k$, and $w_k < 0$. Then:

$$\begin{aligned}
\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) =& \frac{W_{k-1}}{W_n}\left(\frac{1}{W_{k-1}}\sum_{i=1}^{k-1} w_i f(a_i)\right) + \frac{1}{W_n}\sum_{i=k}^{n} w_i f(a_i), \qquad (10)\\
\geq& \frac{W_{k-1}}{W_n} f\left(\frac{1}{W_{k-1}}\sum_{i=1}^{k-1} w_i a_i\right) + \frac{1}{W_n}\sum_{i=k}^{n} w_i f(a_i), \quad \text{by(J)},\\
=& \frac{W_{k-1}}{W_n} f\left(\frac{1}{W_{k-1}}\sum_{i=1}^{k-1} w_i a_i\right) - \frac{W_{k-1}}{W_n} f(a_k)\\
&+ \frac{W_k}{W_n} f(a_k) + \frac{1}{W_n}\sum_{i=k+1}^{n} w_i f(a_i). \qquad (11)
\end{aligned}$$

Now the coefficients $W_k, w_{k+1}, \dots, w_n$ satisfy condition (9) and so we can apply

the induction hypothesis to the last two terms of (11) to get

$$\begin{aligned}\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) \geq & \frac{W_{k-1}}{W_n} f\left(\frac{1}{W_{k-1}}\sum_{i=1}^{k-1} w_i a_i\right) \\ & - \frac{W_{k-1}}{W_n} f(a_k) + f\left(\frac{W_k}{W_n} a_k + \frac{1}{W_n}\sum_{i=k+1}^{n} w_i a_i\right).\end{aligned}$$

Now the three coefficients $W_{k-1}/W_n, -W_{k-1}/W_n, 1$ also satisfy (9) and the result follows by the case $n = 3$.

It remains to consider the case $n = 3$. In this case we can assume without loss in generality that $W_3 > 0, w_1, w_3 > 0, w_2 < 0$ and $a_1 < a_2 < a_3$; all other cases either contradict (9) or reduce to (J) in the case $n = 2$.

Consider then (10) and (11) with $k = 2$; there is no need to apply (J) so they are equal. Then, putting $\tilde{a} = \dfrac{W_2 a_2 + w_3 a_3}{W_3}$, we get that

$$\begin{aligned}\frac{1}{W_3}\sum_{i=1}^{3} w_i f(a_i) &= -\frac{w_1}{W_3}\big(f(a_2) - f(a_1)\big) + \frac{W_2 f(a_2) + w_3 f(a_3)}{W_3} \\ &\geq -\frac{w_1}{W_3}\big(f(a_2) - f(a_1)\big) + f(\tilde{a}), \quad \text{by(J)}.\end{aligned}$$

Hence, putting $\overline{a} = \dfrac{1}{W_3}\displaystyle\sum_{i=1}^{3} w_i a_i$

$$\begin{aligned}\frac{1}{W_3}\sum_{i=1}^{3} w_i f(a_i) - f(\overline{a}) &\geq -\frac{w_1}{W_3}\big(f(a_2) - f(a_1)\big) + f(\tilde{a}) - f(\overline{a}) \\ &= \frac{w_1(a_2 - a_1)}{W_3}\left[\frac{f(\tilde{a}) - f(\overline{a})}{\tilde{a} - \overline{a}} - \frac{f(a_2) - f(a_1)}{a_2 - a_1}\right] \\ &\geq 0, \quad \text{by 4.1 Lemma 2, since } a_2 < \tilde{a} < a_3, \text{ and } a_1 < \overline{a} < a_3.\end{aligned}$$

This completes the proof of the case $n = 3$.

(*ii*) Proof (viii) of 4.2 Theorem 12, gives a particularly simple geometrical induction of the Jensen-Steffensen inequality.

While $D_2(s)$, using the notation of that proof, is positive precisely on the interval $]0, 1[$, see 4.2 Remark (i), the function $D_3(s, t)$ is positive on a region larger than the interior of the triangle T. This is because D_3 is continuous and positive on the whole of T except for the corners.

Precisely $D_3 \geq 0$ in the region bounded by the 0-level curve of D_3 that passes through the corners of T. The region bounded by the 0-level curve depends, in

general, on the values of x, y, z. Thus if $x = y$ it is the strip $0 \leq t \leq 1$, while if $y = z$ it is the strip $0 \leq s + t \leq 1$.

The question to be taken up is to find, if possible, a region larger than T that does not depend on x, y, z, as T does not so depend, and on which $D_3 \geq 0$. The region we are looking for is $S = \bigcap_{\{(x,y,z);x<y<z\}} \{(s,t); D_3(x,y,z;s,t) \geq 0\}$. If T is a proper subset of S then Jensen's inequality will hold for certain negative values of the weights. This is just what Steffensen proved.

It follows from the above that S is a subset of the parallelogram common to the two strips $0 \leq t \leq 1$ and $0 \leq s + t \leq 1$; that is the region:

$$0 \leq s + t \leq 1, \qquad 0 \leq t \leq 1. \tag{P}$$

Note that this parallelogram P, unlike the triangle T, is not symmetric with respect to the variables s, t and so the condition $x \leq y \leq z$ used to prove Steffensen's extension is necessary..

Since, as we have observed, D_3 reduces to a D_2 on each of the sides of T it follows from the observations made about D_2, 4.2 Remarks (i), that $D_3 < 0$ on the extensions of these sides; that is $D_3(s,t) < 0$ if (i) $t = 0$ and $s < 0$ or $s > 1$; (ii) $s = 0$ and $t < 0$ or $t > 1$ (iii) $s + t = 1$ and $t < 0$ or $t > 1$.

In addition considerations of the partial derivatives of D_3 at the corners of T show that $D_3 < 0$ in the regions containing the external angles of T; that is the regions bounded by two rays on which we have just seen that D_3 is negative.

The tangent to the 0-level curve at the origin makes an angle θ_1 with the positive s-axis where $\tan\theta_1 = -\dfrac{\partial D_3/\partial s\,(0,0)}{\partial D_3/\partial t\,(0,0)} = -\dfrac{(y-x)\,(f'(c) - f'(x))}{(z-x)\,(f'(d) - f'(x))}$, here as below $x < c < y$, $x < d < z$; as a result we have that $-1 < \tan\theta_1 < 0$. This implies that the line $s + t = 0$ crosses the 0-level curve at the origin, being on the side of T when $s < 0$.

Similarly the tangent to the 0-level curve at the $(0,1)$ makes an angle θ_2 with the positive s-axis where $\tan\theta_2 = -\dfrac{(y-x)\,(f'(c) - f'(z))}{(z-x)\,(f'(d) - f'(z))}$,, and so $-1 < \tan\theta_2 < 0$. This implies that the line $s + t = 0$ crosses the 0-level curve at this point, being above the curve when $s < 0$.

A similar discussion at the point $(1,0)$ leads to an angle with a tangent that is sometimes positive and sometimes negative depending on the values of x, y, z, since there we have, with the obvious notation that $\tan\theta_3 = -\dfrac{(y-x)\,(f'(c) - f'(y))}{(z-x)\,(f'(d) - f'(y))}$, and so as is to be expected this corner is of no interest to us.

At the first two corners we have locally that D_3 is positive on the sides of P.

We now show that in fact D_3 is positive on these two sides of P.

Put $\phi(s) = D_3(s,1)$ when $\phi'(s) = f(y) - f(x) - f'(-sx+sy+z)(y-x)$ and $\phi''(s) = -f''(-sx+sy+z)(y-x)^2$. So ϕ is concave, zero at $s=0$, with a unique maximum at $s_0 < 0$, where $-s_0x + s_0y + z$ is a mean-value point of f on $[x,y]$.
Now consider $\gamma(s) = D_3(s,-s)$. A similar argument shows that γ is concave, zero at $s=0$ with a unique maximum at $s_1 < 0$ where $x + s_1y - s_1z$ is a mean-value point of f on $[y,z]$
So finally consider

$$D_3(-1,1) = f(x) - f(y) + f(z) - f(x-y+z) = \big(f(x) - f(x+h)\big) - \big(f(y) - f(y+h)\big),$$

where $h = z - y$. Hence by the convexity of f, we get that $D_3(-1,1) > 0$.
So D_3 is positive on the sides of P except at the corners of T and so by the general properties of D_3 we have that $D_3 \geq 0$ on P, being zero only at the corners of T ; in addition D_3 attains its maximum value on one of the sides of P.
In particular we have, on rewriting the inequalities that define P, that if $x \leq y \leq z$ then (J_3) holds, if

$$0 < 1 - s - t < 1, \; 0 < 1 - t < 1,$$

with equality only if $x = y = z$. This is Steffensen's extension of Jensen's inequality in the case $n = 3$.
Since, as we have seen, there is no Steffensen extension in the case $n = 2$, the preceding result is the first step in an inductive proof of the Steffensen theorem. As a result, to see how the induction proceeds we will consider the case $n = 4$. Here the function to look at is,

$$\begin{aligned} D_4(w,x,y,z;s,t,u) = D_4(s,t,u) &= (1-s-t-u)f(w) + sf(x) + tf(y) + uf(z) \\ &\quad - f\left(\overline{1-s-t-u}\, w + sx + ty + uz\right), \end{aligned}$$

with $w < x < y < z$ and

$$0 \leq s+t+u \leq 1, \qquad 0 \leq t+u \leq 1, \qquad 0 \leq u \leq 1. \qquad \text{S}$$

As before the function D_4 attains both its maximum and minimum values on S on the boundary of S. Unlike the case of T, and its higher dimensional analogues, the fundamental simplices, the restriction of D_4 to a face of S does not immediately reduce to a case of the Steffensen inequality for a lower value of n. However as we will see, it is possible on each face to make this reduction in at most two steps.
There are six cases to consider:
(I) $u = 0$; (II) $u = 1$; (III) $t+u = 0$; (IV) $t+u = 1$; (V) $s+t+u = 0$; (VI) $s+t+u = 1$.

Case (I) Here a simple reduction occurs: $D_4(w,x,y,z;s,t,0) = D_3(w,x,y;s,t)$, and $0 \le s+t \le 1$, $0 \le t \le 1$. So on this face $D_4 \ge 0$ by the case $n = 3$.
Case (II) In this case $0 \le -s-t \le 1$, $0 \le -t \le 1$ and we have to show that $D_4(w,x,y,z;s,t,1) \ge 0$. Now,

$$\begin{aligned}(-s-t)f(w)+&sf(x)+tf(y)+f(z)\\ &=(-s-t)f(w)+(s+t)f(x)-tf(x)+tf(y)+f(z),\\ &\ge(-s-t)f(w)+(s+t)f(x)+f(-tx+ty+z), \text{ by the case } n=3,\\ &\ge f\left(\overline{-s-t}\,w+\overline{s+t}\,x+[-tx+ty+z]\right), \text{ by the case }, n=3,\\ &=f\left(\overline{-s-t}\,w+sx+ty+z\right),\end{aligned}$$

which gives this case.
Case (III) Here $0 \le s \le 1$, $0 \le -t \le 1$ and we consider $D_4(w,x,y,z;s,t,-t)$. Now

$$\begin{aligned}(1-s)f(w)+sf(x)+tf(y)-tf(z) &\ge f(\overline{1-s}\,w+sx)+tf(y)-tf(z), \quad \text{by } (\mathrm{J}_2),\\ &\ge f(\overline{1-s}\,w+sx]+ty-tz), \quad \text{by the case } n=3,\\ &=f(\overline{1-s}\,w+sx+ty-tz).\end{aligned}$$

So $D_4(w,x,y,z;s,t,-t) \ge 0$.
Case (IV) Now $0 \le -s \le 1$, $0 \le t \le 1$ and we must prove $D_4(w,x,y,z;s,t,1-t) \ge 0$. Now,

$$\begin{aligned}-sf(w)+sf(x)+tf(y)+&(1-t)f(z)\\ &\ge -sf(w)+sf(x)+f(ty+\overline{1-t}\,z), \text{ by } (\mathrm{J}_2),\\ &\ge f(-sw+sx+[ty)+\overline{1-t}\,z]), \text{ by the case } n=3,\end{aligned}$$

which gives this case.
Case (V) Here $0 \le -s \le 1$, $0 \le -s-t \le 1$ and

$$\begin{aligned}f(w)+sf(x)+&tf(y)+(-s-t)f(z)\\ &=f(w)+sf(x)-sf(y)+(s+t)f(y)+(-s-t)f(z),\\ &\ge f(w+sx-sy)+(s+t)f(y)+(-s-t)f(z), \text{ by the case } n=3,\\ &\ge f([w+sx-sy]+\overline{s+t}\,y+\overline{-s-t}z), \text{ by the case } n=3,\\ &=f(w+sx-sy+ty+\overline{-s-t}z),\end{aligned}$$

which shows that $D_4(w,x,y,z;s,t,-s-t) \ge 0$.
Case (VI) Like Case (I) this case reduces directly to a case where $n = 3$ as $D_4(w,x,y,z;1-t-u,t,u)$ is just $D_3(x,y,z;t,u)$ and $0 \le t+u \le 1$, $0 \le u \le 1$.

These arguments can be extended to higher values of n

□

REMARK (ii) The above inductive proof and the geometric proof are due to Bullen, [*Bullen 1970a, 1998*]. The geometric proof shows that the domain of weights defined by (9) is best possible.

REMARK (iii) Note that as the domain defined by (9) is not invariant under permutations of the order so that the condition that $\underline{a}$ be monotone is necessary; see the comment in proof (*ii*).

REMARK (iv) A different proof can be found in [*MI*] ; see also [*PPT p.57*], [*Lovera; Magnus; Pečarić 1981b, 1984a, 1985a*].

REMARK (v) It was shown in the proof of the last theorem that condition (9) on the weights implies that if $\underline{a}$ in increasing then $\frac{1}{W_n}\sum_{i=1}^{n} w_1 a_i$ lies in the interval $[a_1, a_n]$. The converse is also true as choosing for $\underline{a}$ the n-tuples $(-1, 0, \ldots, 0)$, $(-1, -1, 0, \ldots, 0)$, $\ldots$, $(-1, -1, \ldots, -1, 0), (-1, -1, \ldots, -1)$ shows.

REMARK (vi) For an application of the argument in proof (*ii*) see II 5.8 Remark (v).

4.4 REVERSE AND CONVERSE JENSEN INEQUALITIES We have seen in 4.2 Remark (i) when $(\sim J_2)$ or $(\sim J_3)$ holds. Various reverse inequalities have been proved for general n; we just mention two of these.

THEOREM 21 [REVERSE JENSEN] *If I is an interval in $\mathbb{R}$ on which f is convex, if $n \geq 2$, $\underline{w}$ a real n-tuple, with $W_n > 0$, $w_1 > 0, w_i < 0, 2 \leq i \leq n$, $\underline{a}$ an n-tuple with elements in I and with $\overline{a} = \frac{1}{W_n}\sum_{i=1}^{n} w_i a_i \in I$ then $(\sim J)$ holds. If f is strictly convex then $(\sim J)$ is strict unless $\underline{a}$ is constant.*

□ This is a particular case of 4.2 Theorem 1; use the n-tuple $(\overline{a}, a_2, \ldots, a_n)$, and weights $(W_n, -w_2, \ldots, -w_n)$.

□

REMARK (i) This is is given in [*PPT p.83*], [*Vasić & Pečarić 1979a*]; these authors have given similar reversals for 4.2 Theorem 15 and 4.2 Corollary 16. A special case of this result is that of the inequality between the pseudo-arithmetic and pseudo-geometric means discussed in II 5.8, and a related topic is the Aczél-Lorentz inequality, III 2.5.7.

THEOREM 22 [REVERSE JENSEN-STEFFENSEN] *Let $\underline{a}, I, n$, be as in Theorem 21 with $\underline{w}$ a real n-tuple with $W_n > 0$. Then ($\sim$J) holds for all functions f convex on I and for every monotonic $\underline{a}$ if and only if for some $m, 1 \le m \le n, W_k \le 0, 1 \le k < m$ and $W_n - W_k \le 0, m < k \le n$.*

□ See [*PPT pp.83–84*], [*Pečarić 1981a, 1984b*]. □

The following upper bound for the right-hand side of (J) was proved in [*Lah & Ribarič*] but the proof given is in [*Beesack & Pečarić*]; see also [*PPT p.98*], [*Pečarić 1990b*].

THEOREM 23 *If f is convex on the interval $I = [a, b]$ and $\underline{a}$ and n-tuple with elements in I, and $\underline{w}$ a positive n-tuple, and if $\overline{a}$ is as in Theorem 21 then*

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) \le \frac{b-\overline{a}}{b-a} f(a) + \frac{\overline{a}-a}{b-a} f(b). \tag{12}$$

The right-hand side of (12) increases, respectively decreases, as a function of b, respectively a. In addition if f is strictly convex, there is equality in (12) if and only if $\underline{a}$ is constant.

□ From 4.1 (2) we have that $f(a_i) \le \dfrac{b-a_i}{b-a} f(a) + \dfrac{a_i - a}{b-a} f(b), 1 \le i \le n$. This clearly implies (12).
The case of equality is easily deduced and the monotonicity follows by writing the right-hand side of (12) as $f(a) + (\overline{a} - a)\dfrac{f(b) - f(a)}{b-a}$ or $f(b) - (b - \overline{a})\dfrac{f(b) - f(a)}{b-a}$, and using 4.1 (3). □

It should be noted that (J) and Theorem 23 can be interpreted as follows,[10] see [*Lob*]:

THEOREM 24 *If f is convex on the interval $I = [a, b]$, $\underline{a}$ an n-tuple with elements in I, and $\underline{w}$ a positive n-tuple then the centroid G of the points $\big(a_i, f(a_i)\big)$ with weights $w_i, 1 \le i \le n$, $(\overline{a}, \overline{\alpha})$, lies above the graph of f and below the chord AB where $A = \big(a, f(a)\big), B = \big(b, f(b)\big)$; further if f is strictly convex and $\underline{a}$ is not constant these inclusions are strict.*

This simple geometric fact has been used by Mitrinović & Vasić to obtain an extension of (J); see [*Mitrinović & Vasić 1975*]. Interesting comments on the history of this method of proof, the so-called *centroid method*, can be found in an earlier paper, [*Mitrinović & Vasić 1974*]; see also [*MPF pp.681–695*], [*Aczél & Fenyő 1948a; Beesack 1983; Beesack & Pečarić*].

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[10] The *centroid* of the points $\underline{a}_1,\ldots,\underline{a}_k$ is $(\underline{a}_1+\cdots\underline{a}_k)/k$; the *centroid* of the points $\underline{a}_1,\ldots,\underline{a}_k$ with *weights* $w_1,\ldots,w_k$, where $w_j>0, 1\le j\le k$, is $(w_1\underline{a}_1+\cdots w_k\underline{a}_k)/W_k$. In particular $\overline{\alpha}=\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i)$.

THEOREM 25 [CONVERSE JENSEN INEQUALITY] *If $I = [a,b]$, and if the function f is positive, strictly convex and twice differentiable on I, $\underline{a}$ an n-tuple with elements in I, $n \geq 2$, and $\underline{w}$ a positive n-tuple then*

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) \leq \lambda f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right); \tag{13}$$

where if

$$\phi = (f')^{-1},\ \mu = \frac{f(b)-f(a)}{b-a},\ \nu = \frac{bf(a)-af(b)}{b-a},$$

then λ is the unique solution of the equation

$$f \circ \phi(\frac{\mu}{\lambda}) = \frac{\mu}{\lambda}\phi(\frac{\mu}{\lambda}) + \frac{\nu}{\lambda}. \tag{14}$$

□ Noting by Theorem 4.1 Theorem 4(d) that if f is convex so is

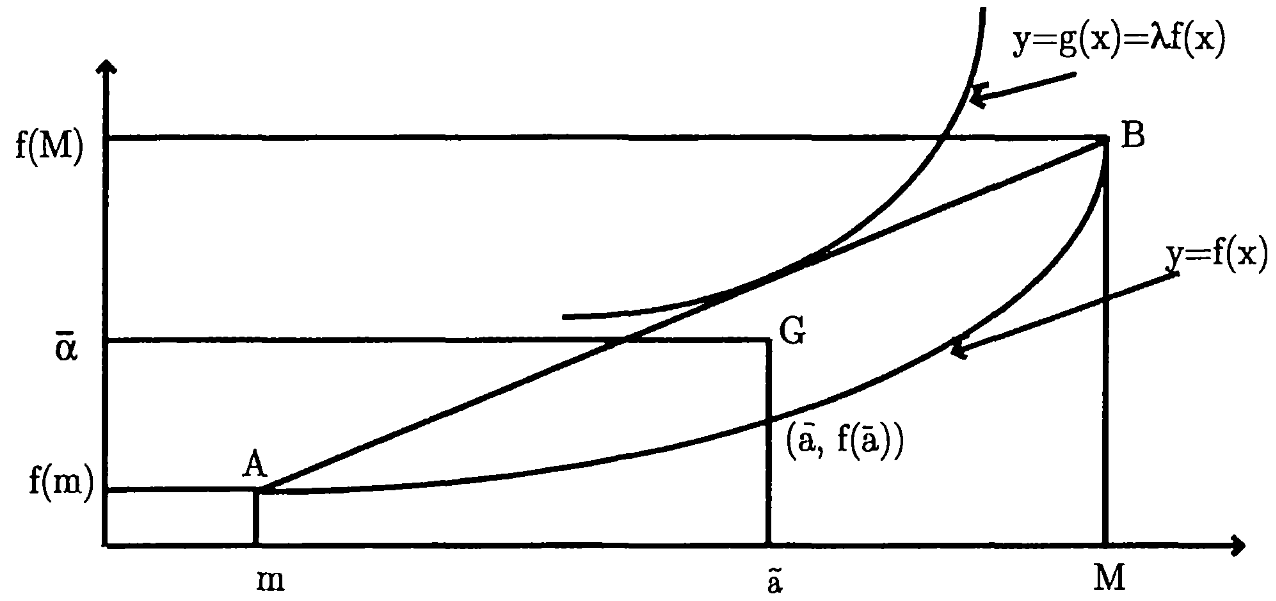

Figure 2

$g = \lambda f, \lambda > 0$, choose λ so that the graph of g touches AB. Then AB lies below the graph of g as also does the centroid G; see Figure 2.

This gives (13), except for the determination of λ.

The equation of AB is $y = \mu x + \nu$ and this line is tangent to $y = g(x) = \lambda f(x)$ if $\lambda f(x) = \mu x + \nu$ and $\lambda f'(x) = \mu$.

Solving these two equations for x leads to the equation $F(x) = 0$ where $F(x) = \mu f(x) - f'(x)(\mu x + \nu)$. The graph of F cuts the axis in exactly one point in I since $F(a)F(b) \leq 0$, and F is strictly decreasing. Hence there is a unique λ that is given by (14). □

REMARK (ii) There is equality in (13) if and only if the point of contact of $y = g(x)$ with AB coincides with the centroid G. Now for G to lie on AB is is both necessary and sufficient that there is an $S \subseteq \{1,2,\ldots,n\}$ such that $a_i = b, i \in S, a_i = a, i \notin S$. If then $\frac{1}{W_n}\sum_{i\in S} w_i = \theta$ there is equality if and only if $f\big(b\theta + a(1-\theta)\big) = f(b)\theta + (1-\theta)f(a)$.

Using the curves $y = \gamma + f(x)$ Mitrinović & Vasić have used the centroid method to obtain the following converse inequality, [*Mitrinović & Vasić 1975*].

THEOREM 26 *Under the same conditions and notations as in Theorem 25*

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) \le \gamma + f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right), \tag{15}$$

where $\gamma = \mu\phi(\mu) + \nu - f \circ \phi(\mu)$.

REMARK (iii) Similar considerations to those in Remark (ii) can be used to obtain the cases of equality in (15).

REMARK (iv) A particularly simple application of this method is Dočev's inequality, II 4.2 Theorem 4.

Pečarić & Beesack have noted that Theorem 23 can be used to generalize Theorems 25 and 26.

THEOREM 27 *Let* $I, f, \underline{a}, \overline{a}, \underline{w}$ *be as in Theorem 23, and let* J *be an interval containing the range of* f, $F : J^2 \mapsto \mathbb{R}$ *increasing in the first variable, then*

$$F\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i),\ f(\overline{a})\right) \le \max_{x\in I}\left\{F\left(\frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)\right),\ f(x)\right\}. \tag{16}$$

The right-hand side of (16) is an increasing function of b, *and a decreasing function of* a.

REMARK (v) Theorems 25 and 26 can be obtained from Theorem 27 by taking $F(x, y) = x/y,\ x - y$ respectively.

The second proof of Jensen's inequality, 4.2 Theorem 12 proof (*ii*), gives a natural proof of a converse of (J); [*Bullen 1998*].

THEOREM 28 [CONVERSE JENSEN INEQUALITY] *If* $f : [a, b] \mapsto \mathbb{R}$ *is twice differentiable with* $f'' \ge 0$ *and if on every subinterval there is a a point where* $f'' > 0$ *then*

$$\begin{aligned}\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) - f\Big(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\Big)& \\ \le (1-t_0)f(a) + t_0 f(b) - f\big(\overline{1-t_0}\,a + t_0 b\big)&,\end{aligned} \tag{17}$$

where a, b *are, respectively the smallest and the largest of the* a_i *with associated non-zero* w_i, *and where* $\overline{1-t_0}\,a + t_0 b$ *is the mean-value point for* f *on* $[a, b]$. *There is equality in (17) only if either all the* $\underline{a}$ *is essentially constant or if all the* w_i *are*

zero except those associated with a and b, and then these have weights $1 - t_0, t_0$ *respectively.*

□ From 4.1 Theorem 6(b) f is strictly convex on $[a, b]$. For the rest of the proof we use the notations of 4.2 Theorem 12 proof (*ii*).
Clearly the argument in that proof shows that $D_2(s)$ attains a unique maximum at $s = s_0$, so that $D_2(s) \le D_2(s_0)$ with equality only if $s = s_0$. So if $0 < s < 1$,

$$\begin{aligned} 0 \le (1-s)f(x) + sf(y) - f\big(\overline{1-s}\,x + sy\big) \\ \le (1-s_0)f(x) + s_0 f(y) - f\big(\overline{1-s_0}\,x + s_0 y\big) \end{aligned}$$

with equality on the left, (J_2), only if $x = y$, or on the right, a converse of (J_2), if $x = y$ or if $s = s_0$. Since $D_2'(s) = f(y) - f(x) - (y-x)f'\big(\overline{1-s}\,x + sy\big)$ and $D_2(s_0) = 0$ we see that $\overline{1-s_0}\,x + s_0 y$ is the mean-value point of f on $[a, b]$. This gives (17) in the case $n = 2$.
Similarly the argument in the same proof shows that $D_3(s)$ attains its maximum value on the boundary of T, and not at the corners where it attains its minimum value, 0. As pointed out in that proof on each edge of T the problem reduces to a case of $n = 2$. So from the above D_3 has a local maximum on each edges and the question is, which is the largest of the three? If $0 < s < 1$ and we considerD_2 as a function of x then $D_2'(x) = (1-s)\left(f'(x) - f'\big(\overline{1-s}\,x + sy\big)\right)$, which is negative; while if we consider D_2 as a function of y then $D_2'(y) = s\left(f'(y) - f'\big(\overline{1-s}\,x + sy\big)\right)$, which is positive. Hence if $x' \le x < y \le y'$ with not both $x' = x$ and $y' = y$ then $D_2(x', y'; s) > D_2(x, y; s)$, $0 < s < 1$; in particular the maximum value of $D_2(x', y'; s)$ is larger than that of $D_2(x, y; s)$.
Hence the maximum of D_3 occurs at $(0, t_0)$ where $\overline{1-t_0}\,x + t_0 z$ is the mean-value point for f on $[x, z]$, since we have taken $x = \min\{x, y, z\}, z = \max\{x, y, z\}$; that is if $0 < s < 1, 1 < t < 1, 0 < s + t < 1$ then

$$\begin{aligned} (1-s-t)f(x) + sf(y) + tf(z) - f\big(\overline{1-s-t}\,x + sy + tz\big) \\ \le (1-t_0)f(x) + t_0 f(z) - f\big(\overline{1-t_0}\,x + t_0 z\big), \end{aligned}$$

with equality only if $x = y = z$, or $s = 0, t = t_0$. This gives (17) in the case $n = 3$.
Clearly the method extends readily to general n. □

REMARK (vi) (J), as well as a converse of Giaccardi, [*Giaccardi 1955*], have been used by several authors to obtain bounds for various functions; see for instance [*Afuwape & Imoru; Kečkić & Vasić*].

REMARK (vii) For a different kind of converse to (J) due to Slater see [*PPT pp.63*]; [*Slater*].

REMARK (viii) Inequality (J) and its converse have been objects of much recent study. Interesting generalizations can be found in [*DI pp.139–142; MPF pp.1–20; PPT pp.43–106*], [*Pečarić*], where there are further references and many details, and in [*Abou-Tair & Sulaiman 1998; Alzer 1991g; Barlow, Marshall & Proschan; Dragomir 1989a,b, 1991, 1992a, 1994a, 1995b; Dragomir & Goh 1996; Dragomir & Ionescu 1994; Dragomir & Milošević 1992, 1994; Dragomir & Toader; Dragomirescu 1989, 1992a,b, 1994; Dragomirescu & Constantin 1990, 1991; Fu, Bao, Zhang Z & Zhang Y; Gatti 1956a; Imoru 1974a, 1975; Lin Y; Marshall & Proschan; Mitrinović & Pečarić 1987; Mond & Pečarić 1993; Needham; Nishi; Pečarić 1979, 1987c, 1990a, 1991a,b; Pečarić & Andrica; Pečarić & Beesack 1986, 1987; Pečarić & Dragomir 1989; Petrović; Pittenger 1990; Rusǐnov; Schaumberger & Kabak 1989,1991; Sirotkina; Takahasi, Tsukuda, Tanahashi & Ogiwara; Vasić & Pečarić 1979c; Vasić & Stanković 1973, 1976; Vincze I*].

4.5 OTHER FORMS OF CONVEXITY

4.5.1 MID-POINT CONVEXITY

DEFINITION 29 *If I is an interval in $\mathbb{R}$ then $f : I \mapsto \mathbb{R}$ is said to be* mid-point convex, or J-convex, on I *if for all $x, y \in I$,*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}. \tag{18}$$

REMARK (i) Since (18) is easily seen to imply (1) for a dense subset of λ it is immediate that a continuous mid-point convex function is convex; the converse is false, see [*RV p.216*]. However very mild restrictions on a mid-point convex function do imply its convexity; see [*RV p. 215*].[11]

THEOREM 30 *If f is mid-point convex on I and bounded at one point of I then it is convex.*

4.5.2 LOG-CONVEXITY

DEFINITION 31 *If I is an interval in $\mathbb{R}$, then a function $f : I \mapsto \mathbb{R}_+^*$ is said to be* log-convex *or* multiplicatively convex *on I if $\log \circ f$ is convex, or, equivalently if for all $x, y \in I$ and all $\lambda, 0 \leq \lambda \leq 1$,*

$$f\left(\lambda x + \overline{1-\lambda} y\right) \leq f^{\lambda}(x) f^{1-\lambda}(y).$$

EXAMPLE (i) The function $x!, x > -1$, is strictly log-con vex; see [*Artin*].

[11] See Notations 9.

LEMMA 32 *(a) A log-convex function is convex*
(b) If $f > 0$ then f is log-convex if and only if $g(x) = e^{ax}f(x)$ is convex for all $a \in \mathbb{R}$.

□ (a) This is an immediate consequence of 4.1 Theorem 4(f).
(b) See [*AI p.19*]. □

EXAMPLE (ii) The converse of (a) is false as $f(x) = x^{3/2}$, $x > 0$, shows.

REMARK (i) See also [*DI p.163; RV pp.18–19*], [*Artin pp.7–14*].

4.5.3 A FUNCTION CONVEX WITH RESPECT TO ANOTHER FUNCTION

DEFINITION 33 *If $I \subseteq \mathbb{R}$ is an interval and $f, g : I \mapsto \mathbb{R}$ are continuous, and g strictly monotonic, $J = g[I]$, also an interval, then f is said to be* (strictly) convex with respect to g *if any of the following equivalent statements hold:*
(a) $f \circ g^{-1}$ is (strictly) convex;
(b) for some function k (strictly) convex on J, $f = k \circ g$;
(c) the curve $x = g(t)$, $y = f(t)$, $t \in I$, is (strictly) convex.

REMARK (i) Note that the curve in (c) is actually a graph; see 4.2 Remark (ii).

REMARK (ii) This idea is a convenient way of expressing a property that occurs naturally; see [*HLP p.75*], [*Cargo 1965; Mikusiński*].

EXAMPLE (i) In this terminology f is log-convex if and only if log is convex with respect to f^{-1}; or in the terminology of the 4.5.2 Definition 31 we could say f is convex with respect to g as g^{-1} is f-convex.

EXAMPLE (ii) f is convex if and only if it is convex with respect to a non-constant affine function; see 4.1 Remark (viii).

EXAMPLE (iii) f is convex with respect to g on I if and only if for all $x_i, 1 \le i \le 3$, three points of I with $x_1 < x_2 < x_3$,

$$\begin{vmatrix} 1 & g(x_1) & f(x_1) \\ 1 & g(x_2) & f(x_2) \\ 1 & g(x_3) & f(x_3) \end{vmatrix} \ge 0.$$

This is a natural extension of 4.1 (2**).

Although the conditions for f to be convex with respect to g follow from conditions for convexity the following condition has been obtained by Mikusiński, and in a different way by Cargo.

THEOREM 34 *If* $g \in \mathcal{C}^2(J)$, *and* $f \in \mathcal{C}^2(I)$, *where* I, J *are as in Definition 33, and if* f', g' *are never zero then a sufficient condition for* f *to be convex with respect to* g *is that*

$$\frac{g''}{g'} \leq \frac{f''}{f'}.$$

REMARK (iii) A geometric interpretation of this property is given in [*Mikusiński*]. If f is convex with respect to g on $I = [a, b]$, g strictly increasing, and $f(a) = g(a)$, $f(b) = g(b)$ then $f(x) \leq g(x)$, $a \leq x \leq b$; see IV 2 Corollary 8. This extends the property of convex functions in 4.1 Remark (iii); see also [*RV pp.240–246*].

EXAMPLE (iv) It is easy to check that $\log x$ is convex with respect to $1 - x^{-1}$ and using the previous remark this implies the left-hand side of 2.2 (9).

4.6 CONVEX FUNCTIONS OF SEVERAL VARIABLES To extend the concept of convex function to higher dimensions we need to make the nature of its domain more precise.

DEFINITION 35 *(a) A set* $U \subseteq \mathbb{R}^n$, $n \geq 1$, *is said to be* convex *if* $\underline{a} \in U$ *and* $\underline{b} \in U$ *implies that* $(1 - \lambda)\underline{a} + \lambda\underline{b} \in U$ *for all* λ, $0 \leq \lambda \leq 1$.
(b) If $U \subseteq \mathbb{R}^n$, $n \geq 1$, *then the* convex hull of A *is the smallest convex set that contains* A.

EXAMPLE (i) If U is a finite set $U = \{\underline{a}_1, \ldots \underline{a}_k\}$, say, then the convex hull is the set $\{\underline{x};\ \underline{x} = \sum_{i=1}^{k} w_i\underline{a}_i,\ \underline{w} \geq 0,\ W_k = 1\}$, the set of all *convex combinations of the points of* U.

EXAMPLE (ii) In $\mathbb{R}$ a set is convex if and only if it is an interval.

EXAMPLE (iii) If $f : I \mapsto \mathbb{R}$ then f is convex if and only if the set $E \subseteq \mathbb{R}^2$, $E = \{(x, y);\ x \in I,\ y \geq f(x)\}$ is convex. This set is called the *epigraph of* f, and the definition easily extends to real-valued functions defined on subsets of $\mathbb{R}^n$, $n \geq 2$.

By replacing the interval I in $\mathbb{R}$ by a convex set U in $\mathbb{R}^n, n \geq 2$, and the points x, y of I by points $\underline{u}, \underline{v}$ of U we can easily extend 4.1 Definition 1 to functions $f : U \to \mathbb{R}$, and 4.5.2 Definition 31 to functions $f : U \to \mathbb{R}_+^*$; [*DI p.63*]. Jensen's inequality, 4.2 Theorem 12, and Theorem 18 extend to this situation with the same proofs, and 4.1 Theorem 4(a), (b) is valid in the following form; see [*RV pp.93, 116–117*]; and 4.5.1 Theorem 30 has a natural extension.

THEOREM 36 *If $f : U \mapsto \mathbb{R}$ is convex on the open convex set U in $\mathbb{R}^n, n \geq 2$, then f is Lipschitz on every compact subset of U and has first order partial derivatives almost everywhere that are continuous on the sets where they exist.*

There is no single criterion for investigating the convexity of functions of several variables. It is therefore useful to define some ideas that will help in special situations, and to list several criteria for convexity.

DEFINITION 37 *(a) A set U in $\mathbb{R}^n$ is called a* cone *if $\underline{u} \in U$ implies that for all $\lambda \geq 0$, we have $\lambda \underline{u} \in U$.*
(b) A function f defined on a cone is homogeneous of degree α *if for all $\lambda > 0$ we have $f(\lambda \underline{u}) = \lambda^\alpha f(\underline{u})$.*
(c) If f is twice differentiable the matrix $H = (f''_{ij})_{1 \leq i,j \leq n}$ is called the Hessian matrix *of f.*
(d) The directional derivative of f at $\underline{u}_0$ in the direction $\underline{v}$ *is*

$$f'_+(\underline{u}_0; \underline{v}) = \lim_{t \to 0+} \frac{f(\underline{u}_0 + t\underline{v}) - f(\underline{u}_0)}{t},$$

when the limit exists.

REMARK (i) A function f that is homogeneous of degree 1 is just said to be *homogeneous.*

REMARK (ii) If f is homogeneous of degree α and is differentiable then the partial derivatives of f are homogeneous of degree $\alpha - 1$ and we have the *Euler's theorem on homogeneous functions* : $u_1 f'_1(\underline{u}) + \cdots + u_n f'_n(\underline{u}) = \alpha f(\underline{u})$.

EXAMPLE (iv) The sets $\mathbb{R}^n_+, (\mathbb{R}^*_+)^n$ are cones.

EXAMPLE (v) If f is differentiable at $\underline{u}_0$ then

$$f'_+(\underline{u}_0; \underline{v}) = \nabla f(\underline{u}_0).\underline{v}.$$

THEOREM 38 *A homogeneous function is convex on the cone U if and only if for all $u, v \in U$,*

$$f(\underline{u} + \underline{v}) \leq f(\underline{u}) + f(\underline{v}), \tag{19}$$

and is strictly convex if and only if (19) holds strictly for all $u, v \in U, \underline{u} \not\sim \underline{v}$.

THEOREM 39 *If f is convex on the open convex set U then $f'_+(\underline{u}_0; \underline{u} - \underline{u}_0)$ exists and*

$$f(\underline{u}) \geq f(\underline{u}_0) + f'_+(\underline{u}_0; \underline{u} - \underline{u}_0). \tag{20}$$

If f is strictly convex and homogeneous then there is equality in (20) if and only if $\underline{u} \sim \underline{u}_0$.
Conversely if f is differentiable on U and (20) holds, strictly, for all $\underline{u}_0, \underline{u} \in U$ then f is convex, strictly convex.

THEOREM 40 *(a) A function is convex if and only if its epigraph is a convex set.*
(b) If f and g are convex, strictly convex, then so is $\alpha f + \beta g$ for all $\alpha, \beta > 0$.
(c) If $f_\alpha, \alpha \in A$, is a family of functions convex on U then $f = \sup_{\alpha \in A}$ is convex on $\{\underline{u};\ \underline{u} \in U, f(\underline{u}) < \infty\}$
(d) If f is convex and monotone in each variable on the parallelepiped $\underline{m} \le \underline{u} \le \underline{M}$, and if each function g_i, $1 \le i \le n$, is convex if f increasing for the ith variable, and concave if f is decreasing for the ith variable, and if $m_i \le g_i \le M_i$, $1 \le i \le n$, then $f(g_1, \ldots g_n)$ is convex.
(e) If f is convex, strictly convex, and homogeneous of degree α, $\alpha > 1$, and positive except possibly at the origin then $f^{1/\alpha}$ is convex, strictly convex, and homogeneous.
(f) Suppose that f is a homogeneous function on a cone in the set $\{\underline{a}; a_n > 0\}$ and convex as a function of $(u_1, \ldots, u_{n-1}, 1)$ then f is convex.
(g) Let $f \in \mathcal{C}^2(U)$, U an open convex set in $\mathbb{R}^n, n \ge 2$. A function f is convex on U if and only if the Hessian matrix H of f is negative semi-definite; if H is positive definite on U then f is strictly convex.

Proofs of the Theorems 38–40 can be found in the standard references; [*RV pp. 94–95, 98, 103, 117*], [*Rockafellar pp.23–27, 30, 32–40, 213–214*], [Soloviov].

REMARK (iii) It is easy to state an analogous result for concave, strictly concave, functions.

REMARK (iv) If $n = 2$ then $H = \begin{pmatrix} f''_{11} & f''_{12} \\ f''_{12} & f_{22} \end{pmatrix}$ and it is non-negative definite if and only if

$$f''_{11} \ge 0, \ [\text{or } f''_{22} \ge 0], \text{ and } f''_{11} f''_{22} - f''^2_{12} \ge 0; \tag{21}$$

H is positive definite if the inequalities (21) are strict; see [*HLP pp.80–81*].
The following examples are important for later applications.

EXAMPLE (vi) Let $\underline{w} \in (\mathbb{R}^*_+)^n$ and $p > 1$ then $\phi(\underline{a}) = \sum_{i=1}^n w_i a_i^p$ is strictly convex on $(\mathbb{R}^*_+)^n$. This follows from the strict convexity of $f(x) = x^p$, $x > 0$, $p > 1$, and Theorem 40 (b).

EXAMPLE (vii) If $\underline{w}$ and p are as in then previous example then $\gamma(\underline{a}) = \phi^{1/p}(\underline{a}) = \left(\sum_{i=1}^n w_i a_i^p\right)^{1/p}$ is strictly convex on $(\mathbb{R}^*_+)^n$. This follows from Example (vi) and Theorem 40 (e) since ϕ is homogeneous of degree p..

EXAMPLE (viii) If $\underline{w}$ is as in then previous examples then $\chi(\underline{a}) = \prod_{i=1}^{n} a_i^{(w_i/W_n)}$ is concave on $(\mathbb{R}_+^*)^n$, strictly if $n > 1$. The concavity is trivial if $n = 1$ so suppose that we have concavity for $n-1$ and write

$$\chi(\underline{a}) = a_n^{(w_n/W_n)} \Big(\prod_{i=1}^{n-1} a_i^{(w_i/W_{n-1})} \Big)^{W_{n-1}/W_n}. \tag{22}$$

Now $g(x) = x^{(w_n/W_n)}, x > 0$ is strictly concave, by the induction hypothesis $\prod_{i=1}^{n-1} a_i^{(w_i/W_{n-1})}$ is concave, and $h(x) = x^{W_{n-1}/W_n}$ is strictly concave. By Theorem 40 (d) the second term on the right-hand side of (22) is strictly concave. So using Theorem 40(f) χ is strictly concave on A; see also [*Rockafellar pp. 27–28*].

EXAMPLE (ix) A further example of a concave function is $p(\underline{a}^{1/m})$, where p is a polynomial that is homogeneous of degree m; see III 2.1 Remark (iii).

Inequality (20) can be extended in the case of homogeneous convex functions to the following inequality that is called the *support inequality*; [*Soloviov*].

THEOREM 41 [SUPPORT INEQUALITY] *If f is a convex homogeneous function of the cone U then for all $\underline{u}, \underline{v} \in U$,*

$$f(\underline{u}) \ge f'_+(\underline{v}; \underline{u}); \tag{23}$$

if f is strictly convex then there is equality in (23) if and only if $\underline{u} \sim \underline{v}$.

□ First note that

$$\begin{aligned} f\big(\underline{v} + \lambda(\underline{u} - \underline{v})\big) &\ge f(\underline{v} + \lambda\underline{u}) - f(\lambda\underline{v}), && \text{by convexity,} \\ &= f(\underline{v} + \lambda\underline{u}) - \lambda f(\underline{v}), && \text{by homogeneity.} \end{aligned}$$

So

$$\begin{aligned} f'_+(\underline{v}; \underline{u} - \underline{v}) &= \lim_{\lambda \to 0+} \frac{f\big(\underline{v} + \lambda(\underline{u} - \underline{v})\big) - f(\underline{v})}{\lambda}, \\ &\ge \lim_{\lambda \to 0+} \frac{f(\underline{v} + \lambda\underline{u}) - \lambda f(\underline{v}) - f(\underline{v})}{\lambda} \\ &= f'_+(\underline{v}; \underline{u}) - f(\underline{v}). \end{aligned}$$

Then (23) follows from (20).
The case of equality follows from that in Theorem 39. □

REMARK (v) If f is concave then ($\sim$23) holds. Further if f is differentiable then (23) is just

$$f(\underline{u}) \ge \nabla f(\underline{v}).\underline{u}. \tag{24}$$

4.7 HIGHER ORDER CONVEXITY Let $a_i, 0 \le i \le n, n \ge 0$, be $(n+1)$ distinct points from the real interval I then if $f : I \mapsto \mathbb{R}$ the *n-th divided difference of f at these points* is [12]

$$[\underline{a}; f] = [\underline{a}; f]_n = [a_0, \ldots, a_n; f] = \sum_{i=0}^{n} \frac{f(a_i)}{\varpi'(a_i)},$$

where $\varpi(x) = \varpi_n(x; \underline{a}) = \prod_{i=0}^{n}(x - a_i)$.

REMARK (i) It is worth noting that

$$[\underline{a}; f]_n = \frac{\det\begin{pmatrix} f(a_0) & f(a_1) & \cdots & f(a_n) \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} a_0^n & a_1^n & \cdots & a_n^n \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}}; \tag{25}$$

see for instance [*Popoviciu 1934*a].

EXAMPLE (i)

$$[\underline{a}; f]_1 = \frac{f(a_1) - f(a_0)}{a_1 - a_0},$$
$$[\underline{a}; f]_2 = \frac{f(a_0)}{(a_0 - a_1)(a_0 - a_2)} + \frac{f(a_1)}{(a_1 - a_2)(a_1 - a_0)} + \frac{f(a_2)}{(a_2 - a_0)(a_2 - a_1)}.$$

REMARK (ii) The last expression shows that the left-hand side of 4.1(2*) is just $[x_1, x_2, x_3; f]_2$.

EXAMPLE (ii) If $f(x) = x^p, p = -1, 0, 1, 2, \ldots$ then

$$[\underline{a}; f]_n = \begin{cases} 0, & \text{if } p = 0, 1 \ldots, n-1; \\ 1, & \text{if } p = n; \\ \displaystyle\sum_{k_0 + \cdots + k_n = p-n} (a_0^{k_0} a_1^{k_1} \ldots a_n^{k_n}), & \text{if } p = n+1, n+2, \ldots; \\ \dfrac{(-1)^n}{a_0 a_1 \ldots a_n}, & \text{if } p = -1. \end{cases}$$

See [*Milne-Thomson pp.7-8*].

12 Note that in this context $\underline{a}$ is an $(n+1)$-tuple.

LEMMA 42 *(a)*

$$[a_0, a_1, a_2; fg]_2 = g(a_0)[a_0, a_1, a_2; f]_2 + [a_1, a_2; f]_1[a_0, a_1; g]_1 + f(a_3)[a_0, a_1, a_3; g]_2$$

(b) if $n \geq 1$ *then*

$$\frac{a_n - a_0}{n}[\underline{a}; f]_n = [\underline{a}'_0; f]_{n-1} - [\underline{a}'_n; f]_{n-1} \tag{26}$$

□ Simple calculations prove these identities. □

REMARK (iii) If f and g have second order derivatives then on taking limits the identity in (a) reduces to $(fg)'' = gf' + 2f'g' + fg''$.

The definition of $[a_0, \dots, a_n; f]$ can be extended to allow for certain of the points to coalesce, the so-called *confluent divided difference*, provided we assume sufficient differentiability properties for f so as to allow the various limits to exist. If $n_i + 1$ of the elements of the n-tuple $\underline{a}$ are equal to a_i, $0 \leq i \leq m$, then

$$[\underline{a}; f]_n = \frac{1}{n_0! \dots n_m!} \frac{\partial^{n_0 + \cdots n_m}}{\partial^{n_0} a_0 \dots \partial^{n_m} a_m}[a_0, \dots, a_m; f]_m.$$

EXAMPLE (iii) For instance it can easily be checked that

$$[a, a, b, b; f] = \frac{f'(b) + f'(a) - 2[a, b; f]}{(b - a)^2}.$$

See [*Milne-Thomson pp.12–19*]; [*Horwitz 1995*].

DEFINITION 43 *If* I *is a real interval then* $f : I \mapsto \mathbb{R}$ *is said to be* n-convex on I, $n \geq 0$, *if for all choices of* $(n + 1)$ *distinct points from* I.

$$[\underline{a}; f]_n \geq 0. \tag{27}$$

If instead ($\sim$*27) holds we say that* f *is* n-concave on I. *Further if (27), respectively* ($\sim$*27), is always strict we say that* f *is* strictly n-convex, respectively strictly n-concave.

REMARK (iv) If $n = 2$ then, by Remark (ii), Definition 42 is equivalent to 4.1 Definition 1 so 2-convex functions are just convex functions. Also from Example (i) 1-convex functions are just increasing functions; and, since $[\underline{a}; f]_0 = f(a_0)$, 0-convex functions are just non-negative functions.

THEOREM 44 *(a) A function is both n-convex and n-concave if and only if it is a polynomial of degree at most $(n-1)$.*
(b) A function f is n-convex, $n \geq 2$ if and only if $f^{(n-2)}$ exists and is convex.

□ A proof of this theorem together with many other details can be found in [*Aumann & Haupt pp.271–291*]. □

REMARK (v) In particular if f is n-convex, $n \geq 2$, and if $0 \leq k \leq n-2$ then $f^{(k)}$ is $(n-k)$-convex; $f_{\pm}^{(n-1)}$ exist, are increasing and $f_{-}^{(n-1)} \leq f_{+}^{(n-1)}$.

LEMMA 45 *(a) If $f^{(n)} \geq 0$ on the interval I then f is n-convex on I, and if every subinterval of I contains a point where $f^{(n)} > 0$ then f if strictly n-convex on I.*
(b) If f is n-convex, $n \geq 2$, and if $\underline{a}, \underline{b}$ are n-tuples with distinct elements and such that $\underline{a} \leq \underline{b}$ then

$$[\underline{a}; f]_{n-1} \leq [\underline{b}; f]_{n-1};$$

when f is strictly n-convex this inequality is strict.
(c) If f is n-convex, $n \geq 2$, it is strictly n-convex unless on some sub-interval f is a polynomial of degree at most $(n-1)$.
(d) If f is a polynomial of degree at least n and if $f^{(n)} \geq 0$, then f is strictly n-convex.
(e) If for all $h \neq 0$ small enough and all $x, a < x < b$, we have for a bounded function f that

$$[x, x+h, \ldots, x+nh; f]_n = \frac{1}{n!h^n} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(x+ih) \geq 0,$$

then f is n-convex on $]a,b[$. If the inequality is always strict then f is strictly n-convex on $]a,b[$.

□ (a) This follows from Theorem 44(b) and the case $n = 2$, 4.1 Theorem 6(b); [*Bullen 1971*a].
(b) This is a consequence of (26); see [*Bullen 1971*a].
(c) and (d) follow from (b) and Theorem 44(a); [*Bullen & Mukhopadhyay p.314*].
(e) Since f is bounded then it is continuous, [*RV p.239*], and for the rest see [*Popoviciu pp.48–49*]. □

EXAMPLE (iv) Using Lemma 45(a) we easily see that e^x is strictly n-convex for all n, and $\log x$ is strictly n-convex if n is odd, and strictly n-concave if n is even,

EXAMPLE (v) Similarly $x^r, x > 0$, is strictly n-convex if sign $\left(\prod_{i=0}^{n-1}(r-i)\right) = 1$. In particular this function is 3-convex if $r > 2$ or $0 < r < 1$, and is strictly 3-concave if $1 < r < 2$ or $r < 0$.

REMARK (vi) The property in (b) generalizes 4.1 (3).

REMARK (vii) The property in (e) generalizes J-convexity, 4.5.1, and a function with this property is said to be *n-convex (J)*, or *weakly n-convex*. Part (e) extends 4.5.1 Theorem 30.

REMARK (viii) An extensive study of higher order convexity can be found in [*Popoviciu*]; see also [*DI pp.190–191; PPT pp.15–17; RV pp.237–240*].

4.8 SCHUR CONVEXITY

DEFINITION 46 *A function $f : I^n \mapsto \mathbb{R}$, where I is an open interval in $\mathbb{R}_+$, is said to be* Schur convex on I^n *if for all $\underline{a}, \underline{b} \in I^n$,*

$$\underline{b} \prec \underline{a} \Longrightarrow f(\underline{b}) \leq f(\underline{a}); \tag{28}$$

if the inequality is always strict when $\underline{b} \prec \underline{a}$ and $\underline{a}$ is not a permutation of $\underline{b}$ then f is said to be strictly Schur convex on I^n.

REMARK (i) If $-f$ is Schur convex, strictly Schur convex, on I^n then f is said to be *Schur concave, strictly Schur concave on I^n*.

REMARK (ii) In proving Schur convexity we can often assume that $n = 2$. This follows from 3.3 Lemma 13; see [*MO p.58*].

LEMMA 47 *(a) If f is Schur convex on I^n then it is symmetric.*
(b) If f_i, $1 \leq i \leq m$, are all Schur convex on I^n and $h : \mathbb{R}^m \mapsto \mathbb{R}$ is increasing in each variable then $h(f_1, \ldots, f_m)$ is Schur convex on I^n.
(c) In particular with the assumptions in (b) $\min\{f_1, \ldots, f_m\}, \max\{f_1, \ldots, f_m\}$, and $\prod_{i=1}^m f_i$ are Schur convex on I^n.
(d) If f is Schur convex and increasing, in each variable, on I^n and if $g : \mathbb{R} \mapsto I$ is convex then $f\big(g(a_1), \ldots, g(a_n)\big)$ is f is Schur convex on I^n

□ All the proofs are straightforward; see [*MO pp.60–63*] □

THEOREM 48 [SCHUR-OSTROWSKI] *If $f : I^n \mapsto \mathbb{R}$, where I is an open interval in $\mathbb{R}_+$, has continuous derivatives and is symmetric then f is Schur convex on I^n if and only if for all $i \neq j$*

$$(a_i - a_j)\big(f_i'(\underline{a}) - f_j'(\underline{a})\big) \geq 0 \text{ for all } \underline{a} \in I^n$$

□ See [*MO pp.56–57*]. □

THEOREM 49 *If $f : I^n \mapsto \mathbb{R}$, where I is an open interval in $\mathbb{R}_+$, is symmetric and convex then it is Schur convex.*

□ By Remark (ii) we can assume that $n = 2$ when for some λ, $0 \le \lambda \le 1$ $b_1 = (1-\lambda)a_1 + \lambda a_2$, $b_2 = \lambda a_1 + (1-\lambda)a_2$; see for instance 3.3 Lemma 13. So

$$\begin{aligned} f(\underline{b}) =& f\big((1-\lambda)a_1 + \lambda a_2, \lambda a_1 + (1-\lambda)a_2\big) \\ =& f\big((1-\lambda)(a_1, a_2) + \lambda(a_2, a_1)\big) \\ &\le (1-\lambda)f(a_1, a_2) + \lambda f(a_2, a_1), \text{ by convexity.} \\ =& f(\underline{a}), \text{ by symmetry.} \end{aligned}$$

□

There is an extensive literature on this subject; see in particular [*AI pp.167-168; BB p 32; DI pp.228–229; EM6 p.75; MO pp.54-82; PPT pp.332-336*].

4.9 MATRIX CONVEXITY If I is a interval in $\mathbb{R}$ then a function $f : I \mapsto \mathbb{R}$ is said to be a *increasing matrix function of order n* if for all $A, B \in \mathcal{H}_n^+$ with eigenvalues in I and $A \le B$ we have that $f(A) \le f(B)$[13].

REMARK (i) The usages decreasing matrix function and monotone function are easily defined and if a function is monotone for all orders it is said to be *operator monotone.*

EXAMPLE (i) If $f(x) = x^p$, $0 < p \le 1$ or $g(x) = \log x$ then f and g are operator increasing on $\mathbb{R}$; see [*DI p.182; MO pp.463–467; RV pp.259–262*].

REMARK (ii) The above definition can be made without the restriction to positive definite matrices but the examples given above need positive semi-definiteness for f, and positive definiteness for g.
If I is a interval in $\mathbb{R}$ then a function $f : I \mapsto \mathbb{R}$ is said to be a a *convex matrix function, of order n,* if for all $A, B \in \mathcal{H}_n^+$ with eigenvalues in I and all $\lambda, 0 \le \lambda \le 1$,

$$f(\overline{1-\lambda}\, A + \lambda B) \le \overline{1-\lambda}\, f(A) + \lambda f(B). \tag{29}$$

REMARK (iii) If the opposite inequality holds then the function is said to be a *concave matrix function, of order n.* A function that is convex, concave, of all orders is said to be *operator convex, concave*; [*DI p.64; MO pp.467–474; RV pp.259–262*].

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[13] The various matrix concepts are defined in Notations 7.

EXAMPLE (ii) If $f(x) = x^2, x^{-1}, 1/\sqrt{x}$ then f is operator convex ; if $f(x) = \sqrt{x}$ then f is operator concave.

REMARK (iv) The comments in Remark (ii) apply here as well.
The topic of operator monotone and operator convex functions has a large literature; in addition to the above references see [*Furuta*], [*Ando 1979; Kubo & Ando*].

II THE ARITHMETIC, GEOMETRIC AND HARMONIC MEANS

This chapter is devoted to the properties and inequalities of the classical arithmetic, geometric and harmonic means. In particular the basic inequality between these means, the Geometric Mean-Arithmetic Mean Inequality, is discussed at length with many proofs being given. Various refinements of this basis inequality are then considered; in particular the Rado-Popoviciu type inequalities and the Nanjundiah inequalities. Converse inequalities are discussed as well as Čebišev's inequality. Some simple properties of the logarithmic and identric means are obtained.

1 Definitions and Simple Properties

1.1 The Arithmetic Mean

Definition 1 *If* $\underline{a} = (a_1, \ldots, a_n)$ *is a positive* n*-tuple then the* arithmetic mean of $\underline{a}$ *is*

$$\mathfrak{A}_n(\underline{a}) = \frac{a_1 + \cdots + a_n}{n}. \tag{1}$$

This mean[1] is the simplest mean and by far the most common; in fact for a non-mathematician this is probably the only concept for averaging a set of numbers. The arithmetic mean of two numbers a and b, $(a+b)/2$, was known and used by the Babylonians in 7000 B.C., [*Wassell*], and occurs in several contexts in the works of the Pythagorean school, sixth-fifth century B.C. For instance in the idea of the *arithmetic proportion*[2] of two numbers, a, b, $0 < a < b$, the number x such that $x - a : b - x :: 1 : 1$, when of course $x = (a+b)/2$.[3] Aristotle, also in the sixth century B.C., used the arithmetic mean but did not give it this name. Another interpretation arises from the picturing of addition as the abutting of two line segments. One then asks what line segment when abutted to itself will

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1 More precisely called *the arithmetic mean with equal weights*; see Remark(x) below.

2 The notion of a *proportion*, between four positive numbers, lengths, areas etc. is somewhat archaic but has a long history, see[*Euclid Book V, Heath vol 1.pp.84–90, 384–391*]. If A,B,C,D are positive numbers then A:B::C:D, read A *is to* B *as* C *is to* D, is equivalent to $A/B = C/D$.

3 In this form the arithmetic mean is one of the ten neo-Pythagorean means; see VI 2.1.4 and 1.2 below.

produce the same length as the abutting of two given line segments.The idea of arithmetic mean is also found in the concept of centroid used by Heron, and earlier by Archimedes in the third century B.C; [*Heath vol.II p.350*].

In the notation introduced in Definition 1 the n-tuple may be replaced by some specific formulation such as $\mathfrak{A}_n(a_1,\ldots,a_n)$, or $\mathfrak{A}_n(a_i,\ 1\le i\le n)$. Further if $n=2$ the suffix is omitted unless it is needed for clarity[4]; thus we will write $\mathfrak{A}(\underline{a};\underline{w})$ etc., when $n=2$. When there is no ambiguity either $\underline{a}$, or the subscript n, or both may be omitted.

CONVENTION 1 If $\underline{a}$ is an n-tuple, $n\ge 2$, and if $1\le m\le n$, we will write, whenever there is no ambiguity,

$$\mathfrak{A}_m(\underline{a})=\frac{a_1+\cdots+a_m}{m}.$$

CONVENTION 2 The various notations introduced above will apply in various contexts throughout this work.

Clearly (1) does not depend on $\underline{a}$ being positive, and many of the properties of $\mathfrak{A}$ can be deduced without this assumption, see below Remarks (v), (x), 1.3.3, 1.3.4, 1.3.9 and 2.4.5 Remark (v). In fact the whole discussion can take place in a very general context, see for instance VI 5 and [*Anderson & Trapp*].

However we have the following convention that will hold throughout the rest of this book unless otherwise indicated.

CONVENTION 3 All n-tuples and sequences will be positive unless specifically stated otherwise; that is if $\underline{a}$ is an n-tuple then unless otherwise stated, $\underline{a}\in(\mathbb{R}_+^*)^n$.

The question of more general n-tuples will be discussed in various places later; for a situation where the elements of $\underline{a}$ are complex see [*Majó Torrent*].[5]

If the elements $\underline{a}$ are in $\mathbb{Q}$ then $\mathfrak{A}_n(\underline{a})\in\mathbb{Q}$, and because the above definition is so elementary there is a certain point in using only the simplest tools to deduce properties of this mean. Of course non-elementary proofs besides having an interest in themselves suggest methods of generalization, and are often simpler than the more elementary approaches.

Some elementary properties of $\mathfrak{A}$ are listed in the following theorem; they are all obtained by easy applications of simple properties of positive real numbers.

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[4] There are many means that are only defined when n=2, see VI 2, and dropping the suffix makes comparisons easier to read. However the usage with n=2, or even n=1, is useful from time to time in inductive arguments.

[5] Convention 3 will not apply in Chapter IV.

THEOREM 2 *If $\underline{h}$ is a real n-tuple, $\underline{a}, \underline{a}+\underline{h}, \underline{b}$ n-tuples, and if $\lambda > 0$ then:*
(Ad) [ADDITIVITY]

$$\mathfrak{A}_n(\underline{a}+\underline{b}) = \mathfrak{A}_n(\underline{a}) + \mathfrak{A}_n(\underline{b});$$

(As)$_m$ [m-ASSOCIATIVITY]

$$\mathfrak{A}_n(\underline{a}) = \mathfrak{A}_n(\mathfrak{A}_m, \ldots \mathfrak{A}_m, a_{m+1}, \ldots, a_n), \quad 1 \le m \le n;$$

(Co) [CONTINUITY]

$$\lim_{\underline{h}\to\underline{0}} \mathfrak{A}_n(\underline{a}+\underline{h}) = \mathfrak{A}_n(\underline{a});$$

(Ho) [HOMOGENEITY]

$$\mathfrak{A}_n(\lambda\underline{a}) = \lambda\mathfrak{A}_m(\underline{a}), \quad \lambda > 0;$$

(In) [INTERNALITY]

$$\min \underline{a} \le \mathfrak{A}_n(\underline{a}) \le \max \underline{a}, \tag{2}$$

with equality if and only if $\underline{a}$ is constant;
(Mo) [MONOTONICITY]

$$\underline{a} \le \underline{b} \implies \mathfrak{A}_n(\underline{a}) \le \mathfrak{A}_n(\underline{b}),$$

with equality if and only if $\underline{a} = \underline{b}$;
(Re) [REFLEXIVITY] *If $\underline{a}$ is constant, $a_i = a$, $1 \le i \le n$, then $\mathfrak{A}_n(\underline{a}) = a$;*
(Sy) [SYMMETRY] *$\mathfrak{A}_n(a_1, \ldots, a_n)$ is not changed if the elements of $\underline{a}$ are permuted .*

REMARK (i) When m-associativity holds for all m, $1 \le m \le n$, as it does for the arithmetic mean, we call it the property of *substitution*.

REMARK (ii) Clearly it is inequality (2) that justifies the name of mean, see [B^2 *p.230*]. While (2) will be called the *property of internality* the full property given in (In) will be called *strict internality.*

REMARK (iii) The word monotonic used in (Mo) is sometimes replaced by *monotone*, or by *isotone*, see [B^2 *p.230*]. The concept can also be considered in a strict form, *strictly monotonic*, etc.

REMARK (iv) The properties listed in Theorem 2 are not independent. For instance (In) is implied by (Mo) and (Re).

REMARK (v) It is useful to note that these properties of the arithmetic means hold if the n-tuple $\underline{a}$ is allowed to be real.
Almost all the means we will discuss in this book satisfy some or all of these conditions. The properties (Co), (Re) and (In) are so basic that we would consider

them essential in any possible definition of a mean; the property (Ho) is a very common property but does not always hold; for examples see IV. Others, such as (Ad) are in some sense characteristic of the arithmetic mean; see VI 6.

REMARK (vi) While (Ad) gives a simple relation between $\mathfrak{A}_n(\underline{a}), \mathfrak{A}_n(\underline{b})$ and $\mathfrak{A}_n(\underline{a}+\underline{b})$ it is much more difficult to obtain one between $\mathfrak{A}_n(\underline{a}), \mathfrak{A}_n(\underline{b})$ and $\mathfrak{A}_n(\underline{a}\,\underline{b})$, but one is given later, see 5.3, and is known as *Čebišev's inequality.*

A very natural extension of Definition 1 is suggested when some of the elements of $\underline{a}$ occur more than once or, from a practical point of view, if some are considered more important than others.

DEFINITION 3 *Given two n-tuples* $\underline{a}$, $\underline{w}$, *the* weighted arithmetic mean of $\underline{a}$ with weight, or weights, $\underline{w}$ *is*

$$\mathfrak{A}_n(\underline{a};\underline{w}) = \frac{w_1a_1 + \cdots + w_na_n}{w_1 + \cdots + w_n} = \frac{1}{W_n}\sum_{i=1}^n w_ia_i. \tag{3}$$

It is easily checked that this more general mean has all the properties listed in Theorem 2 except (Sy). Instead this more general mean has the following property:

(Sy)*[ALMOST SYMMETRY] $\mathfrak{A}_n(\underline{a},\underline{w})$ *is not changed if the* $\underline{a}$ *and* $\underline{w}$ *are permuted simultaneously.*

REMARK (vii) The name arithmetic mean will normally refer to (3), and if we wish to specify (1) it will be referred to as *the arithmetic mean with equal weights.*

REMARK (viii) Using this notation Jensen's inequality, I 4.2 (J), can be written as:

$$f\big(\mathfrak{A}_n(\underline{a};\underline{w})\big) \le \mathfrak{A}_n\big(f(\underline{a});\underline{w}\big). \tag{4}$$

A refinement and strengthening of inequality in (2) is given in the following result

LEMMA 4 *(a) If* $\underline{a},\underline{b}$ *and* $\underline{w}$ *are n-tuples, then*

$$\min\big(\underline{a}\,\underline{b}^{-1}\big) \le \frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{A}_n(\underline{b};\underline{w})} \le \max\big(\underline{a}\,\underline{b}^{-1}\big).$$

(b) If $W_n = 1$ *and* $n \ge 2$, *then*

$$\max\underline{a} - \mathfrak{A}_n(\underline{a};\underline{w}) \ge \frac{\min\underline{w}}{n+1}\sum_{1\le i<j\le n}\big(\sqrt{a_i} - \sqrt{a_j}\big)^2.$$

□ (a) Let $m = \min\big(\underline{a}\,\underline{b}^{-1}\big)$ and $M = \max\big(\underline{a}\,\underline{b}^{-1}\big)$, then $m \le \dfrac{a_i}{b_i} \le M$, $1 \le i \le n$, or $mw_ib_i \le a_iw_i \le Mw_ib_i$, $1 \le i \le n$; summing over i gives the result.

(b) Let $S(\underline{a})$ denote the difference between the left-hand side and the right-hand side of the inequality in (b); we have to show that $S(\underline{a}) \geq 0$. Put $w = \min \underline{w}$, and assume without loss in generality that $\underline{a}$ is decreasing. Now,

$$\begin{aligned}\sum_{1\leq i<j\leq n} \left(\sqrt{a_i} - \sqrt{a_j}\right)^2 =&(n-1)\sum_{i=1}^{n} a_i - 2 \sum_{1\leq i<j\leq n} \sqrt{a_i a_j} \\ \leq&(n-1)\sum_{i=1}^{n} a_i - 2 \sum_{1\leq i<j\leq n} \min(a_i, a_j) \\ =&\sum_{i=1}^{n}(n+1-2i)a_i.\end{aligned}$$

So $S(\underline{a}) = a_1 - \mathfrak{A}_n(\underline{a};\underline{w}')$, where $w_i' = w_i + \dfrac{n+1-2i}{n+1}w$, $1 \leq i \leq n$, and $W_n' = 1$. Hence by (2) $S(\underline{a}) \geq 0$ as was to be shown. □

Remark (ix) Result (a) is due to Cauchy, see [*AI p.204*], and analogous inequalities can be given for the geometric and harmonic means defined in the next section.

Remark (x) Part (b) is due to Alzer and can be extended to real n-tuples $\underline{a}$; [*Alzer 1997d*].

Remark (xi) Another generalization is given in [*Bromwich pp.242, 418–420, 473–474*], [*Bromwich*].

Remark (xii) For an amusing discussion of weighted arithmetic means see [*Falk & Bar-Hillel*].

1.2 The Geometric and Harmonic Means Two other means of a very elementary nature have been in use for a long time. Like the arithmetic mean they arise naturally in many simple algebraic and geometric problems, some of which are to be found in Euclid and in the work of the Pythagorean school. Thus in that school's theory of harmony study is made of the proportions $a\colon \dfrac{2ab}{a+b} :: \dfrac{a+b}{2} \colon b$; further in Euclid we have the study of the proportions $x - a\colon b - x :: a\colon x$ when $x = \sqrt{ab}$, the geometric mean of a and b, and $x - a : b - x :: a : b$ when $x = 2ab/(a+b)$, the harmonic mean; see also 1.1 Footnote 3. The name geometric is probably based on the Greek picturing of multiplication of two numbers as the area of a rectangle. One then asks what number multiplied by itself will produce a square of the same area as the rectangle obtained by multiplying two given numbers; [*Aumann 1935b*].

DEFINITION 5 *Given two n-tuples $\underline{a}$, $\underline{w}$, the* weighted geometric mean, respectively harmonic mean, of $\underline{a}$ with weight, or weights, $\underline{w}$ *is*

$$\mathfrak{G}_n(\underline{a};\underline{w}) = \left(\prod_{i=1}^{n} a_i^{w_i}\right)^{1/W_n}, \tag{5}$$

respectively

$$\mathfrak{H}_n(\underline{a};\underline{w}) = \frac{W_n}{\sum_{i=1}^{n}\left(\frac{w_i}{a_i}\right)}. \tag{6}$$

REMARK (i) The other variations of notation introduced for the arithmetic mean will be used here, and with other means introduced later. Thus if $\underline{w}$ is constant we get the means with equal weights, written $\mathfrak{G}_n(\underline{a}), \mathfrak{H}_n(\underline{a})$; and note should be made of 1.1 Conventions 1, 2 and 3.

REMARK (ii) The replacement of the arithmetic means in inequality 1(4) by other means has been used by some authors to define another form of convexity; thus we would say that a function $f : \mathbb{R}_+^* \mapsto \mathbb{R}$ is *convex relative to the geometric mean*, or just *geometrically convex*, if for all n-tuples $\underline{a}$ and $\underline{w}$;

$$f\big(\mathfrak{G}_n(\underline{a};\underline{w})\big) \le \mathfrak{G}_n\big(f(\underline{a});\underline{w}\big).$$

A function f is geometrically convex if and only if $f \circ \exp$ is log-convex; [*Jarczyk & Matkowski; Kominek & Zgraja; Matkowski & Rätz 1995a,b; Niculescu; Thielman; Toader 1991b*]; see also III 6.3 .

REMARK (iii) The harmonic mean of two numbers with equal weights was originally named the *sub-contrary mean*; it was renamed by Archytas and Hippasus; see [*Eves 1976; Heath vol.I p.85*][6].

The following simple identities should be noted.

$$\mathfrak{H}_n(\underline{a};\underline{w}) = \left(\mathfrak{A}_n(\underline{a}^{-1};\underline{w})\right)^{-1} \quad\text{or}\quad \frac{W_n}{\mathfrak{H}_n(\underline{a};\underline{w})} = \sum_{i=1}^{n}\frac{w_i}{a_i}; \tag{7}$$

$$\mathfrak{G}_n(\underline{a};\underline{w}) = \exp\left(\mathfrak{A}_n(\log\underline{a};\underline{w})\right), \text{ or} \log\left(\mathfrak{G}_n(\underline{a};\underline{w})\right) = \mathfrak{A}_n(\log\underline{a};\underline{w}). \tag{8}$$

Because of the particular simplicity of (7) the properties of the harmonic mean will often not be investigated in detail. In any case all of these means turn up again later as special cases of the more general power means; see III 1.

REMARK (iv) For other relations between the arithmetic and geometric means with some geometric interpretations see [*Jecklin 1962a; Usai 1940a*].

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[6] The frequency of the octaves being in the ratios 1,2,3... the harmonic mean of 1 and 2, namely 4/3, gives the frequency of the fourth, that of 1 and 3, namely 3/2, gives the frequency the fifth etc. Further 8, the number of angles in a cube, is the harmonic mean of the number of edges, 12, and the number of faces, 6. All good reasons for the name change; see [*Heath vol.1 pp.75- 76, 85-86*], [*Wassell*].

Theorem 6 *The geometric and harmonic means have the properties of (Co), (Ho), (Mo),(Re), (Sy*), (Sy) in the case of equal weights, strict internality, and* $(As)_m$ *for all* m*, the property of substitution. Further*

$$\mathfrak{G}_n(\underline{a}\,\underline{b};\underline{w}) = \mathfrak{G}_n(\underline{a};\underline{w})\,\mathfrak{G}_n(\underline{b};\underline{w}); \tag{9}$$

$$\left(\mathfrak{G}_n(\underline{a};\underline{w})\right)^r = \mathfrak{G}_n(\underline{a}^r;\underline{w}),\ r \in \mathbb{R}. \tag{10}$$

Remark (v) A relation between $\mathfrak{G}_n(\underline{a}+\underline{b};\underline{w})$ and $\mathfrak{G}_n(\underline{a};\underline{w})$ and $\mathfrak{G}_n(\underline{b};\underline{w})$ is more difficult to obtain and is proved later, III 3.1.3 (10) and is sometimes called *Hölder's inequality*; see also 1.1 Remark (vi).

Remark (vi) A particularly simple case of (9) is :

$$\mathfrak{G}_n(\underline{a}^{-1};\underline{w}) = \left(\mathfrak{G}_n(\underline{a};\underline{w})\right)^{-1}.$$

Remark (vii) While it is clear that the definition of the geometric mean requires the n-tuple $\underline{a}$ to be positive, and the definition of the harmonic mean requires that no element of $\underline{a}$ be zero extensions have been considered; [*Asimov*].

1.3 Some Interpretations and Applications

1.3.1 A Geometric Interpretation If $0 < a < b$ let $ABCD$ be a trapezium with AB, DC the parallel sides and $AB = b$, $DC = a$, K the intersection of AC and BD, see Figure 1.

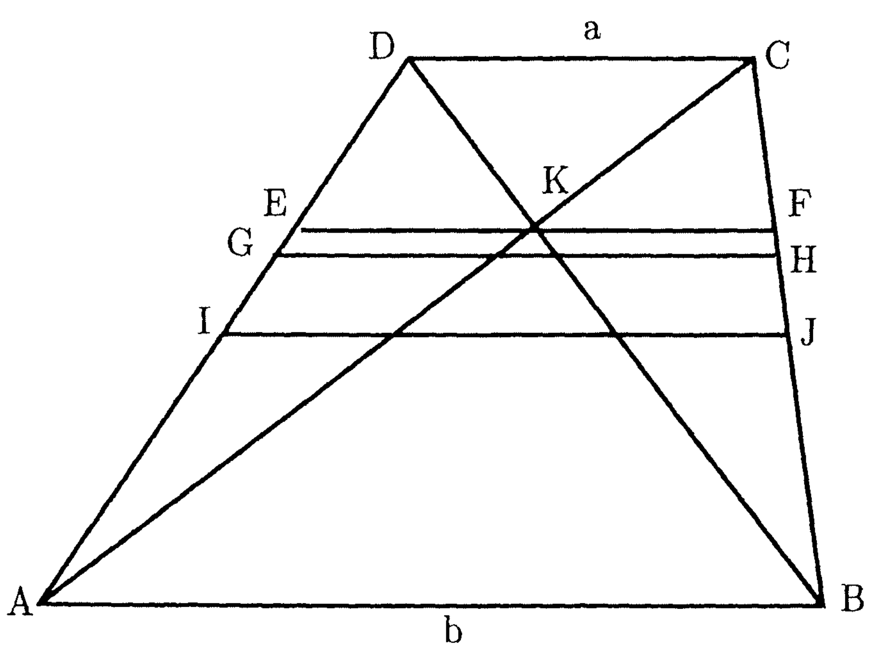

Figure 1

IJ, GH and EKF are parallel to AB; IJ bisects AD and BC; GH divides $ABCD$ into two similar trapezia. Then $IJ = \mathfrak{A}(a,b), GH = \mathfrak{G}(a,b), EF = \mathfrak{H}(a,b)$.
The above diagram suggests the following inequality that can be checked in a few numerical cases by the reader;

$$a = \min\{a,b\} \le \mathfrak{H}(a,b) \le \mathfrak{G}(a,b) \le \mathfrak{A}(a,b) \le b = \max\{a,b\}, \tag{11}$$

with equality if and only if $a = b$. This is an important result, a case of the geometric mean-arithmetic mean inequality; we will return to below in section 2.

REMARK (i) That EF lies above IJ, so that $\mathfrak{H}(a,b) \le \mathfrak{A}(a,b)$, can be seen as follows: consider what happens to EF as DC increases in length from 0 to a.

An extremely interesting series of applications of averages in graph theory and combinatorics is to be found in [*Wilf*].

1.3.2 ARITHMETIC AND HARMONIC MEANS IN TERMS OF ERRORS

Given two positive numbers a, b with $a < b$, and some number x with $a \le x \le b$, if we guess y to be the value of x then the *error* in making this guess is $\epsilon = |y - x|$, while the *relative error* is $\rho = |y - x|/x$.

How should y be chosen so as to minimize the possible error, respectively relative error? A classical theorem of Čebišev gives a method for such problems: *choose y so that the errors, respectively the relative errors, in the two extreme positions are equal.*

In the first case $\epsilon = \min_{a\le y\le b} \max_{a\le x\le b} \left\{|y - x|\right\}$ occurs for the y such that $|y - a| = |y - b|$. That is $y = (a + b)/2 = \mathfrak{A}(a,b)$.

In the second case $\rho = \min_{a\le y\le b} \max_{a\le x\le b} \left\{|y - x|/x\right\}$ occurs for the y such that $|y - a|/a = |y - b|/b$. That is $y = 2ab/(a + b) = \mathfrak{H}(a,b)$.

In words:

the approximation that yields the minimum for the greatest possible value of the error, respectively the relative error, committed in approximating an unknown quantity between two known positive bounds is the arithmetic mean, respectively, harmonic mean, of these bounds.

REMARK (i) This idea, first found in [*Pólya*], has been investigated in detail; see [*Aissen 1968; Beckenbach 1950; Metcalf; Mon, Sheu & Wang C L 1992a*].

REMARK (ii) A similar discussion for a different mean can be found in III 5.1 Remark (i).

Given n real numbers $\underline{a} = (a_1, \ldots, a_n)$ we can ask for the value of x such that the quantity $\sum_{i=1}^{n}(a_i - x)^2$ is minimized. Using calculus, or the fact that a quadratic attains its minimum at the mid-point of its zeros, we see that the required value of x is $\mathfrak{A}_n(\underline{a})$. If the numbers $a_i, 1 \le i \le n$, are observations that are given weights w_i, $1 \le i \le n$, then the best possible value for x is $\mathfrak{A}_n(\underline{a}; \underline{w})$.

1.3.3 AVERAGES IN STATISTICS AND PROBABILITY The use of averages in statistics and probability is too well known to need elaboration; see for instance [*Moroney Chap.4*], [*Norris 1976*], and an article by the Moroney in [*Newman J. p.1457*].

The Italian school of statisticians has made an extensive study of this subject as the many articles in the Bibliography show; see [*Barbensi; Bonferroni 1923-4; Gini 1940, 1949, 1952; Gini, Boldrini Galvani, & Venere; Gini & Galvani; Roghi*], and all the issues of the journal *Metron* founded by Gini, the most prolific member of this school.

If we write $\overline{a} = \mathfrak{A}_n(\underline{a})$ then the *variance* of $\underline{a}$ is $\mathfrak{A}_n\big((\underline{a} - \overline{a}\underline{e})^2\big)$; the square root of this, the *standard deviation* more properly fits into the section on power means, III 1 being in the notation defined there $\mathfrak{Q}_n(\underline{a} - \overline{a}\underline{e})$.

Another property of the arithmetic mean is as follows: the normal distribution is the only probability distribution P such that $D(x) = \prod_{i=1}^{n} P(a_i - x)$ has its maximum at $\mathfrak{A}_n(\underline{a})$; see [*Rényi p.311*] and for an extension see [*Hosszú & Vincze*]. In this application $\underline{a}$ need not be positive.

Gauss stated that the most probable value of a series of numerical observations was their arithmetic mean and proofs of this from more basic hypotheses about probability have been given by various authors; [*Bemporad 1926, 1930; Broggi; Schiaparelli 1868,1875, 1907; Schimmak; Tissèrand*]. The proof given in [*Whittaker & Robinson pp.215–217*] was disputed by Zoch although his objections have in turn been disputed, [*Wertheimer; Zoch 1935, 1937*]. While this statement of Gauss was made as a postulate, the arithmetic mean of any number of observations tends to the true value as the number of observations increases although this is a property of many other means; [*Whittaker & Robinson p.215*].

Other uses of these ideas are made below, see 2.5.4.

1.3.4 Averages in Statics and Dynamics If the n-tuple $\underline{a}$ denotes the coordinates on a line of n particles whose masses are given by the n-tuple $\underline{w}$, then $\mathfrak{A}_n(\underline{a};\underline{w})$ is the coordinate of the centre of mass; $\mathfrak{A}_n(\underline{a}^2;\underline{w})$ is the square of the radius of gyration of the system about the origin; the radius of gyration itself is more properly considered as the power mean $\mathfrak{M}_n^{[2]}(\underline{a};\underline{w})$ defined in III 1; see for instance [*Fowles, Chaps.7,8*].

In this application $\underline{a}$ need not be positive, in fact if we allow the particles to be situated in space $\underline{a}$ can be an n-tuple of vectors.

1.3.5 Extracting Square Roots A very ancient use of the arithmetic and harmonic means is *Heron's method* of extracting square roots; [*Heath vol.II pp.323–326*], [*Chajoth; Pasche 1946, 1948*].

Suppose we wish to find the square root of the positive number x; choose any two numbers a, b with $0 < a < b$ and $ab = x$. Putting $a_0 = a$, $b_0 = b$ define inductively, $a_n = \mathfrak{H}(a_{n-1}, b_{n-1})$, $b_n = \mathfrak{A}(a_{n-1}, b_{n-1})$, $n \in \mathbb{N}^*$. It is easy to check that for all

$n \in \mathbb{N}, a_n b_n = x$ and, using (11), that $a_n < a_{n+1} < b_{n+1} < b_n$. $n \in \mathbb{N}$. Further

$$b_n - a_n < \frac{b_{n-1} - a_{n-1}}{2} < \cdots < \frac{b-a}{2^n}, \quad n \in \mathbb{N}.$$

So $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \sqrt{x}$.

This result can be expressed as follows: *the iteration of the arithmetic and harmonic means of two numbers converges to their compound mean, the geometric mean.*

The iteration of means and compound means are discussed in more detail later; see VI 3.

REMARK (i) Heron's method has been extended to roots of higher order; see VI 3.2.2 (e), [*Georgakis; Nikolaev; Ory*].

REMARK (ii) A discussion of this topic and other simple concepts associated with means can be found in [*Bullen 1979; Carlson 1971*].

REMARK (iii) An iterative method for approximating higher order roots by using geometric means, that is, in effect using square roots to obtain higher order roots, has been given; see [*Vythoulkas*].

1.3.6 CESÁRO MEANS The famous Cesáro means used to sum divergent series, in particular Fourier series, are examples of arithmetic means; [*Hardy 1949, p.96*]. Thus, given a sequence $\underline{a} = (a_0, a_1, \ldots)$ define $A_j^k, k, j \in \mathbb{N}$ as follows:[7] $A_j^0 = A_{j+1}$, $j \in \mathbb{N}$; $A_j^k = \sum_{i=0}^{j} A_i^{k-1}$, $k \in \mathbb{N}^*$. Then the *k-th Cesáro mean of the sequence* $\underline{a}$ is defined by

$$\mathfrak{C}_n^k(\underline{a}) = \frac{A_n^k}{\binom{n+k}{k}}.$$

[Note that if $\underline{a} = \underline{e}_1$ then $A_n^k = \binom{n+k}{k}$.]

Simple calculations lead to,

$$\mathfrak{C}_n^k(\underline{a}) = \frac{1}{\binom{n+k}{k}} \sum_{i=0}^{n} \binom{n-i+k-1}{k-1} A_{i+1} = \mathfrak{A}_{n+1}(\underline{A}; \underline{\tilde{w}}),$$

where $\underline{A} = (A_1, \ldots), \underline{\tilde{w}} = \binom{n-i+k+1}{k+1}$, $0 \le i \le n$.

---

[7] Remember that the sequence starts with a_0 so, with the notation of Notations 6(xi), $A_n = \sum_{i=0}^{n-1} a_i$.

By analogy we can use the geometric or harmonic means in a similar manner; [*Pizzetti 1940*]. For instance: let $G_j^0 = a_j$, $j \in \mathbb{N}$; $G_j^k = \prod_{i=1}^{j} G_i^{k-1}$, $k \in \mathbb{N}^*$, and $\mathfrak{D}_n^k(\underline{a}) = \left(G_n^k\right)^{1/\binom{n+k}{k}}$.

1.3.7 Means in Fair Voting A very interesting application of the three elementary classical means has recently been made to the problem of the fair distribution of seats in the U.S. House of Representatives amongst the fifty states. This problem and this use of these means is discussed in [*Sullivan*], where other references are given. After this paper the whole area of fair voting has received considerable attention; see for instance [*Balinski & Young; Saari*].

1.3.8 Method of Least Squares Given a set of data $\underline{r}_i = (u_i, v_i)$, $1 \le i \le n$, we can ask for the best line $v = mu + c$ that fits this data in the sense that the sum $\sum_{i=1}^n (v_i - mu_i - c)^2$ is minimized. That is the squares of the vertical distances of the data points from the line is minimized. This procedure goes under the name of *the method of least squares.* A simple application of calculus shows that

$$m = \frac{\mathfrak{A}_n(\underline{u}\,\underline{v}) - \mathfrak{A}_n(\underline{u})\mathfrak{A}_n(\underline{v})}{\mathfrak{A}_n(\underline{u}^2) - \mathfrak{A}_n(\underline{u})^2}, \quad c = \frac{\mathfrak{A}_n(\underline{u}^2)\mathfrak{A}_n(\underline{v}) - \mathfrak{A}_n(\underline{u})\mathfrak{A}_n(\underline{u}\,\underline{v})}{\mathfrak{A}_n(\underline{u}^2) - \mathfrak{A}_n(\underline{u})^2},$$

where of course $\underline{u} = (u_1, \ldots, u_n)$, $\underline{v} = (v_1, \ldots, v_n)$. See [*EM5 pp.376–380; Whittaker & Robinson pp.209–259*], [*Encke*].

1.3.9 The Zeros of a Complex Polynomial The following are interesting results on the zeros of a complex polynomial.

Theorem 7 *If $f(z) = a_0 + \cdots + a_n z^n$ is a complex polynomial of degree n, $a_n \ne 0$, with zeros $\underline{z} = (z_1, \ldots, z_n)$ and if $\underline{z}' = (z_1' \ldots z_{n-1}')$ are the zeros of f' then*

$$\mathfrak{A}_n(\underline{z}) = \mathfrak{A}_{n-1}(\underline{z}').$$

□ Elementary algebra tells us that $z_1 + \cdots + z_n = -(a_{n-1}/a_n)$.
Since $f'(z) = a_1 + \cdots + na_n z^{n-1}$, we have by the same reasoning that $z_1' + \cdots + z_{n-1}' = -((n-1)a_{n-1}/na_n)$.
The result is now immediate. □

Theorem 8 *If $f(z) = a_0 + \cdots + a_n z^n$ is a complex polynomial of degree n, $a_n \ne 0$, with zeros $\underline{z} = (z_1, \ldots, z_n)$ then for each zero of f', z' say, there are non-negative weights $\underline{w}$ such that $z' = \mathfrak{A}_n(\underline{z}; \underline{w})$.*

□ The result is trivial if z' is also a zero of f, and in this case we have all weights except one zero.

Assume that this is not the case, z' is not a zero of f, $f(z') \neq 0$. Then:

$$0 = \overline{\frac{f'(z')}{f(z')}} = \sum_{k=1}^{n} \overline{\left(\frac{1}{(z'-z_k)}\right)} = \sum_{k=1}^{n} \frac{z'-z_k}{|z'-z_k|^2}.$$

Solving for z' gives the result with weights $w_k = \dfrac{1}{|z'-z_k|^2}$, $1 \le k \le n$, and the weights are all positive in this case. □

This result is known as the Gauss-Lucas theorem and geometrically says: if the zeros of f are the vertices of an n-gon then any zero of f' lies inside this n-gon; [*Milovanović, Mitrinović & Rassias pp.179–183*].

REMARK (i) For a different result involving the zeros of f and f' see 5.6 below.

REMARK (ii) There are of course many other applications of the arithmetic mean in the literature; [*Dörrie; Jecklin 1971; Martins*].

2 The Geometric Mean-Arithmetic Mean Inequality

2.1 THE STATEMENT OF THE THEOREM Although the inequality between the arithmetic and geometric means[8] in its simplest form, 1.3.1(11), was probably known in antiquity, the general result for weighted means of n numbers seems to have first appeared in print in the nineteenth century, in the notes of Cauchy's course given at the École Royale, [*Cauchy 1821, p.315*], although certain results of Newton are related to this inequality, see I 1.1 Remark(ii) and V 2 Remarks (ii), (v).

THEOREM 1 [THE GEOMETRIC MEAN-ARITHMETIC MEAN INEQUALITY] *Given two n-tuples $\underline{a}$ and $\underline{w}$ then*

$$\mathfrak{G}_n(\underline{a};\underline{w}) \le \mathfrak{A}_n(\underline{a};\underline{w}), \tag{GA}$$

with equality if and only if $\underline{a}$ is constant.

This section is devoted to proofs of (GA); proofs giving sharper results that imply (GA) will be given in later sections, or chapters.

COROLLARY 2 *Given two n-tuples $\underline{a}$ and $\underline{w}$ then*

$$\mathfrak{H}_n(\underline{a};\underline{w}) \le \mathfrak{G}_n(\underline{a};\underline{w}) \le \mathfrak{A}_n(\underline{a};\underline{w}), \tag{1}$$

[8] This result is named in various ways; the arithmetic-geometric mean inequality, the arithmetic mean-geometric mean inequality, etc. We will usually refer to it as (GA).

with equality if and only if $\underline{a}$ is constant.

□ (*i*) Using the first identity in 1.2(7) and (GA) we have that

$$\mathfrak{H}_n(\underline{a};\underline{w}) = \Big(\mathfrak{A}_n(\underline{a}^{-1};\underline{w})\Big)^{-1} \le \Big(\mathfrak{G}_n(\underline{a}^{-1};\underline{w})\Big)^{-1} = \mathfrak{G}_n(\underline{a};\underline{w}).$$

(*ii*) Alternatively following Transon, [*Transon*], we can use the second identity in 1.2(7). This on rewriting and using (GA) gives:

$$\begin{aligned}\frac{1}{\mathfrak{H}_n(\underline{a};\underline{w})} &= \frac{\mathfrak{A}_n(\prod_{i=1,i\neq j}^n a_i,\ 1\le j\le n;\underline{w})}{\prod_{i=1}^n a_i}\\ &\ge \frac{\mathfrak{G}_n(\prod_{i=1,i\neq j}^n a_i,\ 1\le j\le n;\underline{w})}{\prod_{i=1}^n a_i} = \frac{1}{\mathfrak{G}_n(\underline{a};\underline{w})}.\end{aligned}$$

The case of equality is also an easy deduction from that of Theorem 1. □

Remark (i) The apparently weaker inequality

$$\mathfrak{H}_n(\underline{a};\underline{w}) \le \mathfrak{A}_n(\underline{a};\underline{w}), \tag{HA}$$

implies (GA); see 2.4.3 Proof (*xix*).

Remark (ii) If we consider all the n-tuples with given harmonic and arithmetic means we can ask for the range of possible values of their geometric means. In the case of equal weights Post has shown that the extreme values are attained when we choose the n-tuple to have only two values, and one of them is taken $(n-1)$ times; [*Post*].

Remark (iii) For another form of (GA) see III 2.1 Remark (vii).

2.2 Some Preliminary Results Before proving 2.1 Theorem 1 we will consider some results that help to simplify later work.

2.2.1 (GA) with $n = 2$ and Equal Weights We first consider (GA) in its simplest form, 1(11), and give several proofs of this very elementary result; there are no doubt many more.

Lemma 3 *If a and b are positive real numbers then*

$$\sqrt{ab} \le \frac{a+b}{2}, \tag{2}$$

with equality if and only if $a = b$.

Remark (i) First note the interesting fact that when $n = 2$ the geometric mean is itself the geometric mean of the arithmetic and harmonic means; that is

$$\mathfrak{G}(a,b) = \sqrt{\mathfrak{A}(a,b)\mathfrak{H}(a,b)}.$$

Using this we can write inequality (2) in any of the following forms:

$$\mathfrak{G}(a,b) \le \mathfrak{A}(a,b), \quad \mathfrak{H}(a,b) \le \mathfrak{G}(a,b), \quad \mathfrak{H}(a,b) \le \mathfrak{A}(a,b).$$

$\square$ (*i*) The result is immediate from the identity $(a+b)^2 = 4ab + (a-b)^2$. This identity is illustrated by Figure 2, in which $0 < a < b$.

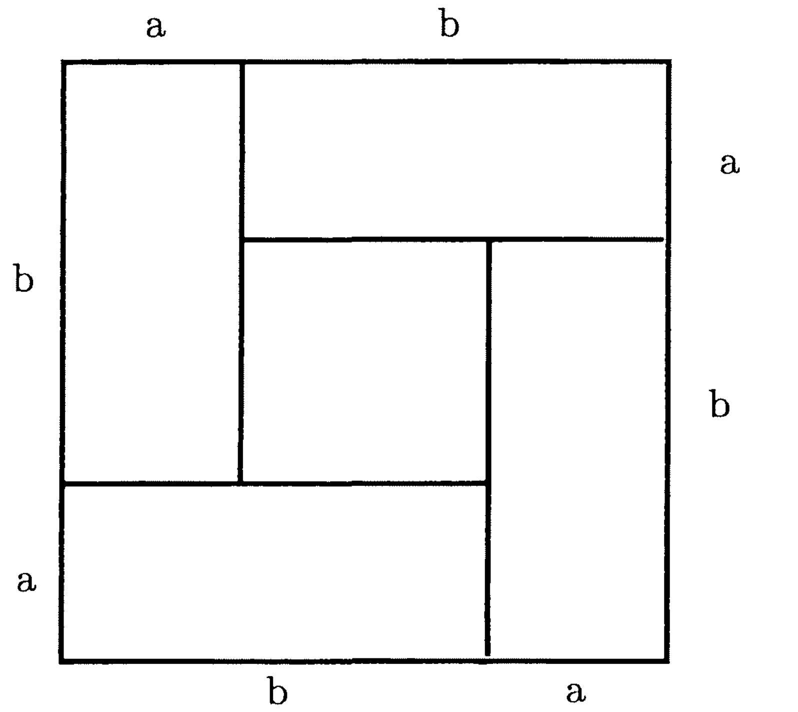

Figure 2

(*ii*) The result follows by noting that $|\sqrt{b} - \sqrt{a}| \ge 0 \Longrightarrow a + b - 2\sqrt{ab} \ge 0$

(*iii*) Since (2) is homogeneous[9] there is no loss in generality in assuming $ab = 1$, or equivalently that $b = 1/a$ when Lemma 3 is equivalent to: $a + \dfrac{1}{a} \ge 2$ with equality if and only if $a = 1$, which is just I 2.2 (20).

(*iv*) A variant of the the previous proof is to note that if $ab = 1, 0 < a < b$, then $0 < a < 1 < b$ and so $(b-1)(1-a) > 0$; expanding this last inequality leads to $a + b > 1 + ab = 2$.

(*v*) The first geometric proof is in [*Heath vol.II pp.363–364; Pappus Book 3, p.51*] and is illustrated by Figure 3.

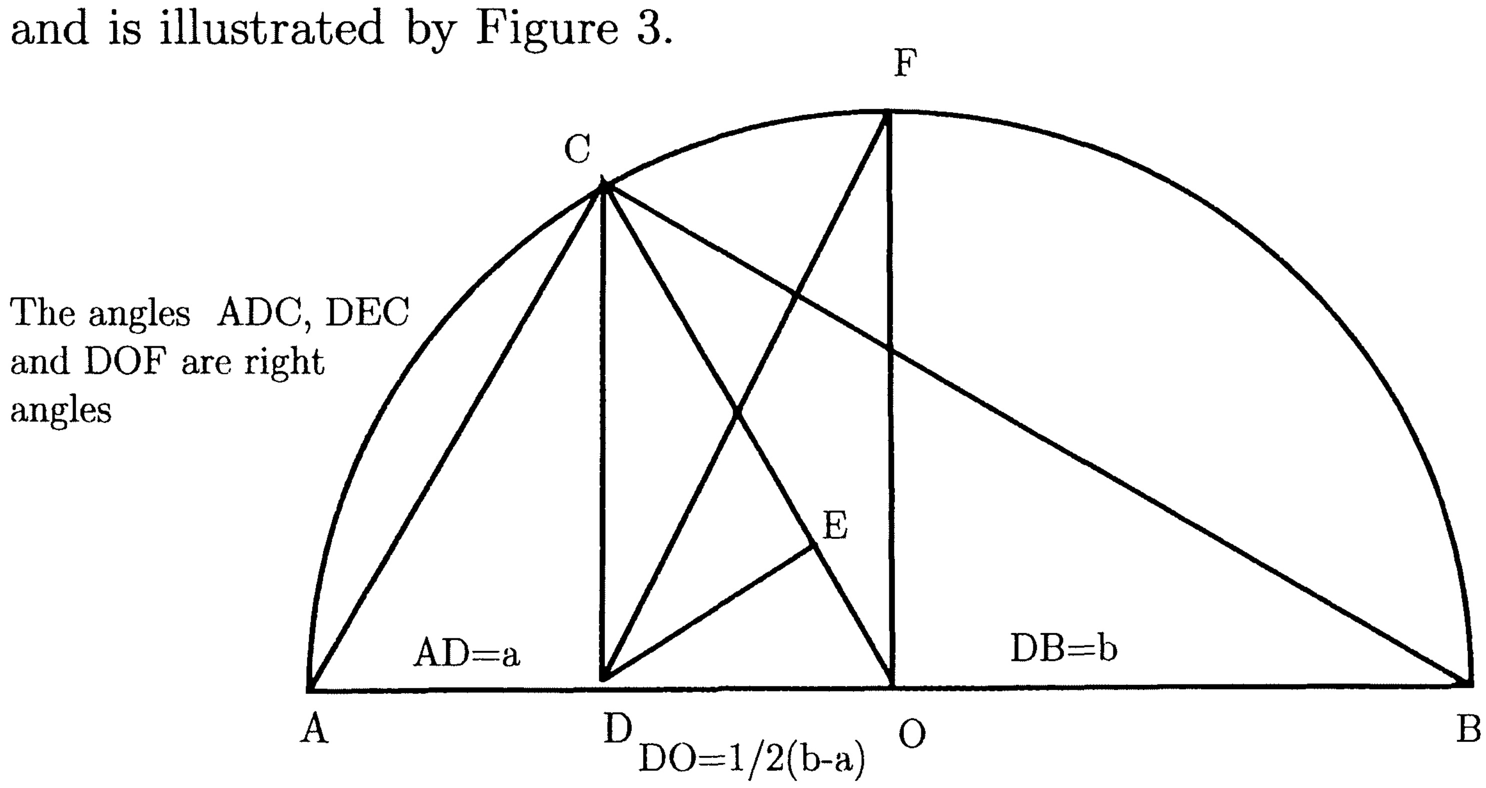

Figure 3

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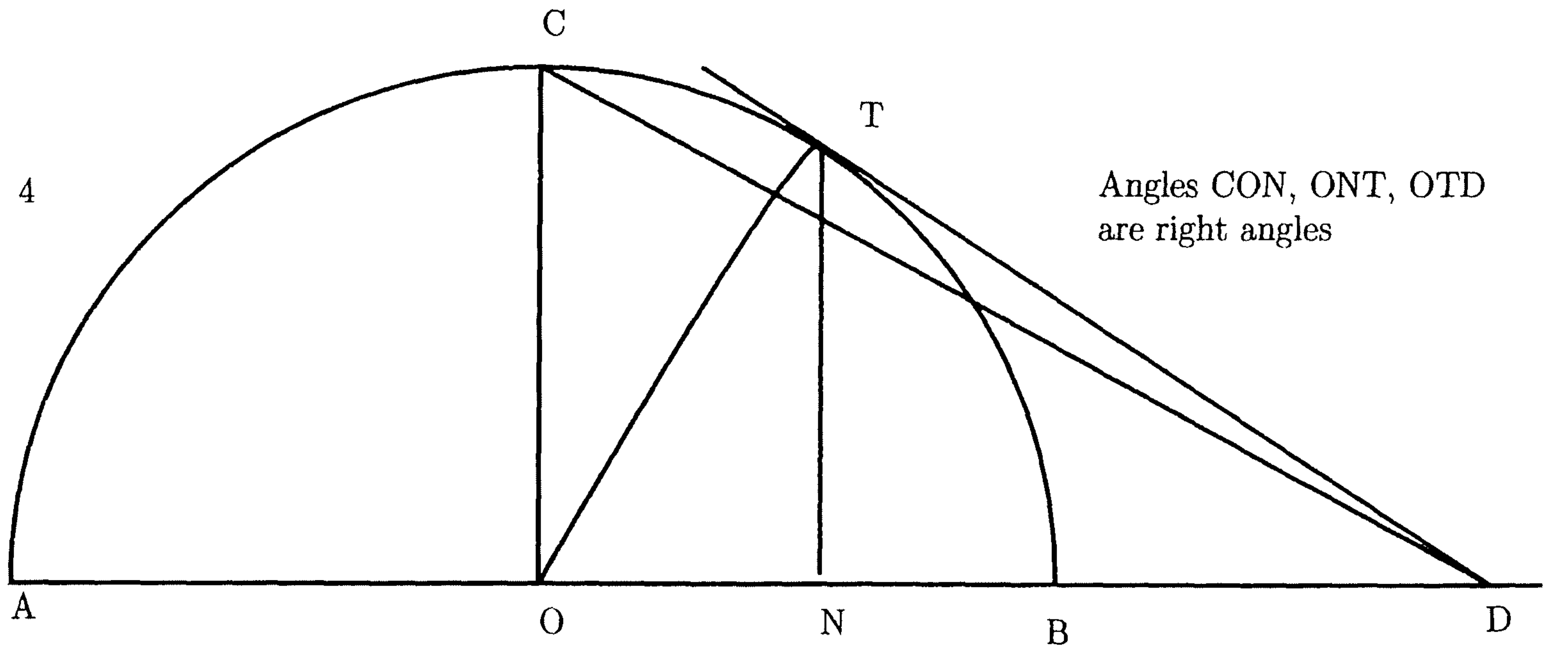
[9] An inequality $P \ge Q$ is said to be *homogeneous* when the function $P - Q$ is homogeneous.

Take any point D on the diameter AB of a semi-circle of centre O, let $AD = a, DB = b$. Construct the right angle ADC, then $CD = \sqrt{ab}$, and $CO = (a+b)/2$. The shortest distance from C to AB is the perpendicular distance so $CD \le CO$ with equality if and only if $D = O$.

If DE is perpendicular to CO it is not difficult to check that $CE = 2ab/(a+b) = \mathfrak{H}(a,b)$. So using a similar argument to the above some have proved that if $a \neq b$ then $\mathfrak{H}(a,b) < \mathfrak{G}(a,b)$; see [*Ercolano 1972, 1973; Gallant; Garfunkel & Plotkin; Grattan-Guinness; Schild; Sullivan*].

(*vi*) A similar proof can be found in [*Ercolano 1972*] but using Figure 4.

Figure 4



Here $AD = b, BD = a$; and then $OD = (a+b)/2, ND = 2ab/(a+b)$, and $TD = \sqrt{ab}$.

(*vii*) Another geometric proof is given in Figure 5.

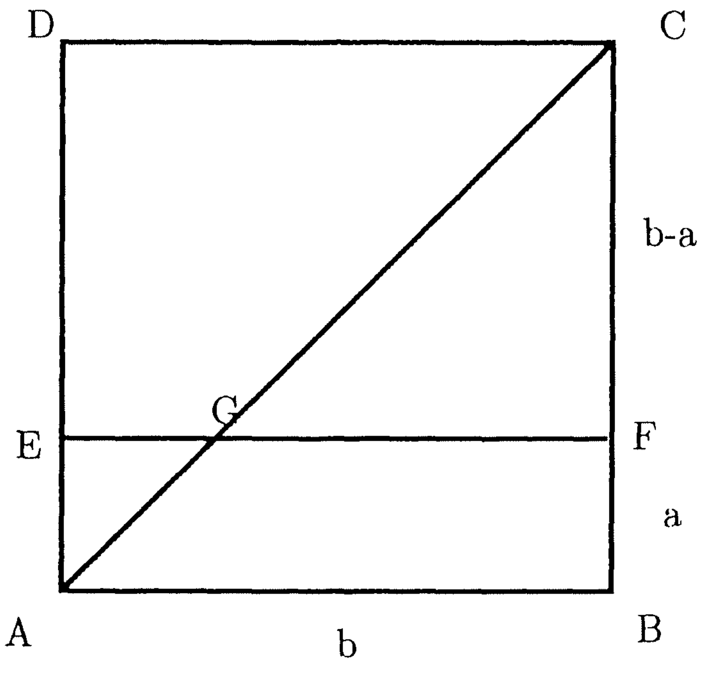


Figure 5

Let $ABCD$ be a square of side b, and let $ABFE$ be a rectangle of sides a and b. Then

$$\begin{aligned} \text{area ABFE} &= \text{area AGE} + \text{area ABFG} \\ &\le \text{area AGE} + \text{area ABC}; \end{aligned}$$

that is

$$ab \le \frac{a^2}{2} + \frac{b^2}{2},$$

which is equivalent to (2). Further equality occurs only when ABC and $ABFG$ have the same area, that is if and only if $a = b$.

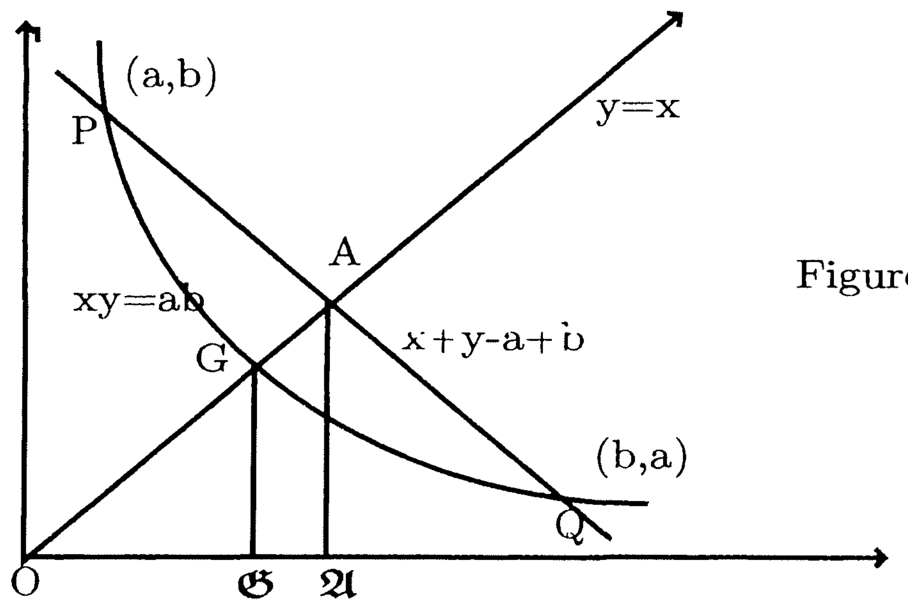


Figure 6

($viii$) The arithmetic mean of a and b is the common value of the co-ordinates of the point where the level curve of $f(x,y) = x + y$ passing through the point (a,b) meets the line $y = x$. This simple observation, together with a similar one for the geometric mean that is obtained when f is replaced by $g(x,y) = xy$ gives a simple proof of Lemma 3 base on the geometry of these curves.

Assume that $0 < a < b$ and let OGA, PGQ, PAQ be the curves $y = x$, $xy = ab$, $x + y = a + b$, respectively, see Figure 6. Then $G = \big(\mathfrak{G}(a,b), \mathfrak{G}(a,b)\big)$ and $A = \big(\mathfrak{A}(a,b), \mathfrak{A}(a,b)\big)$.

Since the function $f(x) = abx^{-1}$, $x > 0$, is convex, I 4.1 Corollary 7(a), the chord PQ, that is the line $x + y = a + b$, lies above the graph of f, I 4.1 Remark (iii), that is above $xy = ab$, and so the point G is to the left of the point A; that is $\mathfrak{G}(a,b) \le \mathfrak{A}(a,b)$.

Alternatively consider the fact that the two curves PAQ, PGQ only meet at P and Q, and that at P the slope of the first is -1, and that of the second is $-a/b$. Since $-a/b > -1$, the curve PGQ lies to the left of PAQ between P and Q.

(ix) The exponential function is strictly convex, I 4.1 Corollary 7(a), so by (J), $\exp\big(\frac{x+y}{2}\big) \le \frac{\exp x + \exp y}{2}$, with equality if and only if $x = y$. Putting $a = e^x, b = e^y$ and noting that the exponential function is strictly increasing completes this proof.

(x) A simple proof is given in [*Hästö*].

If $0 < a < b$ then putting $u^2 = b/a$ then $u > 1$ and $\dfrac{\mathfrak{A}(a,b)}{\mathfrak{G}(a,b)} = \dfrac{\mathfrak{A}(1,u)}{\mathfrak{G}(1,u)} = \dfrac{1}{2}\big(\dfrac{1}{u} + u\big)$ which is not less than 1 by I 2.2 (20).

(xi) See 5.5 Remark (ii); [*Burk 1985, 1987*]. □

Remark (ii) Proofs (iii), (iv)and (ix) will be adapted to prove (GA); see 2.4.2

proof (*xi*), 2.4.3 proof (*xxix*), 2.4.2 proof (*xvi*).

REMARK (iii) Proof (*ii*) which is in [*Eves 1980 p.14*] can be elaborated to give a further inequality; see VI 2.1.4 Theorem 22.

REMARK (iv) Proof (*x*) is an elaboration of proof (*iii*) and proves a little more; not only is $\mathfrak{A}(a,b)/\mathfrak{G}(a,b)$ bigger than 1, but if $a < b$ the ratio is increasing as a function of b and decreasing as a function of a. A similar proof can be given by considering the difference $\mathfrak{A}(a,b) - \mathfrak{G}(a,b) = \dfrac{a}{2}G(u)$ where $G(u) = 1 + 2u(u-1)$.

2.2.2 (GA) WITH $n = 2$, THE GENERAL CASE We now consider (GA) in its next simplest form, part (a) of the following lemma.

LEMMA 4 *(a) If a, b, α and β are positive real numbers with $\alpha + \beta = 1$ then*

$$a^\alpha b^\beta \le \alpha a + \beta b, \tag{3}$$

with equality if and only if $a = b$.
(b) If either $\alpha < 0$ or $\alpha > 1$ then ($\sim$3) holds, with the same case of equality.

□ *(a)* We give eleven proofs of this result.
(*i*) Inequality (3) can be written as $\left(\dfrac{a}{b}\right)^\alpha \le 1 + \alpha\left(\dfrac{a}{b} - 1\right)$, which follows from (B), I 2.1, on putting $a/b = 1 + x$.
(*ii*) It follows from a simple case of 2.2.3 Lemma 5 below that we need only consider α and β rational; the following proof using that assumption is given in [*Aiyar*].
Let $\alpha = p/(p+q)$, $\beta = q/(p+q), p, q \in \mathbb{N}^*$ and, assuming that $0 < a < b$, subdivide $[a, b]$ into $p + q$ equal sub-intervals, each of length $(b - a)/(p + q)$, by the points $x_i, 0 \le i \le p + q$, putting $x_0 = a, x_{p+q} = b$. Then:
$x_i - x_{i-1} = (b-a)/(p+q)$, $1 \le i \le p+q$, $x_{i+1} = (x_i + x_{i+2})/2$, $0 \le i \le p+q-2$, and so by (2)

$$\frac{a}{x_1} < \frac{x_1}{x_2} < \cdots < \frac{x_{p+q-1}}{b}.$$

Now put $\underline{c} = (a/x_1, x_1/x_2, \ldots, x_{q-1}/x_q)$, $\underline{b} = (x_q/x_{q+1}, \ldots, x_{p+q-1}/b)$. Then from the above and the internality of the geometric mean, 1.2 Theorem 6, $\mathfrak{G}_q(\underline{c}) < \mathfrak{G}_p(\underline{b})$; that is

$$\left(\frac{a}{x_q}\right)^{1/q} < \left(\frac{x_q}{b}\right)^{1/p}.$$

Simple calculations show that $x_q = \dfrac{pa + qb}{p+q}$, which on substituting in the last inequality gives the desired result.

A similar proof can be given using $x_i/x_{i-1} = (a/b)^{1/(p+q)}, i \le i \le p+q$, and $\sqrt{x_i x_{i+2}} = x_{i+1}$ to obtain the sub-intervals,

(*iii*) The method of proof (*viii*) of Lemma 3 can be adapted to this more general situation. The weighted arithmetic mean, respectively geometric mean, of a and b is the common value of the co-ordinates of the point where the level curve of $f(x,y) = \alpha x + \beta y$, respectively $g(x,y) = x^\alpha y^\beta$, through $P(a,b)$ meets the line $y = x$.

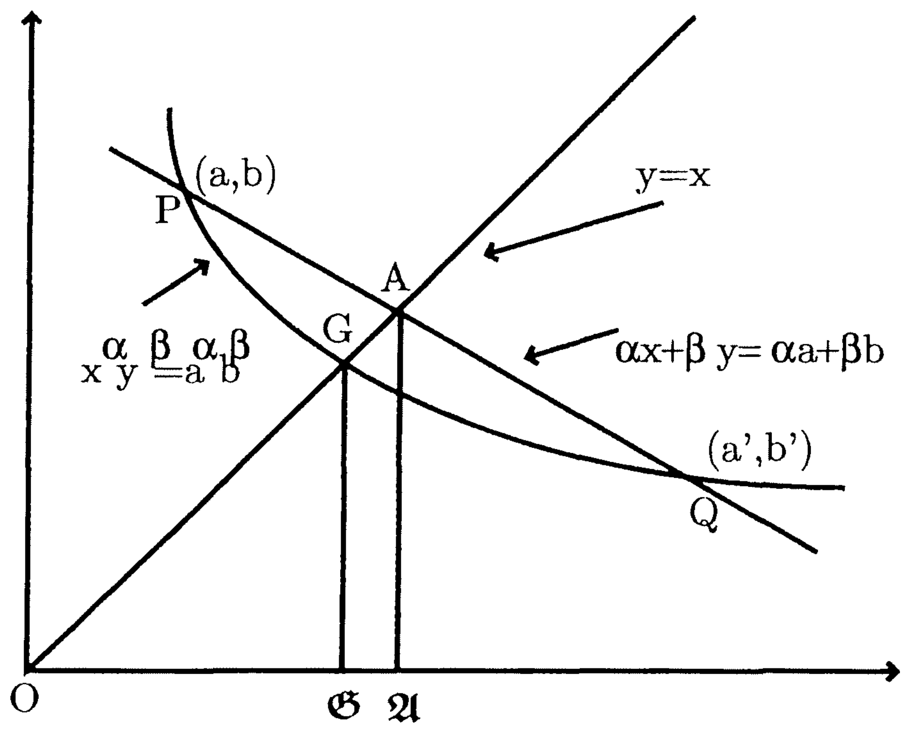


Figure 7

Let OGA, PGQ, PAQ be $y = x, x^\alpha y^\beta = a^\alpha b^\beta, \alpha x + \beta y = \alpha a + \beta b$ respectively, where we assume that $0 < a < b$, see Figure 7. Then $G = \mathfrak{G}(a,b;\alpha,\beta), A = \mathfrak{A}(a,b;\alpha,\beta)$, and it is not difficult to check that PGQ is convex or that it is to the left of PAQ at A, that is $\mathfrak{G}(a,b;\alpha,\beta) \le \mathfrak{A}(a,b;\alpha,\beta)$. In this case however we must verify that the two curves meet as shown in a point $Q(a',b')$. Simple calculations show that the two curves meet in points (x,y) where $h(x) = 0$, with $h(x) = \alpha x^{1/\beta} - (\alpha a + \beta b)x^{\alpha/\beta} + \beta a^{\alpha/\beta} b$. Clearly $h(0) > 0, h(a) = 0$, and for large x, $h(x) > 0$; further h has a unique negative minimum at $x = \mathfrak{A}(a,b)$. So h has two zeros a, a', with $a < \mathfrak{A}(a,b) < a'$.

(*iv*) The strict convexity of the exponential function can be used as in proof (*ix*) of Lemma 3; see [*Mullin*].

(*v*) Consider the function $t(x) = \alpha x^\beta + \beta x^{-\alpha}$. Differentiation shows that if $x > 0$ then $t(x) \ge t(1) = 1$. Substituting $x = a/b$, gives (5), and the case of equality.

(vi) The method used in proof (vii) of Lemma 3 can be generalized.

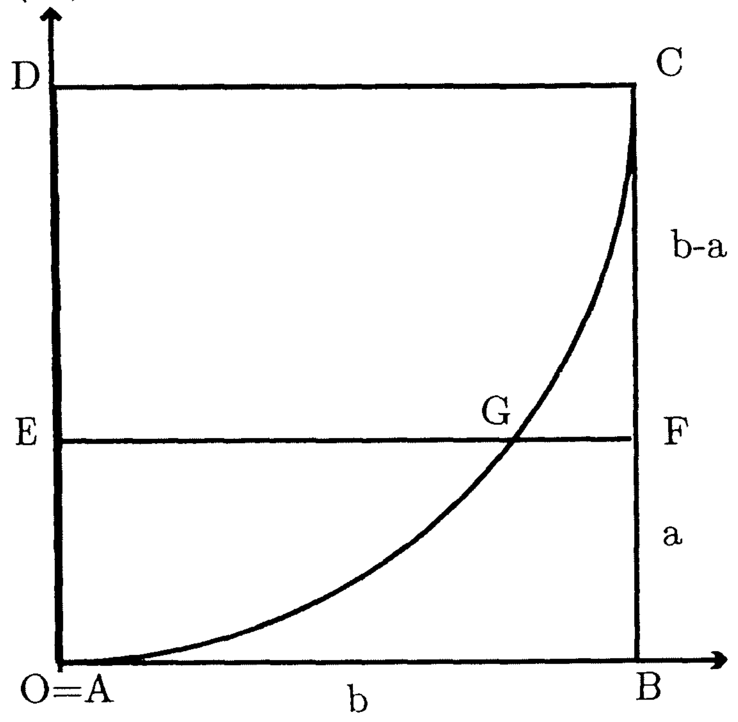


Figure 8

Assume that AB, AD are the co-ordinate axes and that AGC is the curve $y = x^{\alpha/\beta}$, see Figure 8.. Then, as in the proof mentioned, if $AB = b, AE = a$,

$$\begin{aligned}\text{area ABFE} &= \text{area AGE} + \text{area ABFG}\\ &\le \text{area AGE} + \text{area, ABC};\end{aligned}$$

or, by simple calculus

$$ab \le \alpha a^{1/\alpha} + \beta b^{1/\beta}; \tag{4}$$

and a simple change of variable completes the proof.

Substituting $\alpha = 1/p, \beta = 1 - 1/p = 1/p'$, in (4), where p' denotes the conjugate index, see Notations 4, gives:

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}, \tag{5}$$

a form of (3) used later; see III 2.1. Inequality (5) is sometimes called *Young's inequality*, although it is really a very special case of that inequality, [*AI pp.48–49; MPF pp.379–389*].

(vii) This is another proof of the case of rational weights; [*Brown*].

Let m, n be two positive integers and $0 < c < d$, then

$$\frac{c^m - d^m}{c^n - d^n} = \frac{c^{m-1} + c^{m-2}d + \cdots + d^{m-1}}{c^{n-1} + c^{n-2}d + \cdots + d^{n-1}} > \frac{mc^{m-1}}{nd^{n-1}} > \frac{mc^m}{nd^n}.$$

On multiplying this gives: $nd^n(d^m - c^m) > mc^m(d^n - c^n)$. On rewriting this inequality we get $mc^{m+n} + nd^{m+n} > (m+n)c^m d^n$. Now in this last inequality put $a = c^{m+n}, b = d^{m+n}$ to get (3) in the case of rational weights.

($viii$) The restriction to rational weights in the previous proof can be removed, see [*Bullen 1997*].

Consider the distinct power functions $\phi_1(x) = x^u$, $\phi_2(x) = x^v$; where $u \neq v$, $u, v \geq 1$ and $x > 0$. Applying the mean-value theorem of differentiation, see I 2.1 Footnote 1, to both of these functions on the interval $[c, d]$, $c > 0$, we get, on cancelling the common factor $d - c$,

$$\frac{d^u - c^u}{d^v - c^v} = \frac{ue_1^{u-1}}{ve_2^{v-1}},$$

for some e_1, e_2, with $c < e_1, e_2 < d$. Using the last condition and $u \neq v$, $u, v \geq 1$, we get that

$$\frac{d^u - c^u}{d^v - c^v} > \frac{uc^{u-1}}{vd^{v-1}} > \frac{uc^u}{vd^v};$$

or on multiplying out $vd^v(d^u - c^u) > uc^u(d^v - c^v)$. We have assumed that $u \neq v$ but it is easy to check that the last inequality remains valid when $u = v$. The proof now proceeds as in proof *(vii)* with u, v instead of m, n respectively.
If the Cauchy mean-value theorem[10] is used we can take $e_1 = e_2$ and then we need only take $u, v > 0$, although given the fact that we only really use the ratios $u/(u + v), v/(u + v)$ the initial restriction, $u, v \geq 1$, is unimportant.
(ix) The method used in proof *(iii)* of (J), I 4.2 Theorem 12 , can be adapted to give a proof of (3); [*Bullen 1979, 1980*].
Changing notation by putting $x = a, y = b, t = \beta, 1 - t = \alpha$, (3) becomes

$$G(t) = x^{1-t}y^t \geq A(t) = (1 - t)x + ty, \quad 0 \leq t \leq 1. \tag{6}$$

Further since the equality is trivial if $x = y$ we can, without loss in generality, assume $0 < x < y$.
It is easy to see that both of the functions A and G strictly increase, from the value x when $t = 0$ to the value y when $t = 1$.
Simple calculations give: $A'(t) = y - x$, $G'(t) = \big(\log y - \log x\big)G(t)$; and $A''(t) = 0$, $G''(t) = \big(\log y - \log x\big)^2 G(t)$. Then clearly $A' > 0$ and $G' > 0$, which confirms the above remark. Further however $G'' > 0$ so G is strictly convex. Since A is linear and $A(0) = G(0), A(1) = G(1)$, we see that the graph of A is a chord to the graph of G; and G being strictly convex it has a graph that lies below this chord, which is just (6).
An alternative method for this proof is given below in the discussion of (b).

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[10] The *Cauchy, or extended, mean-value theorem* states:*if the functions f,g are continuous on $[a,b]$ and differentiable on $]a,b[$, with g' never zero, then there is a point c, $a<c<b$, such that* $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$; [*CE p.598*].

(x) The following is a variant of proof (v); [*Maligranda 1995*]
If $p \geq 1$ and $a, b > 0$ consider the function

$$f(t) = \frac{1}{p} t^{-1/p'} a + \frac{1}{p'} t^{1/p} b, \quad t > 0,$$

where p' denotes the conjugate index. Simple calculations show that

$$f'(t) = -\frac{t^{-(1+1/p')}}{pp'}(bt - a),$$

so that $f(t) \geq f(a/b)$, with equality if and only if $t = a/b$, that is

$$a^{1/p} b^{1/p'} \leq \frac{1}{p} t^{-1/p'} a + \frac{1}{p'} t^{1/p} b,$$

with equality if and only if $t = a/b$. Now putting $t = 1$ $, \frac{1}{p} = \alpha, \frac{1}{p'} = \beta$ in the last inequality gives (3), and the inequality is strict unless $a = b$.
(xi) See also the proof of VI 5 Theorem 2 (a).

(b) We divide this part of the lemma into two cases.
Case (i) : $\alpha < 0$
Put $a' = a^\beta, b' = b^\beta, \alpha' = -\alpha/\beta, \beta' = 1 - \alpha' = 1/\beta$. Now apply (3) to a', b', α', β'.
Case (ii): $\alpha > 1$
In this case $\beta < 0$ so the previous argument can be easily adapted.
Alternatively we could use (B).
The proof (ix) of (a) gives both cases immediately. With the notation of that proof write $D(t) = A(t) - G(t)$, $t \in \mathbb{R}$. Then we have $D(0) = D(1) = 0$, and $D''(t) < 0$. Hence $D(t) > 0$, $0 < t < 1$, and $D(t) < 0$, $0 > t, \text{or}\, t > 1$. □

REMARK (i) It follows from proof *(i)* of (a), and (b) that Lemma 4 and I 2.1 Theorem 1 are equivalent. In particular proofs (ii)—(vi) of (a) can be used to give alternative proofs of (B).

2.2.3 THE EQUAL WEIGHT CASE SUFFICES We now show that 2.1 Theorem 1 can be deduced from its equal weight case.

LEMMA 5 *It is sufficient to prove Theorem 1 for the case of equal weights.*

□ Once Theorem 1 has been proved for constant $\underline{w}$, simple arithmetic arguments lead immediately to the case of rational weights; then (GA) for general real $\underline{w}$ follows by a limit argument.
To complete the proof it must be shown that if $\underline{a}$ is not constant then (1) is strict.

Suppose that not all of the weights are rational and write $w_i = u_i + v_i$, where $u_i \geq 0$ and $v_i \in \mathbb{Q}_+^*$, $1 \leq i \leq n$. By (GA), $\mathfrak{G}_n(\underline{a};\underline{u}) \leq \mathfrak{A}_n(\underline{a};\underline{u})$; also since $\underline{a}$ is not constant we have by Theorem 1 with rational weights, $\mathfrak{G}_n(\underline{a};\underline{v}) < \mathfrak{A}_n(\underline{a};\underline{v})$. So

$$\begin{aligned}\mathfrak{G}_n(\underline{a};\underline{w}) &= \Big(\mathfrak{G}_n(\underline{a};\underline{u})\Big)^{U_n/W_n}\Big(\mathfrak{G}_n(\underline{a};\underline{v})\Big)^{V_n/W_n}\\ &< \Big(\mathfrak{A}_n(\underline{a};\underline{u})\Big)^{U_n/W_n}\Big(\mathfrak{A}_n(\underline{a};\underline{v})\Big)^{V_n/W_n}\\ &\leq \frac{U_n}{W_n}\mathfrak{A}_n(\underline{a};\underline{u}) + \frac{V_n}{W_n}\mathfrak{A}_n(\underline{a};\underline{v}), \qquad \text{by (5)},\\ &= \mathfrak{A}_n(\underline{a};\underline{w}).\end{aligned}$$

□

Remark (i) This result seems to appear for the first time in [*HLP p.18*]; see also [*Herman, Kučera & Šimša pp.143, 170–171*]. Another proof is given later; see III 3.1.1 Remark (v). The result is generalized in 3.1 Lemma 2.

2.2.4 Cauchy's Backward Induction We now give a famous result of Cauchy, [*Cauchy 1821 p.315*]. It enables the proof of Theorem 1 to be reduced to the case of n-tuples with n belonging to some strictly increasing sequence, usually $n_k = 2^k, k \in \mathbb{N}^*$, and is part of Cauchy's proof of (GA), see below 2.4.1 proof *(ii)*.

Lemma 6 *If Theorem 1 has been proved for a particular positive integer then it is valid for all smaller positive integers.*

□ By Lemma 5 it suffices to consider Theorem 1 in the case of equal weights holds. Assume the result holds when $n = m$, that $k \in \mathbb{N}^*, 1 \leq k < m$ and $\underline{a}$ a positive k-tuple. Define the m-tuple $\underline{b}$ by

$$b_i = \begin{cases} a_i, & \text{if } 1 \leq i \leq k;\\ A = \mathfrak{A}_k(\underline{a};\underline{w}), & \text{if } k < i \leq m.\end{cases}$$

By the hypothesis, $\mathfrak{G}_m(\underline{b};\underline{w}) \leq \mathfrak{A}_m(\underline{b};\underline{w})$; this inequality being strict unless $\underline{b}$ is constant. This inequality is easily seen to be,

$$\Big(\mathfrak{G}_k(\underline{a})\Big)^{W_k/W_n n}(A)^{(1-W_k)/W_n} \leq \mathfrak{A}_m(\underline{b};\underline{w}) = A,$$

or $\mathfrak{G}_k(\underline{a}) \leq A = \mathfrak{A}_k(\underline{a})$, and there is equality if and only if $\underline{a}$ is constant. □

Remark (i) It follows from Lemma 6 that to prove (GA) it is sufficient to show that if $0 < n_1 < n_2 < \cdots$, $\lim_{k\to\infty} n_k = \infty$ then the assumption of the validity of (GA) for $n = n_k$ implies its validity for $n = n_{k+1}$.
We can formalize this as the *Cauchy principle of mathematical induction*; [*Dubeau 1991b*].

THEOREM 7 *If $n_0 \in \mathbb{N}^*$ and $\mathcal{P}(n)$ is a statement about integers $n \geq n_0$ such that:*
(a) $\mathcal{P}(n_0)$ is valid;
(b) if $\mathcal{P}(k)$ is valid for some $k \geq n_0$ then there is an integer $n_k > k$ such that $\mathcal{P}(n_k)$ is valid;
(c) if $\mathcal{P}(k)$ is valid for any $k > n_0$ then $\mathcal{P}(k-1)$ is valid.
Then $\mathcal{P}(n)$ is valid for all $n \geq n_0$.

2.3 SOME GEOMETRICAL INTERPRETATIONS Before turning to the proofs of (GA) we give some particularly simple forms of that inequality that have interesting geometrical applications.

LEMMA 8 *Theorem 1 is equivalent to either of the following.*
(a) If $\underline{a}$ is an n-tuple such that $\prod_{i=1}^n a_i = 1$ then $\sum_{i=1}^n a_i \geq n$, with equality if and only if $\underline{a}$ is constant;
(b) If $\underline{a}$ is an n-tuple such that $\sum_{i=1}^n a_i = 1$ then $\prod_{i=1}^n a_i \leq (1/n)^n$, with equality if and only if $\underline{a}$ is constant.

□ In both cases one implication is trivial.
Assume that (a) holds ; let $\underline{a}$ be any positive n-tuple; define $\underline{b} = \left(\frac{a_1}{\mathfrak{G}}, \dots, \frac{a_n}{\mathfrak{G}}\right)$. Clearly $\prod_{i=1}^n b_i = 1$, and so by (a) $\sum_{i=1}^n b_i \geq n$. This by the definition of $\underline{b}$ is just $\mathfrak{A} \geq \mathfrak{G}$.
Assume that (b) holds ; let $\underline{a}$ be any positive n-tuple; define $\underline{c} = \left(\frac{a_1}{n\mathfrak{A}}, \dots, \frac{a_n}{n\mathfrak{A}}\right)$. Clearly $\sum_{i=1}^n c_i = 1$, and so by (b) $\prod_{i=1}^n c_i \leq (1/n)^n$. This by the definition of $\underline{c}$ is just $\mathfrak{G} \leq \mathfrak{A}$.
Thus either (a) or (b) is sufficient to imply (GA) in the case of equal weights, which by Lemma 5 is sufficient to prove this lemma. □

REMARK (i) The case $n = 2$ of Lemma 8(a) is just I 2.2 (20); for (b) see [*Goursat*]. Both results are classical and proofs can be found in many places; see for instance [*Chrystal pp.52–56*], [*Darboux 1887*].

Both parts of this lemma have simple geometric interpretations.

COROLLARY 9 *(a) Of all n-parallelepipeds of given volume the one with the least perimeter is the n-cube.*
(b) Of all n-parallelepipeds of given perimeter the one with the greatest volume is the n-cube.

COROLLARY 10 *Of all the partitions of the interval $[0, 1]$ into n sub-intervals, the*

partition into equal sub-intervals is the one for which the product of the intervals is the greatest. See [*Pólya 1954 p.129*].

LEMMA 11 *(a) In the case of* $n = 3$ (GA) *is equivalent to the statement: of all the triangles of given perimeter the equilateral triangle has the greatest area.*
(b) In the case of $n = 4$ (GA) *for 4-tuples of numbers the sum of any three of which is greater than the fourth is equivalent to the statement: of all the concyclic quadrilaterals of given perimeter the square has the greatest area.*

□ (a) Let a, b, c be the lengths of the sides of a triangle, $s = (a + b + c)/2$ its semi-perimeter; then by a formula of Heron, [*Heath vol.II pp. 321–323; Melzak 1983b pp.1–3*], its area is $A = \sqrt{s(s-a)(s-b)(s-c)}$. When the triangle is equilateral this area is $A_0 = s^2/3\sqrt{3}$. By (GA) in the case $n = 3$ and equal weights,

$$A = \sqrt{s}\Big(\sqrt[3]{(s-a)(s-b)(s-c)}\Big)^{3/2} \leq \sqrt{s}\left(\frac{(s-a)+(s-b)+(s-c)}{3}\right)^{3/2}$$
$$= \frac{s^2}{3\sqrt{3}} = A_0$$

with equality if and only if $s - a = s - b = s - c$, or $a = b = c$.
Conversely if a_1, a_2, a_3 are any three positive numbers define a, b, c by $a_1 = s - a, a_2 = s - b, a_3 = s - c$, where s, as above, is $(a+b+c)/2$. Then simple calculations show that $s = a_1 + a_2 + a_3$ so $a = s - a_1 = a_2 + a_3, b = a_3 + a_1, c = a_1 + a_2$ are positive and further $a + b - c, b + c - a, c + a - b$ are also positive, being $2a_3, 2a_1 2a_2$ respectively, so that a, b, c are the sides of a triangle. Now using the inequalities above

$$\mathfrak{G}_3(a_1, a_2, a_3) = \Big(\frac{A}{\sqrt{s}}\Big)^{2/3} \leq \Big(\frac{A_0}{\sqrt{s}}\Big)^{2/3} = s/3 = \mathfrak{A}_3(a_1, a_2, a_3)$$

with equality if and only if $a = b = c$, or $a_1 = a_2 = a_3$.
(b)Let a, b, c, d be the sides of a concyclic quadrilateral, $s = (a+b+c+d)/2$ its semi-perimeter; then, by a formula of Brahmagupta, [*Melzak 1983b pp.4–6*], its area is $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$. When the quadrilateral is a square this area is $A_0 = s^2/4$. Noting that the 4-tuple $a_1 = s - a, a_2 = s - b, a_3 = s - c, a_4 = s - d$ has the property required in the hypotheses— the sum of any three is greater than the fourth— apply the equal weight (GA) in this special case to get

$$A = \Big(\sqrt[4]{(s-a)(s-b)(s-c)(s-d)}\Big)^2 \leq \left(\frac{(s-a)+(s-b)+(s-c)+(s-d}{4}\right)^2$$
$$= \frac{s^2}{4} = A_0$$

with equality if and only if $s-a=s-b=s-c=s-d$, or $a=b=c=d$.
Conversely if a_1, a_2, a_3, a_4 are any four positive numbers satisfying the hypothesis that the sum of any three is greater than the fourth define, as in (a), a, b, c, d by $a_1 = s-a, a_2 = s-b, a_3 = s-c, a_4 = s-d$, where s, as above, is $(a+b+c+d)/2$. Then simple calculations show that $2s = a_1+a_2+a_3+a_4$ so $2a = 2s-2a_1 = a_2+a_3+a_4-a_1 > 0$ and similarly b, c and d are positive. Further $a+b+c-d = 2a_4 > 0$, $b+c+d-a = 2a_1 > 0$,$c+d+a-b = 2a_2$, $d+a+b-c = 2a_3$ so that a, b, c, d are the sides of a concyclic quadrilateral, see [*Melzak 1983b pp.8–9*]. Now using the inequalities above

$$\mathfrak{G}_4(a_1, a_2, a_3, a_4) = A^{1/2} \le A_0^{1/2} = s/2 = 2s/4 = \mathfrak{A}_4(a_1, a_2, a_3, a_4),$$

with equality if and only if $a=b=c=d$, or $a_1=a_2=a_3=a_4$.
Noting that Lemma 5 shows that (GA) for equal weights and a given n-tuple implies the general case for the same n-tuple completes the proof. □

REMARK (ii) Extending this to a general concyclic n-gon seems difficult as there seems to be no simple formula for the area of a concyclic n-gon, $n \ge 5$; see [*Robbins*]. The need to phrase part (b) of this lemma in a manner different from part (a) was communicated to me by Mowaffaq Hajja.

REMARK (iii) For further discussions of the results in this section the reader should consult the following: [*Kazarinoff 1961a pp.18–58; Kline p.126*], [*Bioche; Garver; Shisha; Usai 1940b*].

2.4 PROOFS OF THE GEOMETRIC MEAN-ARITHMETIC MEAN INEQUALITY The proofs will, as far as is known, be in the order of their appearance. Seventy four proofs are given and there are undoubtedly as many more in the literature. A survey of proofs given up to 1904 can be found in [*Muirhead 1901/04*] and a survey of several proofs also occurs in [*Colwell & Gillett*].

It is sufficient by Lemma 5 to give a proof for the equal weight case and many proofs do this. Also, given 2.2.1 Lemma 3, or 2.2.2 Lemma 4, it is sufficient to give the inductive step in any proof by induction; see also 2.2.4 Remark (i). Further for a complete proof of the theorem it is clearly sufficient to prove that the inequality is strict for non-constant n-tuples. In addition in an inductive proof we can also assume that no two elements of the the n-tuple $\underline{a}$ are equal, for if two are equal the inequality follows by the induction hypothesis; in fact we may assume that the n-tuple is strictly increasing.
The proofs are divided into sections determined by the publication dates of the four main references [*HLP; BB; AI; MI*], 1934, 1965, 1970 and 1988, respectively,

preceded by the prehistoric proofs. A baker's dozen of proofs that are in journals not seen by the author are listed for completeness in section 2.4.7.

2.4.1 PROOFS PUBLISHED PRIOR TO 1901. PROOFS (*i*)–(*vii*)

(*i*) MACLAURIN CIRCA 1729

This is by far the earliest proof and it appears to be due to Maclaurin who states the result in the form of 2.3 Corollary 10.

□ Suppose that $0 < a_1 \le a_2 \le \cdots \le a_n, a_1 \ne a_n$. If a_1 and a_n are replaced by $(a_1 + a_n)/2$ then $\mathfrak{A}$ is unchanged, but using (2) it is easily see that $\mathfrak{G}$ is increased. If then $\underline{a}$ is varied so as to keep $\mathfrak{A}$ fixed, and of such n-tuples $\underline{a}'$ is the one at which $\mathfrak{G}$ assumes its maximum value, the above argument shows that $\underline{a}'$ must be constant. Hence the maximum of $\mathfrak{G}$ is attained when all the terms of $\underline{a}$ are equal, and this maximum value is equal to $\mathfrak{A}$. □

[*Chrystal p.47*], [*Grebe; Maclaurin*].

Given the date of the proof it is not surprising that the existence of an $\underline{a}'$ at which $\mathfrak{G}$ attains its maximum was taken for granted. This missing step can be supplied in either of the following ways, [*HLP p.19, footnote(a)*].

(a) Use the fact that a continuous function, ϕ, defined on a compact set, K, attains its maximum on that set. In this situation if $\underline{x} = (x_1, \ldots, x_n)$,

$$\phi(\underline{x}) = \Big(\prod_{i=1}^{n} x_1\Big)^{1/n}; \; K = \Big\{\underline{x};\, x_i \ge 0,\, 1 \le i \le n, \text{ and } \sum_{i=1}^{n} x_i = n\mathfrak{A}\Big\}.$$

(b) After k steps in Maclaurin's proof let us denote the resulting n-tuple by $\underline{a}^{(k)}$, and assume it is increasing. Then clearly

$$a_1^{(k)} \le a_1^{(k+1)} \le \cdots \le a_n^{(k+1)} \le a_n^{(k)}.$$

Hence $\lim_{k\to\infty} a_1^k$, and $\lim_{k\to\infty} a_n^k$ both exist, say with values a, A respectively. It is not difficult to see that after n steps that maximum difference in the sequence has been reduced by at least one half; that is

$$a_n^{(n)} - a_1^{(n)} \le \frac{a_n - a_1}{2},$$

and so $\lim_{k\to\infty} \big(a_n^{(k)} - a_1^{(k)}\big) = 0$, or $A = a$. This argument is due to Hardy.

(*ii*) CAUCHY 1821

This elementary proof depends on a sophisticated induction argument, see 2.2.4 Lemma 6, and consists of proving (GA) for all integers n of the form 2^k, $k \in \mathbb{N}^*$.

□ Assume the result is known for $k=m$ and let $\underline{a}=(a_1,\ldots,a_{2^m})$, $\underline{b}=(b_1,\ldots,b_{2^m})$ and $\underline{c}=(c_1,\ldots,c_{2^{m+1}})=(\underline{a},\underline{b})=(a_1,\ldots,a_{2^m},b_1,\ldots,b_{2^m})$.
Now

$$\begin{aligned}\mathfrak{G}_{2^{m+1}}(\underline{c}) &= \sqrt{\mathfrak{G}_{2^m}(\underline{a})\mathfrak{G}_{2^m}(\underline{b})},\\ &\le \sqrt{\mathfrak{A}_{2^m}(\underline{a})\mathfrak{A}_{2^m}(\underline{b})}, \text{ by the induction hypothesis} \qquad (7)\\ &\le \mathfrak{A}\big(\mathfrak{A}_{2^m}(\underline{a}),\mathfrak{A}_{2^m}(\underline{b})\big), \text{ by (GA) in case } n=2, \text{ Lemma 3,} \qquad (8)\\ &= \mathfrak{A}_{2^{m+1}}(\underline{c}).\end{aligned}$$

Equality occurs at (7), by the induction hypothesis, if and only if $\underline{a}$, and $\underline{b}$ are constant, a and b respectively say; and then at (8) there is equality if and only if $a=b$; that is we get equality if and only if $\underline{c}$ is constant. □

[*Cauchy 1821 p.315; Pólya & Szegö p.64*], [*Chajoth; Forder*].

REMARK (i) A variant of this proof can be found in [*Boutroux*]; Bellman has given a particularly lucid version, [*Bellman 1954*].

(*iii*) LIOUVILLE 1839

□ Assume (GA) for all integers less than n and put $\underline{a}=(a_1,\ldots,a_{n-1},x)$, and let $f(x)=\mathfrak{A}_n^n(\underline{a})-\mathfrak{G}_n^n(\underline{a})$. Then

$$f'(x)=\mathfrak{A}_n^{n-1}(\underline{a})-\mathfrak{G}_{n-1}^{n-1}(\underline{a})=\Big(\frac{n-1}{n}\mathfrak{A}_{n-1}(\underline{a})+\frac{x}{n}\Big)^{n-1}-\mathfrak{G}_{n-1}^{n-1}(\underline{a}).$$

This shows that f' is increasing; further f' vanishes when $x=x'$, where $x'=n\mathfrak{G}_{n-1}(\underline{a})-(n-1)\mathfrak{A}_n(\underline{a})$. Thus f has a unique minimum at x', and the value of this minimum is $f(x')=(n-1)\mathfrak{G}_{n-1}^{n-1}(\underline{a})\big(\mathfrak{A}_{n-1}(\underline{a})-\mathfrak{G}_{n-1}(\underline{a})\big)$, and by the induction hypothesis $f(x')\ge 0$, .
Hence $f\ge 0$, which completes the proof except for the case of equality. In order that $f(x)=0$ we need both that $x=x'$ and $f(x')=0$. By the induction hypothesis $f(x')=0$ if and only if $a_1=\cdots=a_{n-1}=a$, say; but then $x'=a$ and the proof is complete. □

[*Liouville*].

REMARK (ii) This early proof does not seem to have been noticed until it appeared in the book by Mitrinović & Vasić, [*Mitrinović & Vasić pp.28–29*]. As a result many other authors discovered it independently; see, for instance, [*Buch; Lawrence; Yosida*]. Like proof (*i*) this is not an elementary proof. The method can be used to obtain more precise results; see below 3.1 Theorem 1, and [*Rüthing 1982*].

(*iv*) THACKER 1851.

This inductive proof is based on ($\sim$B).

□

$$\begin{aligned}\mathfrak{A}_n^n(\underline{a}) &= a_1^n\Big(1+\frac{a_2+\cdots a_n+(1-n)a_1}{na_1}\Big)^n,\\ &\ge a_1\Big(\frac{a_2+\cdots+a_n}{n-1}\Big)^{n-1}, \text{ by I 2.2(11), or by}\\ &\qquad (\sim\text{B}),\text{ I 2.1 Theorem 1, with}\ \alpha=\frac{n}{n-1},\\ &\ge a_1\mathfrak{G}_{n-1}^{n-1}(\underline{a})=\mathfrak{G}_n^n(\underline{a}), \text{using the induction hypothesis.}\end{aligned}$$

Further, from I 2.1 Theorem 1 and the induction hypothesis this inequality is strict unless $\underline{a}$ is constant. □

[*Thacker*].

(*v*) HURWITZ 1891

□ If f is any function of n variables put $P\big(f(a_1,\ldots,a_n)\big)=\sum! f(a_{i_1},\ldots,a_{i_n})$. So for instance:

$$\text{if } f(a_1,\ldots,a_n)=a_1a_2\cdots a_n \text{ then } P(a_1a_2\cdots a_n)=n!(a_1a_2\cdots a_n)=n!\mathfrak{G}_n(\underline{a}^n);$$
$$\text{if } f(a_1,\ldots,a_n)=a_1^n \text{ then } P(a_1^n)=(n-1)!(a_1^n+a_2^n\cdots+a_n^n)=n!\mathfrak{A}_n(\underline{a}^n).$$

Define ϕ_k, $1\le k\le n-1$, by $\phi_k=P\big((a_1^{n-k}-a_2^{n-k})(a_1-a_2)a_3a_4\cdots a_{k+1}\big)$. Then, as is easily seen, $\phi_k=P\big((a_1^{n-k-1}+\cdots+a_2^{n-k-1})(a_1-a_2)^2a_3a_4\cdots a_{k+1}\big)$, and so if $\underline{a}$ is positive and not constant, $\phi_k>0, 1\le k\le n-1$ while if $\underline{a}$ is constant then $\phi_k=0, 1\le k\le n-1$. Also

$$\phi_k=2\Big(P\big(a_1^{n-k+1}a_2\cdots a_k\big)-P\big(a_1^{n-k}a_2\cdots a_{k+1}\big)\Big)$$

and so, summing over k we get $\sum_{k=1}^{n-1}\phi_k=2\big(P(a_1^n)-P(a_1\cdots a_n)\big)$. Hence, from the initial comments,

$$\mathfrak{A}_n(\underline{a}^n)-\mathfrak{G}_n(\underline{a}^n)=\frac{1}{2(n!)}\sum_{k=1}^{n-1}\phi_k\ge 0,$$

with equality if and only if $\underline{a}$ is constant. □

[*Hurwitz 1891*].

REMARK (iii) This proof is interesting in that it is the first to give an exact value for the difference $\mathfrak{A}_n(\underline{a}^n)-\mathfrak{G}_n(\underline{a}^n)$.

REMARK (iv) In [*BB p.8*] it is pointed out that this proof contains the germ of a technique that Hurwitz was to use later in his famous paper, [*Hurwitz 1897*],

on the generation of invariants by integration over groups. For further discussion along these lines the reader is referred to the papers [*Motzkin 1965a,b*].

(*vi*) CRAWFORD 1900

This is a variant of proof (*i*), more sophisticated but elementary.

□ As in proof (*i*) let the n-tuple $\underline{a}$ be such that $0 < a_1 \le \cdots \le a_n, a_1 \ne a_n$ and define $\underline{a}'$ by changing a_1 to $\mathfrak{A}_n(\underline{a}) = \mathfrak{A}$, and a_n to $a_1 + a_n - \mathfrak{A}$, the rest of the $a'_i = a_i$. Then $\mathfrak{A}_n(\underline{a}') = \mathfrak{A}$ and

$$\begin{aligned}\mathfrak{G}_n^n(\underline{a}') =&\mathfrak{G}_n^n(\underline{a}) - \prod_{i=2}^{n-1} a_i(\mathfrak{A} - a_1)(a_n - \mathfrak{A})\\ >&\mathfrak{G}_n^n(\underline{a}), \quad \text{the second term being positive by internality.}\end{aligned}$$

After at most $(n-1)$ repetitions of this process we arrive at a constant n-tuple $\underline{a}''$, and this gives a proof of (GA). □

[*Briggs & Bryan p.185; Hardy 1948 p.32; Pólya 1954 p.247; Sturm p.3*], [*Crawford; Fletcher; Muirhead 1900/01, 1901/1904, 1906*].

REMARK (v) A similar proof has been constructed by defining $a'_1 = \mathfrak{G}_n(\underline{a}) = \mathfrak{G}$, and $a_n = a_1a_2/\mathfrak{G}$. The idea in this proof has been as a basis of a general method of forming inequalities, [*Kečkić*].

(*vii*) CHRYSTAL 1900

□ Again let the n-tuple $\underline{a}$ be increasing and non-constant, and assume (GA) for all integers less than n.

$$\mathfrak{A}_n(\underline{a}) = \frac{n-1}{n}\mathfrak{A}_{n-1}(\underline{a}) + \frac{a_n}{n} = \mathfrak{A}_{n-1}(\underline{a}) + \frac{a_n - \mathfrak{A}_{n-1}(\underline{a})}{n}. \tag{9}$$

By internality $a_n - \mathfrak{A}_{n-1}(\underline{a}) > 0$ and so from the right-hand side of (9)

$$\begin{aligned}\mathfrak{A}_n^n(\underline{a}) >&\mathfrak{A}_{n-1}^n(\underline{a}) + \big(a_n - \mathfrak{A}_{n-1}(\underline{a})\big)\mathfrak{A}_{n-1}^{n-1}(\underline{a}), \text{ by } (\sim B),\\ =&a_n\mathfrak{A}_{n-1}^{n-1}(\underline{a})\\ \ge&a_n\mathfrak{G}_{n-1}^{n-1}(\underline{a}), \quad \text{by the induction hypothesis,}\\ =&\mathfrak{G}_n^n(\underline{a}).\end{aligned}$$

□

[*Chrystal vol.II*], [*Muirhead 1901/04; Oberschelp; Popović; Wigert*].

REMARK (vi) If the inductive hypothesis is used on the middle expression in (9) then we get

$$\mathfrak{A}_n^n(\underline{a}) > \left(\mathfrak{G}_{n-1}(\underline{a}) + \frac{a_n - \mathfrak{G}_{n-1}(\underline{a})}{n}\right)^n.$$

and the above argument again gives (GA); see [*Weber pp.689–690*], [*Tweedie*].

2.4.2 PROOFS PUBLISHED BETWEEN 1901 AND 1934. PROOFS (*viii*)-(*xvi*)

(*viii*) MUIRHEAD 1900/1901

See V 6 Remark (iii).

(*ix*) DOUGALL 1905

Following ideas of Muirhead, Dougall made a study of various identities from which (GA) becomes apparent. In particular he gives the following proof of (GA).

□ With the notation of I 1.1(1) and Example (i) , if $\underline{a}$ is a non-constant n-tuple the polynomial

$$\prod_{i=1}^{n}(x+a_i) = \sum_{i=0}^{n}\binom{n}{i}d_i x^i$$

has distinct roots so inequality I 1.1(4) is strict, that is

$$d_{n-r}^{1/r} > d_{n-r-1}^{1/r+1}, \quad 1 \le r \le n.$$

In particular $d_{n-1} > d_0^{1/n}$, which is (GA). □

[*Dougall*].

REMARK (i) In fact Dougall gives an exact formula for $d_1^n - d_n$; such a formula also occurs in [*Jolliffe; Muirhead, 1900/01*]. A similar proof can be found in [*Green*].

(*x*) PÓLYA 1910

□ If $\underline{a}$ is a non-constant n-tuple define $\underline{b}$ by $a_i = (1+b_i)\mathfrak{A}_n(\underline{a};\underline{w}), 1 \le i \le n$. Then $\underline{b}$ is not the zero n-tuple and $\sum_{i=1}^{n} w_i b_i = 0$, and

$$\begin{aligned}\mathfrak{G}_n(\underline{a};\underline{w}) =& \mathfrak{A}_n(\underline{a};\underline{w})\mathfrak{G}_n(1+\underline{b};\underline{w})\\ <& \mathfrak{A}_n(\underline{a};\underline{w})\exp\Big(\sum_{i=1}^{n} w_i b_i\Big), \text{ by inequality I 2.2(8),}\\ =& \mathfrak{A}_n(\underline{a};\underline{w}).\end{aligned}$$

□

[*HLP p.103*], [*Alexanderson*], [*Lidstone; Wetzel*].

REMARK (ii) This proof is in [*HLP*] but is assigned this date by Alexanderson. The equal weight case was rediscovered by Lidstone.

REMARK (iii) An extension of the idea in this proof is given below in 2.4.5 proof (*xlii*).

(xi) DÖRRIE 1921

□ If $\underline{a}$ is a non-constant n-tuple, $n \geq 3$, and $\prod_{i=1}^{n} a_i = 1$, then at least one element is bigger than 1 and at least one is less than 1; assume $a_2 < 1 < a_1$. Then

$$\begin{aligned}\sum_{i=1}^{n} a_i =& a_1 + a_2 + \sum_{i=3}^{n} a_i \\ >& 1 + a_1 a_2 + \sum_{i=3}^{n} a_i, \quad \text{see 2.1 Lemma 3 proof } (iv), \\ \geq& 1 + (n-1) = n \quad \text{by the induction hypothesis,}\end{aligned}$$

and by 2.3 Lemma 8(a), this is equivalent to (GA). □

[*Dörrie pp.37–39; Korovkin p.7*], [*Ehlers; Heymann; Kreis 1946*].

(xii) CARR 1926

□ The following is a simple inductive proof.

$$\begin{aligned}\mathfrak{G}_n(\underline{a}) =& \left(\mathfrak{G}_n^{1/n-1}(\underline{a}) a_n^{n-2/n-1}\right)^{1/n-1} \mathfrak{G}_{n-1}^{n-2/n-1}(\underline{a}) \\ \leq& \frac{1}{n-1}\mathfrak{G}_n^{1/n-1}(\underline{a}) a_n^{n-2/n-1} + \frac{n-2}{n-1}\mathfrak{G}_{n-1}(\underline{a}), \quad \text{by (5), the case } n=2, \\ \leq& \frac{1}{(n-1)^2}\mathfrak{G}_n(\underline{a}) + \frac{n-2}{(n-2)^2} a_n + \frac{n-2}{n-1}\mathfrak{G}_{n-1}(\underline{a}), \quad \text{by (5) again,} \\ \leq& \frac{1}{(n-1)^2}\mathfrak{G}_n(\underline{a}) + \frac{n-2}{(n-2)^2} a_n + \frac{n-2}{n-1}\mathfrak{A}_{n-1}(\underline{a}), \text{ by the induction hypothesis,} \\ =& \frac{1}{(n-1)^2}\mathfrak{G}_n(\underline{a}) + \left(1 - \frac{1}{(n-1)^2}\right)\mathfrak{A}_n(\underline{a}).\end{aligned}$$

Simple calculations leads to $\mathfrak{G}_n(\underline{a}) \leq \mathfrak{A}_n(\underline{a})$. □

[*Carr*].

$(xiii)$ STEFFENSEN 1930

We first have the following simple lemma.

LEMMA 12 *If $\underline{a}$, $\underline{b}$ are increasing n-tuples and $\underline{a} \leq \underline{b}$, then $\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right)$ is not decreased by interchanging a_k and b_k; further it is increased unless either $a_k = b_k$, or for all $i \neq k, a_i = b_i$.*

□ This is immediate from the identity

$$\begin{aligned}&\Big(\sum_{\substack{i=1\\i\neq k}}^{n} a_i + b_k\Big)\Big(\sum_{\substack{i=1\\i\neq k}}^{n} b_i + a_k\Big) \\ &\quad= \Big(\sum_{i=1}^{n} a_i\Big)\Big(\sum_{i=1}^{n} b_i\Big) + (b_k - a_k)\Big(\sum_{\substack{i=1\\i\neq k}}^{n} b_i - \sum_{\substack{i=1\\i\neq k}}^{n} a_i\Big).\end{aligned}$$

□

Using Lemma 12 we have the following proof of (GA).

□ Suppose that $\underline{a}$ is non-constant increasing n-tuple, then

$$
\begin{aligned}
n^n\mathfrak{G}_n^n(\underline{a}) =&(\overbrace{a_1+\cdots+a_1}^{n\,terms})(a_2+\cdots+a_2)\ \ldots\ (a_n+\cdots+a_n)\\
<&(a_1+a_2+\cdots+a_n)(a_1+a_2+\cdots+a_2)\ \ldots\ (a_1+a_n+\cdots+a_n).
\end{aligned}
$$

by Lemma 12. The result follows by a repetition of this argument. □

[*Steffensen 1930, 1931*].

(xiv) NAGELL 1932

□ Assuming that $\underline{a}$ is a decreasing n-tuple define $\Delta = n\big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\big)$ and $\Delta_i = a_i^{1/n} - a_n^{1/n}$, $1 \le i \le n$. Then

$$
\begin{aligned}
\Delta =& \sum_{j=2}^{n} a_n^{(n-j)/n}\left(\binom{n}{j}\sum_{i=1}^{n}\Delta_i^j - n\sum_j{}^{!}\Delta_{i_1}\cdots\Delta_{i_j}\right)\\
=&\sum_{j=2}^{n} a_n^{(n-j)/n}\binom{n}{j}\left(\left(\binom{n-1}{j-1} - \binom{n-2}{j-1}\sum_{i=1}^{n}\Delta_i^j\right)\right.\\
&\left.\qquad + \sum_j{}^{!}\left(\sum_{k=1}^{j}\Delta_{i_k}^j - j\prod_{k=1}^{j}\Delta_{i_k}\right)\right)\\
\ge&\sum_{j=2}^{n} a_n^{(n-j)/n}\binom{n}{j}\binom{n-2}{j-2}\sum_{i=1}^{n}\Delta_i^j,\ \text{using induction on the terms of}\ \sum_j{}^{!}\\
\ge&0.
\end{aligned}
$$

□

[*Nagell; Solberg*].

(xv) HARDY, LITTLEWOOD & PÓLYA 1934

□ Suppose that (GA) has been proved for integers less than n, then

$$
\begin{aligned}
\mathfrak{G}_n(\underline{a};\underline{w}) =&\mathfrak{G}_{n-1}(\underline{a};\underline{w})^{W_{n-1}/W_n}a_n^{w_n/W_n}\\
\le&\frac{W_{n-1}}{W_n}\mathfrak{G}_{n-1}(\underline{a};\underline{w}) + \frac{w_n}{W_n}a_n,\ \text{by (5)}\\
\le&\frac{W_{n-1}}{W_n}\mathfrak{A}_{n-1}(\underline{a};\underline{w}) + \frac{w_n}{W_n}a_n,\ \text{by the induction hypothesis,}\\
=&\mathfrak{A}_n(\underline{a};\underline{w}),
\end{aligned}
$$

The case of equality follow from 2.2.2 Lemma 4 and the induction hypothesis. □

[*HLP p.38*].

REMARK (iv) A similar proof can be given starting with $\mathfrak{A}_n(\underline{a};\underline{w})$. The proof is an adaption of a proof of (J); see I 4.2 Theorem 12 proof (i).

REMARK (v) Other proofs of (J) and of the Jensen-Steffensen inequality can be adapted to give a variant of this proof; see proof 2.4.5 (lii) and [*Magnus*].

REMARK (vi) Even in the case of equal weights appeal must be made to (5) rather than (2). However in the equal weight case the appeal to (5) can be avoided by using the the inequality 1.2 (8); see [*Georgakis*].

(xvi) HARDY, LITTLEWOOD & PÓLYA 1934

□ Using the strict convexity of the exponential function we have:

$$\begin{aligned}\mathfrak{G}_n(\underline{a};\underline{w}) &= \exp\big(\mathfrak{A}_n(\log \underline{a};\underline{w})\big), \text{ by } 1.2(8)\\ &<\mathfrak{A}_n(\exp\circ\log \underline{a};\underline{w}), \text{ by (J), see } 1.1(4),\\ &=\mathfrak{A}_n(\underline{a};\underline{w}).\end{aligned}$$

□

[*HLP p.78*], [*Barton; Flanders*].

REMARK (vii) A similar proof can be given using the strict concavity of the logarithmic function, [*Gentle 1977; Steffensen 1919*].

2.4.3 PROOFS PUBLISHED BETWEEN 1935 AND 1965. PROOFS $(xvii)$–$(xxxi)$

$(xvii)$ BOHR 1935

□ Consider the expansion of e^{xy} and $\dfrac{x^k y^k}{k!}$ in powers of y. Quite trivially no coefficient of the first expansion is less than the corresponding coefficient of the second. It is then quite easy to see that the same is true of the expansions of the two functions $\exp\big(y\sum_{i=1}^n x_i\big)$ and $\dfrac{\big(\prod_{i=1}^n x_i\big)^k y^{nk}}{(k!)^n}$, both obtained as products of functions of the above type.
Comparing the coefficients of y^{nk} in the two equations we get that

$$\frac{\mathfrak{A}_n(\underline{x})}{\mathfrak{G}_n(\underline{x})} \ge \frac{1}{n}\left(\frac{(nk)!}{(k!)^n}\right)^{1/kn}.$$

Now, using *Stirling's formula* in the form $m! \sim m^m e^{-m}\sqrt{2\pi m}$, the right-hand side tends to 1 as $k \to \infty$. □

[*Bohr*].

REMARK (i) Bohr's proof does not give the case of equality, but the following simple argument by Dinghas completes the proof; [*Dinghas 1962/3*]. Consider

the non-negative function $d(\underline{x}) = \mathfrak{A}_n(\underline{x}) - \mathfrak{G}_n(\underline{x})$ then $\partial d/\partial x_i = (1 - \mathfrak{G}_n/x_i)/n$, $1 \le i \le n$. Now assume that for a non-constant $\underline{x}$ we have $d(\underline{x}) = 0$. Choose i so that $x_i = \min \underline{x}$, then $\partial d/\partial x_i$ is negative and so if x_i is increased slightly d becomes negative, which is a contradiction.

(*xviii*) DEHN 1941

A particularly simple geometrical proof has been given by Dehn based on some observations about angles and chords in a circle. In particular if the angles α_i, $1 \le i \le n$, are not all equal and from the centre of a circle cut off chords of lengths a_i, $1 \le i \le n$ respectively then $\mathfrak{A}_n(\underline{a})$ is less than the length of the chord cut off by the angle $\mathfrak{A}_n(\underline{\alpha})$; for full details the reader is referred to the reference. [*Dehn*].

(*xix*) WALSH 1943

We first prove (HA).

LEMMA 13 *If $\underline{a}$ and $\underline{w}$ are n-tuples then*

$$\mathfrak{H}_n(\underline{a};\underline{w}) \le \mathfrak{A}_n(\underline{a};\underline{w}), \tag{HA}$$

or equivalently

$$\left(\sum_{i=1}^{n} w_i a_i\right)\left(\sum_{i=1}^{n} \frac{w_i}{a_i}\right) \ge W_n^2, \tag{10}$$

with equality in either inequality if and only if $\underline{a}$ is constant.

□ The proof is by induction, the case $n = 1$ being trivial. The left-hand side of (10) is equal to

$$\begin{aligned}
&\left(\sum_{i=1}^{n-1} w_i a_i + w_n a_n\right)\left(\sum_{i=1}^{n-1} \frac{w_i}{a_i} + \frac{w_n}{a_n}\right)\\
&\qquad \ge W_{n-1}^2 + w_n \sum_{i=1}^{n-1} w_i\left(\frac{a_n}{a_i} + \frac{a_i}{a_n}\right) + w_n^2, \text{ by the induction hypothesis,}\\
&\qquad \ge W_{n-1}^2 + 2w_n W_{n-1} + w_n^2, \text{ by I 2.2 (20),}\\
&\qquad = W_n^2.
\end{aligned}$$

The case of equality is easily obtained. □

For another proof of (10) see III 2.2 Example(ii).

Now we proceed to Walsh's proof of the equal weight case of (GA) using (10).

□ Assume that $\underline{a}$ is not constant.

It is easily seen that

$$n\sum_{i=1}^{n} a_i^n - \left(\sum_{i=1}^{n} a_i^{n-1}\right)\left(\sum_{i=1}^{n} a_i\right)$$
$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} (a_j^{n-1} - a_i^{n-1})(a_j - a_i) > 0.$$

Hence

$$\frac{\sum_{i=1}^n a_i^n}{\sum_{i=1}^n a_i^{n-1}} > \frac{1}{n}\sum_{i=1}^{n} a_i; \tag{12}$$

and by the induction hypothesis

$$\sum_{\substack{i=1\\ i\neq k}}^{n} a_i^{n-1} \geq (n-1)\prod_{\substack{i=1\\ i\neq k}}^{n} a_i,\ 1 \leq k \leq n,$$

with at most one of these inequalities not being strict.
Adding these n inequalities and dividing by n gives

$$\sum_{i=1}^{n} a_i^{n-1} > \left(\prod_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right). \tag{10}$$

Multiplying (11) and (12) gives

$$\sum_{i=1}^{n} a_i^n > \frac{1}{n}\left(\prod_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right)$$
$$> n\prod_{i=1}^{n} a_i, \text{ by the equal weight case of (10).}$$

This completes the proof of the equal weight case of (GA), and the case of equality is easily obtained from that in (10).
[*Walsh*].

(xx) NANJUNDIAH 1952

□ For given n-tuples $\underline{a}, \underline{w}$ define the n-tuples $\underline{b}, \underline{c}$ as follows:

$$b_i = \frac{W_i}{w_i}a_i - \frac{W_{i-1}}{w_i}a_{i-1},\ c_i = \frac{a_i^{W_i/w_i}}{a_{i-1}^{W_{i-1}/w_i}},\ 1 \leq i \leq n,$$

where $w_0 = 0$ and $a_0 = 1$.
Then by the symmetric form of ($\sim$B), I 2.1(3), $\underline{c} \geq \underline{b}$, with equality only if $\underline{a}$ is constant. Easy calculations show that $\mathfrak{A}_i(\underline{b}; \underline{w}) = a_i = \mathfrak{G}_i(\underline{c}; \underline{w}),\ 1 \leq i \leq n$. Hence, using the monotonicity of the arithmetic mean,

$$\mathfrak{A}_n(\underline{c}; \underline{w}) - \mathfrak{G}_n(\underline{c}; \underline{w}) \geq \mathfrak{A}_n(\underline{b}; \underline{w}) - \mathfrak{G}_n(\underline{c}; \underline{w}) = 0.$$

This completes the proof since clearly we can define $\underline{a}$ from $\underline{c}$, so $\underline{c}$ can be an arbitrary n-tuple; the case of equality is immediate. □

[*Nanjundiah 1952*].

REMARK (ii) This method is the basis of other important inequalities; see 3.1 Theorem proof (v) and 3.4.

(xxi) JACOBSTHAL 1952

This is a simple inductive proof that uses the polynomial in I 1.2(a); the same method is used later for a stronger result; see 3.1 Theorem 1, proof (i).

□

$$\begin{aligned}
\mathfrak{A}_n(\underline{a}) =& \frac{\mathfrak{G}_{n-1}(\underline{a})}{n}\left(\left(\frac{\mathfrak{G}_n(\underline{a})}{\mathfrak{G}_{n-1}(\underline{a})}\right)^n + (n-1)\frac{\mathfrak{A}_{n-1}(\underline{a})}{\mathfrak{G}_{n-1}(\underline{a})}\right)\\
\ge& \frac{\mathfrak{G}_{n-1}(\underline{a})}{n}\left(\left(\frac{\mathfrak{G}_n(\underline{a})}{\mathfrak{G}_{n-1}(\underline{a})}\right)^n + n-1\right), \text{ by the induction hypothesis,}\\
\ge& \frac{\mathfrak{G}_{n-1}(\underline{a})}{n}\left(n\frac{\mathfrak{G}_n(\underline{a})}{\mathfrak{G}_{n-1}(\underline{a})}\right), \text{ by I 1.2(7),}\\
=& \mathfrak{G}_n(\underline{a}).
\end{aligned}$$

□

[*Climescu; Jacobsthal*].

($xxii$) DEVIDÉ 1956

□ If $\underline{a}$ is a non-constant n-tuple define the n-tuple $\underline{b}$ by $a_i + b_i = \mathfrak{A}_n(\underline{a})$, when $B_n = 0$. The following identities are easily deduced.

$$B_r(b_{r+1} + \cdots + b_n) = -B_r^2,\ 1 \le r \le n-1, \tag{13}$$

and order $\underline{a}$ to obtain the following properties:

$$B_n = 0; \quad b_1 b_2 < 0; \quad B_r b_{r+1} \le 0,\ 2 \le r \le n-1. \tag{14}$$

This is possible since from (13) not all $B_r b_j$, $r+1 \le j \le n$, can be positive. The following identities can also be checked:

$$B_r b_{r+1} = \Big(\mathfrak{A}_n(\underline{a}) - B_r\Big)a_{r+1} - \mathfrak{A}_n(\underline{a})\Big(\mathfrak{A}_n(\underline{a}) - B_{r+1}\Big),\ 1 \le r \le n-1,$$

and so

$$B_r b_{r+1} \mathfrak{A}_n^{r-1}(\underline{a}) \prod_{i=r+2}^{n+1} a_i = C_r - C_{r+1}\ 1 \le r \le n-1,$$

where

$$a_{n+1} = 1, \quad \text{and} \quad C_r = \mathfrak{A}_n^{r-1}(\underline{a})\big(\mathfrak{A}_n(\underline{a}) - B_r\big) \prod_{i=r+1}^{n+1} a_i,\ 1 \le r \le n-1.$$

Adding over these identities gives

$$\mathfrak{G}_n^n(\underline{a}) - \mathfrak{A}_n^n(\underline{a}) = \sum_{r=1}^{n-1} B_r b_{r+1} \mathfrak{A}_n^{r-1}(\underline{a}) \prod_{i=r+2}^{n+1} a_i;$$

and the right-hand side is negative by (14). □

[*Devidé*].

Remark (iii) This last identity should be compared with that in 2.4.1 proof (v).

Remark (iv) A n-tuple $\underline{b}$ is the same as $\underline{x}$ used in 2.4.5 proof ($xxxviii$).

($xxiii$) Blanuša 1956

If $\underline{a}$ is an increasing n-tuple then (GA) is equivalent to

$$\Big(\sum_{i=0}^{n-1}(n-i)\tilde{\Delta} a_i\Big)^n \ge n^n \prod_{i=0}^{n-1}\Big(\sum_{j=0}^{i}\tilde{\Delta} a_i\Big), \tag{15}$$

where $\tilde{\Delta} a_0 = a_1$.

The case $n = 1$ of this inequality is trivial. So assume (15) holds as stated, with $n > 1$.

Then, since

$$\Big(\sum_{i=0}^{n-1}(n+1)(n-i)\tilde{\Delta} a_i\Big)^n \sum_{i=0}^{n}(n+1)n\tilde{\Delta} a_i \ge n^{n+1}(n+1)^{n+1}\prod_{i=0}^{n}\Big(\sum_{j=0}^{i}\tilde{\Delta} a_i\Big),$$

it suffices, for the induction, that is to show (19) holds with n replaced by $(n+1)$, to prove that

$$\Big(\sum_{i=0}^{n-1}(n+1)(n-i)\tilde{\Delta} a_i\Big)^n \sum_{i=0}^{n}(n+1)n\tilde{\Delta} a_i \le \Big(\sum_{i=0}^{n} n(n+1-i)\tilde{\Delta} a_i\Big)^{n+1}. \tag{16}$$

Put $\beta = \sum_{i=0}^{n-1}(n+1)(n-i)\tilde{\Delta} a_i$, $\gamma = \sum_{i=0}^{n} i\tilde{\Delta} a_i$ and

$$\alpha = \alpha(j) = \sum_{i=0}^{n}(n+1)n\tilde{\Delta} a_i - j\gamma,\ 0 \le j \le n.$$

Then noting that $(\alpha-\gamma)(\beta+\gamma)\ge\alpha\beta$, we get

$$\beta+\gamma\ge\frac{\alpha(j)}{\alpha(j+1)}\beta,\ 0\le j\le n-1.$$

Multiplying these inequalities gives

$$\alpha(n)(\beta+\gamma)^n\ge\alpha(0)\beta^n,$$

which is (16). □

[*Blanuša*].

(*xxiv*) BELLMAN 1957

This proof uses the methods of dynamic programming.

□ Consider the problem of maximizing $\prod_{i=1}^n a_i$ subject to the condition that $\sum_{i=1}^n a_i=a$; see above 2.3 Lemma 8(b). Let this maximum be $g(n;a)$.
If we pick $a_n\ge0$, it remains to obtain the $a_i\ge0, 1\le i\le n-1$, that maximize $\prod_{i=1}^{n-1}a_i$ subject to the condition $\sum_{i=1}^{n-1}a_i=a-a_n$. Then $g(1;a)=a$ and if $n\ge2$ $g(n;\underline{a})=\max_{0\le a_n\le a}\{a_ng(n-1;a-a_n)\}$. Putting $a_i=ab_i, 1\le i\le n$, leads to $g(n;a)=a^ng(n;1)$; and

$$g(n;1)=g(n-1;1)\max_{0\le y\le1}\{y(1-y)^{n-1}\}=\frac{(n-1)^{n-1}}{n^n}g(n-1;1).$$

Hence, using $g(1;1)=1$ we get that $g(n;a)=\big(a/n\big)^n$, which completes the proof. □

[*MPF pp.697–708; BB p.6*], [*Bellman 1957; Mitrinović & Vasić p.25*], [*Bellman 1957a,b; Dubeau 1990b,c; Iwamoto; Wang C L 1979a–d, 1980a, 1981a,b, 1982, 1984a*].

(*xxv*) MITRINOVIĆ 1958

□ Let $\underline{a}$ be an $(n+1)$-tuple and assume (GA) for integers less than $n+1$. Put

$$A=\frac{a_{n+1}+(n-1)\mathfrak{A}_{n+1}(\underline{a})}{n},\quad G=\Big(a_{n+1}\mathfrak{A}_{n+1}^{n-1}(\underline{a})\Big)^{1/n};$$

then by the induction hypothesis, $A\ge G$. Further

$$\begin{aligned}\mathfrak{A}_{n+1}(\underline{a})=&\frac{A+\mathfrak{A}_n(\underline{a})}{2}\ge\sqrt{A\mathfrak{A}_n(\underline{a})},\text{ by the induction hypothesis,}\\ \ge&\sqrt{G\mathfrak{G}_n(\underline{a})},\text{ by the above observation, and the induction hypothesis,}\\ =&\Big(\mathfrak{G}_{n+1}^{n+1}(\underline{a})\mathfrak{A}_{n+1}^{n-1}(\underline{a};\underline{w})\Big)^{1/2n}.\end{aligned}$$

On rearranging this completes the proof. The case of equality is immediate. □

[*Mitrinović 1964 pp.232–233*].

(*xxvi*) CLIMESCU 1958

□ Put $x = a/b$ in I 1.2(7) to get:

$$nb^{n+1} + a^{n+1} \geq (n+1)b^n a, \tag{17}$$

with equality if and only if $a = b$. Now let $\underline{a}$ be an $(n+1)$-tuple, then

$$\begin{aligned}\mathfrak{G}_{n+1}(\underline{a}^{n+1}) =&\mathfrak{G}_n^n(\underline{a})a_{n+1}\\ \leq&\frac{n}{n+1}\mathfrak{G}_n^{n+1}(\underline{a}) + \frac{1}{n+1}a_{n+1}^{n+1}, \text{ by (17)},\\ =&\frac{n}{n+1}\mathfrak{G}_n(\underline{a}^{n+1}) + \frac{1}{n+1}a_{n+1}^{n+1}\\ \leq&\frac{n}{n+1}\mathfrak{A}_n(\underline{a}^{n+1}) + \frac{1}{n+1}a_{n+1}^{n+1}, \text{ by the induction hypothesis},\\ =&\mathfrak{A}_{n+1}(\underline{a}^{n+1}).\end{aligned}$$

The case of equality follows easily. □

[*Climescu*].

(*xxvii*) NEWMAN 1960

This is a proof of 2.3 Lemma 8(a).

□ Let $\underline{a}$ be a non-constant $(n+1)$-tuple with $\prod_{i=1}^{n+1} a_i = 1$. Since not all the elements of $\underline{a}$ are equal, we may suppose without loss in generality that $a_{n+1} \neq 1$. Then ,

$$\begin{aligned}\sum_{i=1}^{n+1} a_i \geq& n\Big(\prod_{i=1}^{n} a_i\Big)^{1/n} + a_{n+1}, \text{ by the induction hypothesis},\\ =&\frac{n}{a_{n+1}^{1/n}} + a_{n+1}\\ >&n+1, \text{ by } (\sim\text{B}) \text{ with } \alpha = -1/n \text{ and } 1+x = a_{n+1}.\end{aligned}$$

□

[*Newman D J*].

(*xxviii*) DIANANDA 1960

This is a modification of Cauchy's proof, 2.4.1 proof (*ii*).

□

$$\begin{aligned}\mathfrak{G}_n(\underline{a}) &= \sqrt{\mathfrak{G}_{n-1}(\underline{a})a_n^{1/n-1}\mathfrak{G}_n^{n-2/n-1}(\underline{a})}\\ &\le \frac{1}{2}\Big(\mathfrak{G}_{n-1}(\underline{a}) + a_n^{1/n-1}\mathfrak{G}_n^{n-2/n-1}(\underline{a})\Big), \text{ by 2.2.1 (2), } \big(n=2 \text{ equal weight (GA)}\big),\\ &\le \frac{1}{2}\Big(\mathfrak{A}_{n-1}(\underline{a}) + \frac{a_n}{n-1} + \frac{n-2}{n-1}\mathfrak{G}_n(\underline{a})\Big), \text{ by 2.2.2 (5), } \big(n=2 \text{ (GA)}\big),\\ &\qquad\qquad \text{and the induction hypothesis,}\\ &= \frac{n}{2(n-1)}\mathfrak{A}_n(\underline{a}) + \frac{n-2}{2(n-1)}\mathfrak{G}_n(\underline{a}),\end{aligned}$$

which on simplification gives the result. □

[*Diananda 1960*].

(*xxix*) KOROVKIN 1961

In Korovkin's book I 2.2 Lemma 6 is proved using (GA). We use the lemma to give a proof of (GA) in the form of 2.3 Lemma 8(a).

□ Let $\underline{a}$ be an n-tuple with $\prod_{i=1}^n a_i = 1$ and set $x_k = \prod_{i=k}^n a_i$, $1 \le k \le n$. Then the left-hand side of I 2.2 (24) is just $\sum_{i=1}^n a_i$ and so by that inequality $\sum_{i=1}^n a_i \ge n$ as had to be proved. □

[*Korovkin p.8*].

(*xxx*) MOHR 1964

□ Let $\underline{a}$ be a non-constant increasing n-tuple and first we introduce some notation: $\underline{a}^{(0)} = \underline{a}$; $\underline{a}^{(1)} = \big(\underline{a}^{(0)}\big)^{(1)} = (a_1^{(1)}, \ldots, a_n^{(1)})$, where $a_i^{(1)} = \mathfrak{A}_{n-1}(\underline{a}_i')$, where $\underline{a}_i'$ is defined in Notations 6(v); and having defined $\underline{a}^{(0)}, \ldots \underline{a}^{(k-1)}$ then $\underline{a}^{(k)} = \left(\underline{a}^{(k-1)}\right)^{(1)}$.

In other words the terms of $\underline{a}^{(k)}$ are the arithmetic means of the terms of $\underline{a}^{(k-1)}$ taken $(n-1)$ at a time.

The following identities can easily be established:

$$\begin{aligned}\mathfrak{A}_n(\underline{a}^{(k)}) &= \mathfrak{A}_n(\underline{a}^{(0)}),\ k = 0, 1, \ldots;\\ a_i^{(k)} &= \mathfrak{A}_n(\underline{a}^{(0)}) + (-1)^{k-1}\frac{\mathfrak{A}_n(\underline{a}^{(0)}) - a_i}{(n-1)^k},\ 1 \le i \le n,\ k = 1, 2, \ldots,\\ &\le \mathfrak{A}_n(\underline{a}^{(0)}) + \frac{a_n - a_1}{(n-1)^k},\ k = 1, 2, \ldots, \text{ by internality.}\end{aligned}$$

Again by internality,

$$\mathfrak{G}_n(\underline{a}^{(k)}) \le \mathfrak{A}_n(\underline{a}^{(0)}) + \frac{a_n - a_1}{(n-1)^k},\ k = 1, 2, \ldots. \tag{18}$$

Now assume (GA) for all integers less than n, and we have immediately that

$$\begin{aligned} a_i^{(k)} &\geq \mathfrak{G}_{n-1}\big((\underline{a}^{(k-1)})_i'\big) \\ &= \Big(\mathfrak{G}_n(\underline{a}^{(k-1)})\Big)^{n/n-1}\big(a_i^{(k-1)}\big)^{-1/(n-1)}, \ k=1,2,\ldots. \end{aligned}$$

On multiplying these inequalities we obtain

$$\mathfrak{G}_n(\underline{a}^{(k)}) \geq \mathfrak{G}_n(\underline{a}^{(k-1)}), \ k=1,2,\ldots. \tag{19}$$

In addition since $a_1 \neq a_n$,

$$\mathfrak{G}_n(\underline{a}^{(1)}) > \mathfrak{G}_n(\underline{a}^{(0)}).$$

From (18) and (19),

$$\mathfrak{G}_n(\underline{a}^{(m)}) \leq \mathfrak{A}_n(\underline{a}^{(0)}) + \frac{a_n - a_1}{(n-1)^k}, \ k = m, m+1, \ldots. \tag{20}$$

Using (18), choose a k so that

$$\mathfrak{G}_n(\underline{a}^{(1)}) - \frac{a_n - a_1}{(n-1)^k} > \mathfrak{G}_n(\underline{a}^{(0)}).$$

This last inequality and (20) with $m = 1$ completes the proof. □

[*Mohr 1964*].

Remark (v) In fact this proves a little more:

$$\mathfrak{G}_n(\underline{a}^{(m)}) \leq \mathfrak{A}_n(\underline{a}), \ m = 0, 1, \ldots$$

(*xxxi*) Beckenbach & Bellman 1965

This is another calculus proof of 2.3 Lemma 10(b).

□ The object is to find the minimum of the function $\sum_{i=1}^n a_i$ on the compact set $\{\underline{a}; \underline{a} \geq 0, \prod_{i=1}^n a_i = 1\}$.

Using the Lagrange multiplier[11] approach, consider $f(\underline{a}) = \prod_{i=1}^n a_i - \lambda \sum_{i=1}^n a_i$, when $\dfrac{\partial f}{\partial a_j} = \prod_{\substack{i=1 \\ i \neq j}} a_i - \lambda, 1 \leq j \leq n$.

So if $\dfrac{\partial f}{\partial a_1} = \cdots = \dfrac{\partial f}{\partial a_n}$ we must have $a_i = \cdots = a_n$.

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[11] If f and g, are functions of n variables, and we wish to find the extrema of f subject to condition g=0 the auxiliary function $\tilde{f}(\underline{x},\lambda)=f(\underline{x})+\lambda g(\underline{x})$ of $(n+1)$ variables is introduced and the problem reduces to finding the turning points of $\tilde{f}$. The extra variable λ is called a *Lagrange multiplier* and the procedure is called the *method of Lagrange multipliers*; [*CE p.1015; EM5 p. 336*].

This gives n as the unique minimum of $\sum_{i=1}^{n} a_i$ and proves 2.3 Lemma 10(b). $\square$

[BB p.5], *[Amir-Moéz; Rodenberg]*.

2.4.4 PROOFS PUBLISHED BETWEEN 1966 AND 1970. PROOFS *(xxxii)*–*(xxxvii)*

(xxxii) DZYADYK 1966

$\square$ The following two identities are given by Dzyadyk:

$$\frac{n\mathfrak{G}_n(\underline{a}) + a_{n+1}}{n+1} = \mathfrak{G}_{n+1}(\underline{a}) + \frac{\mathfrak{G}_n(\underline{a})}{n+1}\, p_n\left(\frac{\mathfrak{G}_{n+1}(\underline{a})}{\mathfrak{G}_n(\underline{a})}\right); \tag{21}$$

$$\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a}) = \frac{1}{n}\sum_{k=1}^{n} \mathfrak{G}_{k-1}(\underline{a})\, p_{k-1}\left(\frac{\mathfrak{G}_k(\underline{a})}{\mathfrak{G}_{k-1}(\underline{a})}\right), \tag{22}$$

where p_n is the polynomial of I 1.2(a) .

By I 1.2(6) the identity (21) implies that $\dfrac{n\mathfrak{G}_n(\underline{a}) + a_{n+1}}{n+1} \ge \mathfrak{G}_{n+1}(\underline{a})$ and, by the induction hypothesis, the left-hand side of this expression is not greater than $\dfrac{n\mathfrak{A}_n(\underline{a}) + a_{n+1}}{n+1} = \mathfrak{A}_{n+1}(\underline{a})$. This proves (GA), and for equality we must have $a_1 = \cdots = a_n$, by the induction hypothesis, and $\mathfrak{G}_n(\underline{a}) = \mathfrak{G}_{n+1}(\underline{a})$, by I 1.2(7). This implies that $a_1 = \cdots = a_{n+1}$.

Identity (22), again using I 1.2(7), immediately implies (GA) and the case of equality. $\square$

[Dzyadyk].

(xxxiii) GUHA 1967

Guha uses the following lemma to give a simple proof of (GA).

LEMMA 14 *If $p \ge q \ge 0$, $x \ge y \ge 0$ then*

$$(px + y + a)(x + qy + a) \ge \big((p+1)x + a\big)\big((q+1)y + a\big),$$

with equality if and only if $x = y$.

$\square$ This is an immediate consequence of the identity

$$(px + y + a)(x + qy + a) - \big((p+1)x + a\big)\big((q+1)y + a\big) = (px - qy)(x - y).$$

$\square$

□ Repeatedly using Lemma 14 gives:

$$\begin{aligned}\Big(n\mathfrak{A}_n(\underline{a})\Big)^n &= \overbrace{(a_1+\cdots+a_n)\ldots\ldots(a_1+\cdots+a_n)}^{n\text{ factors}}\\ &\geq(2a_1+a_3\cdots+a_n)(2a_2+a_3+\cdots+a_n)\overbrace{(a_1+\cdots+a_n)\ldots\ldots(a_1+\cdots+a_n)}^{n-2\text{ factors}}\\ &\geq\ldots\ldots\\ &\geq na_1(2a_2+a_3+\cdots+a_n)(a_2+2a_3+\cdots+a_n)\ldots\ldots(a_2+\cdots+a_{n-1}+2a_n)\\ &\geq\cdots\cdots\geq\overbrace{(na_1)\ldots(na_n)}^{n\text{ factors}}=n^n\mathfrak{G}_n(\underline{a})^n.\end{aligned}$$

The case of equality follows from that of the lemma. □

[*Guha*].

(*xxxiv*) Gaines 1967

The following result is well-known, [*DI p.75*].

Lemma 15 [Schur] *If $\lambda_i, 1\leq i\leq n$, are the eigenvalues of the complex matrix $A=(a_{ij})_{1\leq i,j\leq n}$ then $\sum_{i=1}^n|\lambda_i|^2\leq\sum_{i,j=1}^n|a_{ij}|^2$ with equality if and only if $A^*A=AA^*$.*

□ If

$$A=\begin{pmatrix}0 & a_1 & 0 & \cdots & 0\\ 0 & 0 & a_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & a_{n-1}\\ a_n & 0 & 0 & \cdots & 0\end{pmatrix}$$

the eigenvalues are all equal to $\mathfrak{G}_n(\underline{a})$ and so by Lemma 15, $\sum_{i=1}^n a_i^2\geq n\mathfrak{G}_n(\underline{a}^2)$, which is equivalent to (GA). □

[*Gaines*].

(*xxxv*) O'Shea 1968

O'Shea gives the following lemma.

Lemma 16 *Let $\underline{a}$ be a non-constant decreasing n-tuple and if $1\leq m\leq n$ define $a_{n+m}=a_n, 1\leq m\leq n$. Then*

$$\sum_{i=1}^n\Big(a_i^m-\prod_{j=1}^m a_{i+j}\Big)\geq 0; \tag{23}$$

further if $m>1$ this inequality is strict.

□ We first remark that if $m<n$ the last term of the sum in (23) is negative. Further if any term (23) is negative so is the succeeding term.

To see this suppose that for some $i_0 < n$ and $m < n$, $a_{i_0}^m - \prod_{j=1}^m a_{i_0+j} < 0$. Then obviously $i_0 + m > n$, but also since $a_1 \ge \cdots \ge a_n$, $a_1 > a_n$ we must have that $i_0 + m < 2n$. This implies that $a_{i_0+m+1} = a_{i_0+m+1-n}$. Hence since $n \ge i_0 + 1 > i_0 + 1 + m - n$,

$$\frac{\prod_{j=1}^m a_{i_0+1+j}}{\prod_{j=1}^m a_{i_0+j}} = \frac{a_{i_0+1+m}}{a_{i_0+1}} = \frac{a_{i_0+m+1-n}}{a_{i_0+1}} \ge 1.$$

So

$$a_{i_0+1}^m \le a_{i_0}^m < \prod_{j=1}^m a_{i_0+j} \le \prod_{j=1}^m a_{i_0+1+j},$$

or

$$a_{i_0+1}^m - \prod_{j=1}^m a_{i_0+1+j} < 0.$$

The proof of the lemma is by induction on m and it is obvious that (23) holds with equality if $m = 1$.

Let r-th term be the first negative term when we can write (23) as

$$\sum_{i=1}^{r-1}\Big(a_i^m - \prod_{j=1}^m a_{i+j}\Big) \ge \sum_{i=r}^{n}\Big(\prod_{j=1}^m a_{i+j} - a_i^m\Big). \tag{24}$$

where all the terms in both sums are non-negative.

Note that $a_r \le a_i, i < r$, $a_r \ge a_i, i > r$ with at least one of these inequalities being strict.

Suppose that (24) holds for some $m, 1 \le m < n$, and multiply this inequality by a_r to get, using the above remark,

$$\sum_{i=1}^{r-1} a_i\Big(a_i^m - \prod_{j=1}^m a_{i+j}\Big) > \sum_{i=r}^{n} a_i\Big(\prod_{j=1}^m a_{i+j} - a_i^m\Big),$$

which is just (23) with m replaced by $(m + 1)$. This completes the proof of the lemma. □

□ To prove (GA) using Lemma 16 note that is just the case $m = n$ of that lemma. □

[*O'Shea*].

(*xxxvi*) DINGHAS 1968

In various papers Dinghas has obtained several identities from which (GA) follows. Some are listed below; for proofs the reader is referred to the original papers.

$$(a) \qquad n\Big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\Big) = \sum_{i=2}^{n}\Big(a_i^{1/i} - \mathfrak{G}_i(\underline{a}^{1/i})\Big)^2 P_{i-2}\big(a_i^{1/i}, \mathfrak{G}_{i-1}(\underline{a}^{1/i})\big),$$

where $P_0(x,y)=1, \quad P_{i-2}(x,y)=\sum_{j=2}^{i}(j-1)x^{i-j}y^{j-2},\ 3\le i\le n.$

$$(b)\quad \mathfrak{A}_n^n(\underline{a})-\mathfrak{G}_n^n(\underline{a})=\sum_{i=2}^{n}\left(\frac{\prod_{j=i+1}^{n}a_j}{i^2}\right)\Big(a_i-\mathfrak{A}_{i-1}(\underline{a})\Big)^2 P_{i-2}\big(\mathfrak{A}_i(\underline{a}),\mathfrak{A}_{i-1}(\underline{a})\big),$$

where P_{i-2} is as in (a).

$$(c)\qquad \frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{G}_n(\underline{a};\underline{w})}=\exp\Big(\sum_{i,j=1}^{n}w_iw_j(a_i-a_j)^2F\big(a_i,a_j,\mathfrak{A}_n(\underline{a};\underline{w})\big)\Big),$$

where $W_n=1$, and $F(x,y,z)=\displaystyle\int_0^\infty\frac{1}{(x+u)(y+u)(z+u)}\,\mathrm{d}u.$

$$(d)\qquad \frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{G}_n(\underline{a};\underline{w})}=\exp\Big(\sum_{i=1}^{n}w_i\big(a_i-\mathfrak{A}_n(\underline{a};\underline{w})\big)^2J\big(a_i,\mathfrak{A}_n(\underline{a};\underline{w})\big)\Big),$$

where $W_n=1$ and $J(x,y)=\displaystyle\int_0^\infty\frac{u}{(1+u)(x+y+u)^2}\,\mathrm{d}u.$

[*Dinghas, 1943/44, 1948, 1953, 1962/63, 1963, 1966, 1968*].

(*xxxvii*) MITRINOVIĆ & VASIĆ 1968

This is a very simple proof of 2.3 Lemma 8(a) that can easily be modified to give a direct proof of the general weight case.

□ Let $\underline{a}$ be as in 2.3 Lemma 8(a), then by right-hand inequality in I 2.2(9),

$$\sum_{i=1}^{n}\log a_i-\sum_{i=1}^{n}a_i+n\le 0,\quad\text{or}\quad \sum_{i=1}^{n}a_i\ge n+\log\Big(\prod_{i=1}^{n}a_i\Big)=n.$$

□

[*Mitrinović & Vasić 1968, p.27*].

2.4.5 PROOFS PUBLISHED BETWEEN 1971 AND 1988 PROOFS (*xxxviii*)–(*lxii*)

(*xxxviii*) YUŽAKOV 1971

□ Let $\underline{a}$ be a non-constant n-tuple with $A=\mathfrak{A}_n(\underline{a})$, and $x_i=a_i-A, 1\le i\le n$, when $\sum_{i=1}^{n}x_i=0$. Assume without loss in generality that $x_2<0<x_1$; then $a_1a_2=(A+x_1)(A+x_2)<A(A+x_1+x_2)$. Hence, assuming (GA) for positive integers less than n,

$$\big((A+x_1+x_2)a_3\dots a_n\big)^{1/(n-1)}\le\frac{(A+x_1+x_2)+a_3+\cdots+a_n}{n-1}=A$$

which, on using the above inequality

$$A > \Big(\frac{a_1a_2}{A}a_3\ldots a_n\Big)^{1/(n-1)},$$

which completes the proof of (GA). □

[*Yužakov*].

REMARK (i) Quantities similar to the x_i are used in 2.4.3 proof (*xxii*).

(*xxxix*) SEGRE

See VI 4.5 Example (iii).

[*Segre*].

(*xl*) CHONG K M 1976

This is based on a result of Chong K M I 3.3 Corollary 17.

□ Assume that $\prod_{i=1}^{n} a_i = 1$ and put $a_i = \dfrac{\prod_{j=1}^{n} a_j}{\prod_{j=i+1}^{n} a_j} = \dfrac{c_i'}{c_i}, 1 \le j \le n-1$, and $a_n = \dfrac{a_n}{\prod_{i=1}^{n} a_i} = \dfrac{c_n'}{c_n}$. Then $\underline{c}'$ is a rearrangement of $\underline{c}$, and so by the I 3.3 Corollary 17,

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \frac{c_i'}{c_i} \ge n.$$

[*Chong K M 1976c*].

(*xli*) CHONG K M 1976

This proof is based on I 3.3 Theorem 18, and the following lemma of Chong K M.

LEMMA 17 *If $\underline{a}$ is an n-tuple and $\underline{A} = \big(\mathfrak{A}_n(\underline{a}), \mathfrak{A}_n(\underline{a}), \ldots, \mathfrak{A}_n(\underline{a})\big)$ then $\underline{A} \prec \underline{a}$.*

□ Assume, without loss in generality, that $\underline{a}$ is decreasing, when if $1 \le k \le n-1$ $k\sum_{i=k+1}^{n} a_i \le (n-k)\sum_{i-1}^{k} a_i$, or

$$k\mathfrak{A}_n(\underline{a}) \le \sum_{i=1}^{k} a_i, \ 1 \le k \le n-1.$$

Since of course $n\mathfrak{A}_n(\underline{a}) = \sum_{i=1}^{n} a_i$, we get that $\underline{A} \prec \underline{a}$. □

□ Now by Lemma 17 and I 3.3 Theorem 18

$$\prod_{i=1}^{n} \mathfrak{A}_n(\underline{a}) = \mathfrak{A}_n^n(\underline{a};\underline{w}) \ge \prod_{i=1}^{n} a_i,$$

which is just (GA). □

[*Chong K M 1976c, 1979*].

(*xlii*) MYERS 1976

This is based on the idea in 2.4.2 proof (x).

LEMMA 18 *If $\underline{x}$ is a real n-tuple with $\sum_{i=1}^{n} x_i = 0$ then $1 \geq \prod_{i=1}^{n}(1+x_i)$ with equality if and only if $\underline{x}$ is constant.*

□ If $n = 2$ then $x_2 = -x_1$ and so $(1+x_1)(1+x_2) = 1 - x_1^2 \leq 1$, with equality if and only if $x_1 = x_2$.
Now assume the lemma has been proved for all integers less than n, $n > 2$ and that $\underline{x}$ is not constant. Then not all $x_i, 1 \leq i \leq n$, are zero, and assume without loss in generality that $x_{n-1} < 0 < x_n$. Then,

$$\begin{aligned}\prod_{i=1}^{n}(1+x_i) \leq &(1+x_{n-1}+x_n)\prod_{i=1}^{n-2}(1+x_i), \text{ since } x_{n-1}x_n < 0,\\ \leq &1, \text{ by the induction hypothesis.}\end{aligned}$$

This completes the proof of the lemma. □

□ Now let $\underline{a}$ be a non-constant n-tuple, with $\underline{b}$ defined as in 2.4.2 proof (x), that is $a_i = (1+b_i)\mathfrak{A}_n(\underline{a}), 1 \leq i \leq n$, when, as in the earlier proof, $\sum_{i=1}^{n} b_i = 0$, and not all of the terms of $\underline{b}$ are zero. Then

$$\begin{aligned}\mathfrak{G}_n(\underline{a}) = &\mathfrak{G}_n(1+\underline{b})\mathfrak{A}_n(\underline{a})\\ < &\mathfrak{A}_n(\underline{a}), \quad \text{by Lemma 18.}\end{aligned}$$

□

[*Myers*].

REMARK (ii) A simple proof Lemma 18 can be based on I 2.2(8). The same method will prove: $\displaystyle\sum_{i=1}^{n} x_i y_i = 0 \Longrightarrow 1 \geq \prod_{i=1}^{n}(1+x_i)^{y_i}$, *with equality if and only if $\underline{x}$ is constant.*

(xliii) CHONG K M 1976

□ Let $\underline{a}$ be a non-constant n-tuple and assume without loss in generality that $a_1 = \min \underline{a} < a_n = \max \underline{a}$. Further assume that (GA) has been proved for integers less than n. Then

$$\begin{aligned}\mathfrak{A}_n^n(\underline{a}) = &\mathfrak{A}_n(\underline{a})\mathfrak{A}_{n-1}^{n-1}\big(a_1 + a_n - \mathfrak{A}_n(\underline{a}), a_2, \ldots, a_{n-1}\big),\\ \geq &\mathfrak{A}_n(\underline{a})\big(a_1 + a_n - \mathfrak{A}_n(\underline{a})\big)a_2 \ldots a_{n-1}, \text{ by the induction hypothesis,}\\ = &\big(a_1 a_n + (a_n - \mathfrak{A}_n(\underline{a}))(\mathfrak{A}_n(\underline{a}) - a_1)\big)a_2 \ldots a_{n-1} > a_1 a_2 \ldots a_n,\end{aligned}$$

which is (GA). □

[*Chong K M 1976b*].

REMARK (iii) Chong has given a similar proof using $\mathfrak{G}_n(\underline{a})$; [*Chong K M 1976a*].

(*xliv*) BELLMAN 1976

Bellman has given a proof of (GA) based on developing identities for $\mathfrak{A}_{2^k}(\underline{a}) - \mathfrak{G}_{2^k}(\underline{a})$ and the use of 2.2.4 Remark (1).

□ First note that

$$a_1^2 + a_2^2 = 2a_1a_2 + (a_1 - a_2)^2, \tag{25}$$

and so

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = 2(a_1a_2 + a_3a_4) + (a_1 - a_2)^2 + (a_3 - a_4)^2. \tag{26}$$

Substituting a_1^2 for a_1, a_2^2 for a_2 etc. in (26) and using (25) we get

$$a_1^4 + a_2^4 + a_3^4 + a_4^4 = 4a_1a_2a_3a_4 + 2(a_1a_2 - a_3a_4)^2 + (a_1^2 - a_2^2)^2 + (a_3^2 - a_4^2)^2. \tag{27}$$

Now add (27) to a similar identity using a_5, a_6, a_7, a_8 and replace a_1 by a_1^2 etc, to obtain a similar identity in eight variable with $a_1^8 + \cdots + a_8^8$ on the left, $8a_1a_2 \ldots a_8$ the first term on the right, and all the other terms on the right being non-negative. In this way we obtain an identity of the form

$$\sum_{i=1}^{2^k} a_i^{2^k} = 2^k \prod_{i=1}^{2^k} a_i + \text{non-negative terms},$$

from which (GA) for n-tuples with $n = 2^k$, $k = 1, 2, \ldots$ is immediate. □

[*Bellman 1976*].

(*xlv*) SCHAUMBERGER & KABAK 1977

This is a particularly simple inductive proof.

□ Clearly $\sum_{\substack{i,j=1\\ j\neq i}}^{n+1} (a_i^n - a_j^n)(a_i - a_j) \geq 0$. So,

$$\begin{aligned} n\sum_{j=1}^{n+1} a_j^{n+1} &\geq \Big(\sum_{i=1}^{n+1} a_i \sum_{\substack{j=1\\ j\neq i}}^{n+1} a_j^n\Big) \\ &\geq \Big(\sum_{i=1}^{n+1} a_i n \prod_{\substack{j=1\\ j\neq i}}^{n+1} a_j\Big), \text{ by the induction hypothesis,} \\ &= n(n+1)\prod_{j=1}^{n+1} a_j. \end{aligned}$$

This gives (GA) and the case of equality is easily obtained. □

[*Schaumberger & Kabak 1977*].

(*xlvi*) 1977[12]

□ We may suppose without loss in generality that $\mathfrak{G}_n(\underline{a};\underline{w}) = e$, and $W_n = 1$. By I 2.2(7) we have $a_i \ge e\log a_i, 1 \le i \le n$, so

$$\mathfrak{A}_n(\underline{a};\underline{w}) = \sum_{i=1}^{n} w_i a_i \ge e\log\big(\prod_{i=1}^{n} a_i^{w_i}\big) = e = \mathfrak{G}_n(\underline{a};\underline{w}).$$

The case of equality is easily obtained. □

(*xlvii*) Climescu 1977[13]

□ Assume, as we may, that the $\underline{a}$ is an $(n+1)$-tuple with $\prod_{i=1}^{n+1} a_i = 1$. If then we have proved (GA) for the integer n we have that

$$a_1 + \cdots + a_n \ge n(a_1 \ldots a_n)^{1/n} \tag{28}$$

Adding over n similar inequalities we obtain

$$a_1 + \cdots + a_{n+1} \ge \sum_{i=1}^{n+1}\Big(\prod_{\substack{j=1\\ j\ne i}}^{n+1} a_j\Big)^{1/n} = \sum_{i=1}^{n+1} a_i^{-1/n}. \tag{29}$$

Using the right-hand side of (35) as the left-hand side of (34) and repeating the argument, and then iterating this we arrive after r steps at

$$a_1 + \cdots + a_{n+1} \ge \sum_{i=1}^{n+1} a_i^{-r/n}.$$

Letting $r \to \infty$ gives $a_1 + \cdots + a_{n+1} \ge n+1$, which by 2.3 Lemma 8(b) completes the proof. □

Remark (iv) Of course this proof does not give the case of equality.

(*xlviii*) Mijalković 1977 [14]

□ Assume that $\mathfrak{G}_n(\underline{a}) = 1$, and that (GA) is known for positive integers less than n.

$$\begin{aligned}\mathfrak{A}_n(\underline{a}) &= \frac{a_n}{n} + \frac{n-1}{n}\mathfrak{A}_{n-1}(\underline{a})\\ &\ge \frac{a_n}{n} + \frac{n-1}{n}\mathfrak{G}_{n-1}(\underline{a}), \text{ by the induction hypothesis,}\\ &= \frac{a_n}{n} + \frac{n-1}{n}a_n^{-1/n-1} \ge 1, \text{ by 2.2.2 (5), } \big(n = 2 \text{ (GA)}\big).\end{aligned}$$ □

12 I have no source for this proof other than [*MI*].

13 I have no proper reference for this proof.

14 This proof is unpublished.

($xlix$) SCHAUMBERGER 1978

The following is a very simple calculus proof.

□ Put $x = a_n$ and $f(x) = n\mathfrak{G}_n(a_1, \ldots, a_{n-1}, x; \underline{w}) - w_n x$, and assume (GA) for integers less than n. Then f has a unique maximum at $x = x_0 = \mathfrak{G}_{n-1}(\underline{a}; \underline{w})$ and as a result $f(x) \le f(x_0)$. This inequality is $W_{n-1}\mathfrak{G}_{n-1}(\underline{a}; \underline{w}) \ge W_n\mathfrak{G}_n(\underline{a}; \underline{w}) - w_n a_n$, from which (GA) follows by the induction hypothesis. □

[*Anderson D 1979a; Schaumberger 1978, 1990a*].

(l) LANDSBERG 1978

The following interesting proof is based on the laws of thermodynamics. There was some criticism of this method of proving mathematical theorems, criticism that was itself objected to and the method extended; see the references and III 3.1.1 Theorem 1 proof (x)

□ Let $a_i, 1 \le i \le n$, be the temperatures of n identical heat reservoirs each having heat capacity c. Put the reservoirs in contact with each other and let them come to an equilibrium temperature A, say.
The first law of thermodynamics, conservation of energy, tells us that $A = \mathfrak{A}_n(\underline{a})$. The second law of thermodynamics implies a gain of entropy, that is to say $cn\log\bigl(A/\mathfrak{G}_n(\underline{a})\bigr) \ge 0$. This implies that $A = \mathfrak{A}_n(\underline{a}) \ge \mathfrak{G}_n(\underline{a})$; further there is equality if and only if there is zero entropy gain, if and only if all the initial temperatures are the same. □

[*Abriata; Deakin & Troup; Landsberg 1978,1980a,b,1985; Sidhu*].

(li) ZEMGALIS 1979

This is a simple inductive proof of (GA) in the form of 2.3 Lemma 8(a).

□ Assume (GA) has been proved for all integers $k, 1 \le k \le n$, and that $\prod_{i=1}^{n+1} a_i = 1$ where without loss in generality we can assume $a_n = \min \underline{a}$, $a_{n+1} = \max \underline{a}$, when $a_n \le 1 \le a_{n+1}$. Then $a_n a_{n+1} + 1 < a_n + a_{n+1}$, see 2.1 Lemma 3 proof($iv$).
Now:

$$\begin{aligned} n+1 =& n\mathfrak{G}_{n+1}(\underline{a}) + 1 = n\mathfrak{G}_n(a_1, \ldots, a_{n-1}, a_n a_{n+1}) + 1 \\ \le& (a_1 + \cdots + a_{n-1} + a_n a_{n+1}) + 1, \text{ by the induction hypothesis,} \\ <& a_1 + \cdots + a_{n+1}; \text{ by the above remark.} \end{aligned}$$

The case of equality is immediate. □

[*Herman, Kučera & Šimša pp.143–144, 151*], [*Zemgalis*].

(lii) BULLEN 1979

The proof (ix) of 2.2.2 Lemma 6, the $n = 2$, general weight case, can be extended to give a geometrical inductive proof along the lines of a proof of (J), I 4.2 Theorem 12 proof (iii). The induction is obvious from the following proof of the $n = 3$ case, where the notation is that used in the quoted proofs of earlier theorems.

□ Consider the function

$$D_3(s,t) = \mathfrak{A}_3(x,y,z;1-s-t,s,t) - \mathfrak{G}_3(x,y,z;1-s-t,s,t),$$

defined on the triangle $T = \{(s,t);\ 0 \le s \le 1,\ 0 \le t \le 1,\ 0 \le s+t \le 1\}$. The proof of (GA) in this case consists in showing that except at the corners of T, $D_3 > 0$, unless $x = y = z$. We can assume, without loss in generality, that $0 < x \le y \le z$; further, we can assume that $0 < x < y < z$; for if any two of x, y, z are equal then D_3 reduces to a case $n = 2$ of (GA) and is known to be positive unless the remaining two are equal. Thus if $y = z$, and we put $u = s + t$

$$D_3(s,t) = D(u) = \mathfrak{A}(x,y;1-u,u) - \mathfrak{G}(x,y;1-u,u),\ 0 \le u \le 1; \qquad (30)$$

and $D(u) > 0$, $u \ne 0, 1$, see proof (ix) of 2.2.2 Lemma 6.

Being continuous the minimum value of D_3 is attained somewhere on T and if it is at an interior point then it is at a point satisfying

$$\begin{aligned}\frac{\partial D_3}{\partial s} &= (y-x) - \mathfrak{G}_3 \log(y/x) = 0,\\ \frac{\partial D_3}{\partial t} &= (z-x) - \mathfrak{G}_3 \log(z/x) = 0.\end{aligned}$$

That is at some point (s_0, t_0) such that

$$\frac{y-x}{\log y - \log x} = \frac{z-x}{\log z - \log x} = \mathfrak{G}_3(x,y,z;1-s_0-t_0,s_0,t_0). \qquad (31)$$

However the logarithmic function is strictly concave so since $y < z$ the first ratio in (31) is strictly less than the second[15], I 4.1 Lemma 2; so there is no such point (s_0, t_0), the minimum of D_3 must occur on the boundary of T. However on a side of T, D_3 is of the form D in (30), and is then positive except at the end points of the side. Hence $D_3(s,t) > 0$ except at the corners of T.

In the general case suppose that we have (GA) for positive integers less than n. If $\underline{t} = (t_i, \ldots, t_{n-1})$, $\underline{w} = (1 - \sum_{i=1}^{n-1} t, \underline{t})$, and $D_n(\underline{t}) = \mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})$, defined on the simplex $T = \{\underline{t};\ 0 \le \underline{w} \le 1\}$. Then as before we can assume that $\underline{a}$ is a strictly increasing n-tuple, for otherwise we are reduced to the case $n-1$, and as

15 The quantities in the two ratios in (37) are the logarithmic means $\mathfrak{L}(x,y)$, $\mathfrak{L}(x,z)$; see 5.5, VI 2.1.1.

before D_n must attain its minimum on a face of T where it is positive except at the corners by the induction hypothesis. $\square$

[*Bullen 1979, 1980*].

(*liii*) CUSMARIU 1981

In trying to extend the geometric proof (v) of 2.2.1 Lemma 3 Cusmariu obtained the following proof of (GA).

$\square$ Consider the following $n \times n$ matrices: I the unit matrix, J the matrix with entries all 1, and S the matrix whose elements $a_{i,i+1} = 1, 1 \le i \le n-1$, $a_{n,1} = 1$, and the rest are zero. If now $V = (I+S)/2 = (v_{ij})_{1 \le i,j \le n}$, then V is doubly stochastic and so $\lim_{m\to\infty} V^m = \dfrac{1}{n}J$. Alternatively this can be obtained directly. Now if $\underline{a}$ is a non-constant n-tuple and for $m \in \mathbb{N}$ let $V^m = (v_{ij}^{(m)})_{1 \le i,j \le n}$ and define

$$f_m = f(\underline{a}, V^m) = \frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{n} a_j^{v_{ij}^{(m)}}.$$

In particular $f_0 = \mathfrak{A}_n(\underline{a})$ and $\lim_{m\to\infty} f_m = \mathfrak{G}_n(\underline{a})$.
Further

$$\begin{aligned} f_m =& \frac{1}{2}\big(f(\underline{a}, V^m) + f(\underline{a}, SV^m)\big)\\ >& \sqrt{f(\underline{a}, V^m) f(\underline{a}, SV^m)}, \text{ by 2.2.1 (2),}\big(\text{(GA)with } n=2 \text{ and equal weights}\big),\\ =& f(\underline{a}, V^{m+1}) = f_{m+1}. \end{aligned}$$

These facts imply (GA). $\square$

[*Cusmariu*].

(*liv*) VAN DER HOEK 1981

This is a simple inductive proof of 2.3 Lemma 8(a).

$\square$ Assume that $\prod_{i=1}^{n+1} a_i = 1$ and put $s = a_{n+1}^{1/n}$ when $\prod_{i=1}^{n}(sa_i) = 1$ and so by the induction hypothesis $\sum_{i=1}^{n}(sa_i) \ge n$. Hence, by I 1.2(6),

$$\sum_{i=1}^{n+1} a_i = \frac{1}{s}\sum_{i=1}^{n}(sa_i) + s^n \ge \frac{n}{s} + s^n > n+1. \qquad \square$$

[*van der Hoek*].

(*lv*) CHONG K M 1981

$\square$ Let $\underline{a}$ an n-tuple with $a_1 \le a_n \le a_{n-1} \le \cdots \le a_3 \le a_2$, $a_1 \ne a_2$, and for some r, $1 \le r \le n-1$, define $u = a_1^{1/n} a_{r+1}^{(n-1)/n}$, $v = a_1^{r/n} a_{r+1}^{1/n} \dots a_n^{1/n}$, $x = \left(\dfrac{a_{r+1}}{a_1}\right)^{1/n}$. Then $u \ge v$ and applying I 2.2 (22) we get that

$$a_{r+1} + P_{r+1} \ge a_1^{1/n} a_{r+1}^{(n-1)/n} + P_r,$$

where $P_r = a_1^{r/n} a_{r+1}^{1/n} \ldots a_n^{1/n}$, $1 \le r \le n-1$, and $A_n = a_1$; this inequality is strict when $r = 1$. Sum these inequalities over $1 \le r \le n-1$, to get

$$
\begin{aligned}
n\mathfrak{A}_n(\underline{a}) &> \mathfrak{G}_n(\underline{a}) + \sum_{r=2}^{n} a_1^{1/n} a_r^{(n-1)/n} \\
&\ge \mathfrak{G}_n(\underline{a}) + (n-1)\Big(\prod_{r=2}^{n} a_1^{1/n} a_r^{(n-1)/n}\Big)^{1/(n-1)}, \text{ by the induction hypothesis,} \\
&= n\mathfrak{G}_n(\underline{a}).
\end{aligned}
$$

This gives (GA). □

[*Chong K M 1981*].

(*lvi*) RÜTHING 1982

Rüthing has given several inductive proofs using the symmetric form of (B), the inequalities I 2.1(2).

□ All the proofs use the inductive hypothesis, $\mathfrak{G}_{n-1} \le \mathfrak{A}_{n-1}$, and an inequality obtained from one or other of the inequalities I 2.1(2) by substituting $\alpha = n$ and various expressions for a and b.

Thus using the left inequality with:
(a) $a = a_n^{1/(n-1)}$, $b = \mathfrak{A}_n^{1/(n-1)}(\underline{a})$, gives $a_n\mathfrak{A}_{n-1}^{n-1} \le \mathfrak{A}_n^n$; (b) $a = a_n^{1/(n-1)}$, $b = \mathfrak{G}_n^{1/(n-1)}(\underline{a})$, gives $\mathfrak{G}_n \le \dfrac{1}{n}\big((n-1)\mathfrak{G}_{n-1} + a_n\big)$;
and using the right inequality with:
(c) $a = \mathfrak{A}_{n-1}^{1/n}(\underline{a})$, $b = a_n^{1/n}$, gives the inequality in (a); (d) $a = \mathfrak{A}_{n-1}$, $b = \mathfrak{A}_n$, also gives the inequality in (a); (e) $a = \mathfrak{G}_{n-1}^{1/n}(\underline{a})$, $b = a_n^{1/n}$, gives the inequality in (b); (f) $a = \mathfrak{G}_{n-1}(\underline{a})$, $b = \mathfrak{G}_n(\underline{a})$ also gives the inequality in (b). □

[*Rüthing 1982*].

(*lvii*) WELLSTEIN 1982

Taking $f(\underline{a}) = \prod_{i=1}^{n} a_i$ in I 4.2 Theorem 13 gives the equal weight case of (GA).
[*Wellstein*]

(*lviii*) SCHAUMBERGER 1985

□ Assume that $W_n = 1$. Then by the right-hand inequality of I 2.2(9),

$$\frac{w_i a_i}{\mathfrak{G}_n} \ge w_i + w_i \log(a_i/\mathfrak{G}_n) = w_i + \log(a_i^{w_i}/\mathfrak{G}_n^{w_i}),\ 1 \le i \le n.$$

Summing over i gives

$$\frac{\mathfrak{A}_n}{\mathfrak{G}_n} \ge W_n + \log(\mathfrak{G}_n/(\mathfrak{G}_n) = 1.$$

The case of equality is immediate. □

[*Schaumberger 1985a, 1988*].

(*lix*) Soloviov 1986

□ By I 4.6 Example (viii) $\chi(\underline{a}) = \mathfrak{G}_n(\underline{a};\underline{w})$ is strictly concave on the cone $(\mathbb{R}^n_+)^*$, and, assuming that $W_n = 1$, $\nabla\chi(\underline{e}) = \underline{w}$.
(GA) now follows by an application of the support inequality, I 4.6 (~24), with $\underline{u} = \underline{a}, \underline{v} = \underline{e}$. □

[*Soloviov*].

(*lx*) Teng 1987

This proof of the equal weight case depends on the following lemma that is another generalization of I 2.2(20).

Lemma 19 *The equal weight case of* (GA) *is equivalent to the inequality*

$$\sum_{i=1}^{n} a_i + \frac{1}{\prod_{i=1}^{n} a_i} \ge n+1, \tag{32}$$

with equality if and only if $\underline{a} = \underline{e}$.

□ Given (GA) the left-hand side of (32) is greater than or equal to

$$(n+1)\left(\prod_{i=1}^{n} a_i\Bigl(\frac{1}{\prod_{i=1}^{n} a_i}\Bigr)\right)^{1/(n+1)} = n+1;$$

there is equality if and only if $a_1 = \cdots = a_n = \dfrac{1}{\prod_{i=1}^{n} a_i}$, that is, if and only if $a_1 = \cdots = a_n = 1$. Now given (32) and a positive real α we have that

$$\sum_{i=1}^{n-1} \frac{a_i}{\alpha} + \frac{1}{\prod_{i=1}^{n-1} \dfrac{a_i}{\alpha}} \ge n,$$

that is

$$\sum_{i=1}^{n-1} a_i + \alpha^n \frac{1}{\prod_{i=1}^{n-1} a_i} \ge \alpha n.$$

Taking $\alpha = (\prod_{i=1}^{n} a_i)^{1/n}$ in the last inequality gives (GA). The case of equality is immediate. □

□ As a result of Lemma 19 to prove (GA) we need to prove (32),which we do using a short version of Teng's proof due to Hering; [*Hering 1990*].
The case $n = 1$ is just I 2.2(20) so suppose (32) holds for n, $n \ge 1$. Then

$$\begin{aligned}\sum_{i=1}^{n+1} a_i + \frac{1}{\prod_{i=1}^{n+1} a_i} &= \sum_{i=1}^{n} a_i + \frac{1}{\prod_{i=1}^{n} a_i} + 1 + a_{n+1} - 1 + \frac{1}{\prod_{i=1}^{n} a_i}\Bigl(\frac{1}{a_{n+1}} - 1\Bigr)\\ &\ge n{+}1 + 1 + (a_{n+1} - 1)\Bigl(1 - \frac{1}{\prod_{i=1}^{n+1} a_i}\Bigr),\end{aligned} \tag{33}$$

by the induction hypothesis. Now either $\prod_{i=1}^{n+1} a_i \geq 1$ when at least one $a_i \geq 1$, or $\prod_{i=1}^{n+1} a_i < 1$ when at least one $a_i < 1$. In either case we can, without loss in generality, assume that this a_i is a_{n+1}. Hence the last term on the right-hand side of (33) is non-negative and the proof of (32) is completed. □

[*Teng*].

(*lxi*) MINASSIAN 1988

This proof of the equal weight case is a mixture of induction and the use of calculus.

□ Let $n \geq 2, \underline{a} = (a_1, \ldots, a_{n-1}, x)$, $s = a_1 + a_2 + \cdots + a_{n-1}$, $p = a_1 a_2 \ldots a_{n-1}$ and define $f(x) = n^n\big(\mathfrak{A}_n^n(\underline{a}) - \mathfrak{G}_n^n(\underline{a})\big), x > 0$. Then
$f(x) = (x+s)^n - n^n px$, $f'(x) = n(x+s)^{n-1} - n^n p$, $f''(x) = n(n-1)(x+s)^{n-2}$.
Hence f is seen to have a minimum at $x = x_0 = np^{1/(n-1)} - s$, and simple calculations give that $f(x_0) = n^n p\big(s - (n-1)p^{1/(n-1)}\big)$. Now the induction hypothesis is $s \geq (n-1)p^{1/(n-1)}$ with equality if and only if $a_1 = \cdots = a_{n-1}$— a property that clearly holds when n=2, 3. So by the induction hypothesis $f(x_0) \geq 0$, with equality if and only if $x = x_0$. □

[*Minassian*].

REMARK (v) Minassian points out that this argument allows for consideration of certain real $\underline{a}$ if (GA) is written in the form $\mathfrak{A}_n^n(\underline{a}; \underline{w}) \geq \mathfrak{G}_n^n(\underline{a}; \underline{w})$.

(*lxii*) YU 1988

□ In the right-hand inequality of I 2.2 (9) put $x = a_i/\mathfrak{G}_n(\underline{a}; \underline{w})$, and multiply by w_i, $1 \leq i \leq n$, and assume that $W_n = 1$.

This gives $w_i \log a_i/\mathfrak{G}_n(\underline{a}; \underline{w}) \leq w_i a_i - w_i, \quad 1 \leq i \leq n$. Adding over i gives (GA), and the inequality is strict unless $\underline{a}$ is constant by I 2.2 (9). □

[*Yu*].

REMARK (vi) This proof is similar to proof (*lviii*).

2.4.6 PROOFS PUBLISHED AFTER 1988. PROOFS (*lxiii*)–(*lxxiv*)

(*lxiii*) SCHAUMBERGER 1989

□ Put $x = a_i e/\mathfrak{G}_n(\underline{a}), 1 \leq i \leq n$, in I 2.2 (7) and multiply to get

$$e^{\left(\frac{e\sum_{i=1}^n a_i}{\mathfrak{G}_n(\underline{a})}\right)} \geq \left(\frac{e^n \prod_{i=1}^n a_i}{\mathfrak{G}_n^n(\underline{a})}\right)^e = e^{ne}.$$

On simplifying this gives the equal weight case of (GA). The case of equality is immediate from that of I 2.2 (7) . □

[*Schaumberger 1989*].

(lxiv) BEN-TAL, CHARNES & TEBOULLE 1989.
See VI 4.6 Remark(vii).
[*Ben-Tal, Charnes & Teboulle*].

(lxv) ALZER 1991.
See 5.5 where an interesting proof is given using Čebišev's inequality.
[*Alzer 1991*a].

(lxvi) SCHAUMBERGER & BENCZE 1993
Assume that we have the following inequality: *if $\underline{x}$ is an increasing non-negative n-tuple then*

$$x_n^n \geq x_1(2x_2 - x_1)(3x_3 - 2x_2)\cdots(nx_n - \overline{n-1}x_{n-1}), \tag{34}$$

with equality if and only if $\underline{x}$ is constant. Now, in this inequality, put $a_1 = x_1, a_2 = 2x_2 - x_1, a_3 = 3x_3 - 2x_2 \ldots$ to get the equal weight case of (GA), together with the case of equality.
To prove inequality (34) note that it is trivial if $n = 1$ and assume that it is valid for some $n \geq 1$; then

$$\begin{aligned} x_1(2x_2 - x_1)(3x_3 - 2x_2)\cdots&(nx_n - \overline{n-1}x_{n-1})\big(\overline{n+1}x_{n+1} - nx_n\big) \\ &\leq x_n^n(\overline{n+1}x_{n+1} - nx_n), \quad \text{by the induction hypothesis,} \\ &= x_n^{n+1}\Big(\overline{n+1}\frac{x_{n+1}}{x_n} - n\Big) \leq x_{n+1}^{n+1}, \quad \text{by I 1.2(7).} \end{aligned}$$

The case of equality follows from that of I 1.2(7).
An alternative proof of inequality (34) is given in 3.7 Example (i).
[*Schaumberger & Bencze 1993*].

(lxvii) ALZER 1996
□ Assume $W_n = 1$ and that $\underline{a}$ is an increasing n-tuple. Then there is a k, $1 \leq k < n$, such that $a_k \leq \mathfrak{G}_n(\underline{a}; \underline{w}) \leq a_{k+1}$. So

$$\frac{\mathfrak{A}_n(\underline{a}; \underline{w})}{\mathfrak{G}_n(\underline{a}; \underline{w})} - 1 = \sum_{i=k+1}^{n} w_i \int_{a_i}^{\mathfrak{G}_n} (t^{-1} - \mathfrak{G}_n^{-1})\,dt + \sum_{i=1}^{k} w_i \int_{\mathfrak{G}_n}^{a_i} (\mathfrak{G}_n^{-1} - t^{-1})\,dt.$$

Since both sums on the right-hand side contain only non-negative terms we get that the left-hand side is non-negative, which is (GA). Further the right-hand side is zero if and only if $a_i = \mathfrak{G}_n, 1 \leq i \leq n$, which gives the case of equality. □
[*Alzer 1996*a].

(*lxviii*) LUCHT 1995

□ Let $\underline{a}, \underline{w}$ be n-tuples, and assume that $W_n = 1$ and $\underline{a}$ is not constant; let $x > 0$, then by the right-hand inequality of I 2.2 (9) we have for each i, $1 \le i \le n$,

$$w_i \frac{a_i}{x} - w_i \ge w_i \log \frac{a_i}{x};$$

further this inequality is strict unless $a_i/x = 1$. Adding and noting that at least one $a_i/x \ne 1$, we get

$$\mathfrak{A}_n(\underline{a};\underline{w}) > x + x \log \frac{\mathfrak{G}_n(\underline{a};\underline{w})}{x}.$$

Taking $x = \mathfrak{G}_n(\underline{a};\underline{w})$ gives (GA).
[*Lucht*].

REMARK (i) This should be compared to 2.4.5 proofs (*lviii*), (*lxii*).

(*lxix*) UŠAKOV 1995

In an interesting paper Ušakov developed a unified method of proving inequalities based on the following simple lemma.

LEMMA 20 *Suppose that* $M \subseteq \mathbb{R}$ *and* $F : M \mapsto \mathbb{R}$, *where* $F(x) = \sum_{i=1}^n f_i(x)$, *then*

$$\sum_{i=1}^n \inf_{x \in M} f_i(x) \le \inf_{x \in M} F(x), \quad \sum_{i=1}^n \sup_{x \in M} f_i(x) \ge \sup_{x \in M} F(x)$$

with equality in either inequality if and only if all the extrema occur at the same value of x.

To prove (GA) assume let $\underline{a}$ and $\underline{w}$ be n-tuples with $W_n = 1$.
Let $f_i(x) = w_i a_i(x - \log x)$, $1 \le i \le n, x \in M = \mathbb{R}_+^*$ then f_i has a minimum at $x_i = 1/a_i$, $1 \le i \le n$, giving $\sum_{i=1}^n \inf_{x \in M} f_i(x) = 1 + \log\bigl(\mathfrak{G}_n(\underline{a};\underline{w})\bigr)$.
$F(x) = \mathfrak{A}_n(\underline{a};\underline{w})x - \log x$ which has a minimum at $x = 1/\mathfrak{A}_n(\underline{a};\underline{w})$ giving $\inf_{x \in M} F(x) = 1 + \log\bigl(\mathfrak{A}_n(\underline{a};\underline{w})\bigr)$. Using the lemma we get (GA) together with the case of equality.
[*Sándor & Szabó; Pečarić & Varošonec; Ušakov*].

(*lxx*) HRIMIC 2000

□ Let $\underline{a}$ be an n-tuple and define the n-tuple $\underline{b}$ by, $b_1 = \dfrac{a_1}{\mathfrak{G}_n(\underline{a})}$, $b_2 = \dfrac{a_1 a_2}{\mathfrak{G}_n^2(\underline{a})}, \ldots$ $\ldots, b_n = \dfrac{a_1 a_2 \ldots a_n}{\mathfrak{G}_n^n(\underline{a})} = 1$. A simple application of I 3.3 Corollary 17 gives,

$$n \le b_1 \frac{1}{b_n} + b_2 \frac{1}{b_1} + \cdots + b_n \frac{1}{b_{n-1}} = \frac{a_1}{\mathfrak{G}_n(\underline{a})} + \frac{a_2}{\mathfrak{G}_n(\underline{a})} + \cdots + \frac{a_n}{\mathfrak{G}_n(\underline{a})},$$

from which (GA) is immediate. □

[*Hrimic*].

(*lxxi*) SCHAUMBERGER 2000

This is another proof that uses the the right-hand inequality I 2.2 (9); see also 2.4.4 proof (*xxxvii*), 2.4.5 proofs (*lviii*), (*lxii*) and(*lxviii*).

□

$$\begin{aligned}\frac{n\mathfrak{A}_n(\underline{a}) - n\mathfrak{G}_n(\underline{a})}{\mathfrak{G}_n(\underline{a})} &= \sum_{i=1}^{n}\left(\frac{a_i}{\mathfrak{G}_n(\underline{a})} - 1\right) \\ &\ge \sum_{i=1}^{n}\log\left(\frac{a_i}{\mathfrak{G}_n(\underline{a})}\right), \text{ by the the right-hand inequality I 2.2 (9),} \\ &= \log\prod_{i=1}^{n}\left(\frac{a_i}{\mathfrak{G}_n(\underline{a})}\right) = 0.\end{aligned}$$

□

[*Schaumberger 2000*].

(*lxxii*) GAO P 2001

□ Assume without loss in generality that $W_n = 1$ and $\underline{a}$ is an increasing n-tuple with $a_1 \ne a_n$ and write $\underline{x}_i = (\overbrace{x, \ldots, x}^{i \text{ terms}}, a_{i+1}, \ldots a_n)$ and define the functions $D_i(x) = \mathfrak{A}_n(\underline{x}_i; \underline{w}) - \mathfrak{G}_n(\underline{x}_i; \underline{w})$, $,0 < x < a_{i+1}$, $1 \le i \le n$.
Then

$$D_i'(x) = W_i\left(1 - \frac{\mathfrak{G}_n(\underline{x}_i; \underline{w})}{x}\right) < 0, \text{by strict internality, 1.2 Theorem 6.}$$

Hence D_i is strictly decreasing and so

$$D_1(a_1) \ge D_1(a_2) = D_2(a_2) \ge D_2(a_3) = D_3(a_3) \ge \cdots \ge D_n(a_n) = 0,$$

with at least one of these inequalities being strict; but $D_1(a_1) > 0$ is just (GA). □

[*Gao P*].

(*lxxiii*) ROOIN 2001

□ This proof proves the equal weight case of (GA) for rational n-tuples $\underline{a}$. It is then easy to see that we can assume this n-tuple consists of integers and even that $\mathfrak{A}_n(\underline{a}) = nk$ where $k \in \mathbb{N}^*$, $k \ge 2$.

Now the set $A = \{\underline{a};\ \underline{a} \in (\mathbb{N}^*)^n \text{ and } \mathfrak{A}_n(\underline{a}) = nk\}$, for a given positive integer k is finite. Hence A contains an $\underline{a}'$ for which the product of its elements is maximum; that is

$$\forall \underline{a} \in A \quad \prod_{i=1}^{n} a_i \le \prod_{i=1}^{n} a_i'.$$

Suppose that $\underline{a}'$ is not constant then there exist two elements of $\underline{a}'$, without loss in generality $a_{1'}$ and a'_2, such that $a'_1 < k < a'_2$, which implies that $a'_2 - a'_1 \geq 2$, and $a'_2 - a'_1 - 1 > 0$. Now $\underline{a}'' = (a'_1 + 1, a'_2 - 1, a'_3, \ldots, a'_n) \in A$ and

$$\begin{aligned}\prod_{i=1}^{n} a''_i =&(a'_1 + 1)(a'_2 - 1)a'_3 \ldots a'_n = (a'_1 a'_2 + a'_2 - a'_1 - 1)a'_3 \ldots a'_n \\ &> \prod_{i=1}^{n} a'_i.\end{aligned}$$

This is a contradiction and so $\underline{a}'$ is constant. Hence $a'_1 = \cdots = a'_n = k$ and this implies (GA). □

REMARK (ii) It is not immediate how to use this result to get (GA) for all positive n-tuples.

[*Rooin 2001c*].

(*lxxiv*) HÄSTÖ 2002

This proof by induction is an elaboration of 2.2.1 Theorem 3 proof (x), in the form suggested there in Remark (vi), and is related to Liouville's proof, 2.4.1 proof (*iii*).

□ Assume without loss in generality than $\underline{a}$ an n-tuple such that $a_1 < \cdots < a_n$. Let $x = a_n$ and put $f(x) = \mathfrak{A}_n(\underline{a}; \underline{w}) - \mathfrak{G}_n(\underline{a}; \underline{w})$ when

$$f'(x) = \frac{w_n}{W_n}\left(1 - \left(\frac{\mathfrak{G}_{n-1}(\underline{a}; \underline{w})}{x}\right)^{W_{n-1}/W_n}\right)$$

Hence f is increasing if $x > \mathfrak{G}_{n-1}(\underline{a}; \underline{w})$, in particular by the internality of the geometric mean if $x \geq a_{n-1}$. Hence $f(x) > f(a_{n-1}) = \mathfrak{A}_{n-1}(\underline{a}'_n; \underline{w}') - \mathfrak{G}_{n-1}(\underline{a}'_n; \underline{w}')$ where $\underline{w}$ is the $n-1$-tuple $(w_1, \ldots, w_{n-2}, w_{n-1} + w_n)$. The result follows from the induction hypothesis. □

[*Hästo*].

REMARK (iii) As in the proof in 2.2.1 this gives a little more as it shows $\mathfrak{A}_n(\underline{a}; \underline{w}) - \mathfrak{G}_n(\underline{a}; \underline{w})$ is increasing as a function of any term exceeding the geometric mean of the remaining terms, and is decreasing as function of any term that is less than the geometric mean of the remaining terms. A similar proof, that also gives a little more, can be based on the ratio $\mathfrak{A}_n(\underline{a}; \underline{w})/\mathfrak{G}_n(\underline{a}; \underline{w})$.

REMARK (iv) Reference should be made to a slightly different use of the function f in proof (*iii*) of 3 .1 Theorem 1.

2.4.7 PROOFS PUBLISHED IN JOURNALS NOT AVAILABLE TO THE AUTHOR

These are listed in chronological order.

[*Schlömilch 1858a, 1859*], see III 3.1.1 Remark (iv); [*Unferdinger 1867; Tait 1867/69; Schaumberger 1971, 1973, 1975a, 1985b, 1987, 1990b, 1991, 1995; Moldenhauer; Pełczyński 1992*].

2.5 APPLICATIONS OF THE GEOMETRIC MEAN-ARITHMETIC MEAN INEQUALITY We have already seen a simple but effective use of (GA) in Heron's method, 1.3.5 above. Another application was given by Kepler. He noted that an ellipse with semi-axes a and b has the same area as a circle with radius $\sqrt{ab}$. Hence from the classical isoperimetric property the perimeter of the ellipse is bigger than the circumference of that circle, $2\pi\sqrt{ab}$. Using (GA) Kepler gave $\pi(a+b)$ as an approximation for the perimeter of the ellipse[16]. We give some further uses below; of course there are many more, see for instance [*Monsky; Ness 1967*].

EXAMPLE (i) *Determine all the polynomials* $x^n+a_{n-1}x^{n-1}+\cdots+a_0$, *with* $a_i = \pm 1$, $0 \le i \le n-1$, *having all real zeros.* □ Suppose that the zeros are $x_i, 1 \le i \le n$, then $\sum_{i=1}^n x_i^2 = a_{n-1}^2 - 2a_{n-2} = 1 \pm 2 = 3$, since the sum must be positive; further $\prod_{i=1}^n x_i^2 = a_0^2 = 1$. By (GA) $1 \le 3/n$, so $n \le 3$. It is now easy to look at the finite number of possible polynomials particularly since when $n = 3$ the equality case of (GA) implies that the zeros are all ± 1, which gives $a_1 = -1$; [*Boyd*]. □

EXAMPLE (ii) The proof of a surprising result due to Simons, [*Simons*], is based on (GA). *Let $\underline{a}$ be an n-tuple and $\underline{\alpha}$ a real n-tuple all of whose terms are distinct and define $f(x) = \sum_{i=1}^n a_i x^{\alpha_i}$, $x > 0$. If $g(x) = ax^\alpha$ is chosen to give a good approximation to f at the point $x = x_0$ in the sense that $f(x_0) = g(x_0)$ and $f'(x_0) = g'(x_0)$ then $f(x) \ge g(x)$ with equality if and only if $x = x_0$.*

□ The conditions imply that $\alpha = \sum_{i=1}^n a_i\alpha_i x_0^{\alpha_i}/f(x_0)$ and $a = f(x_0)x_0^{-\alpha}$. Hence

$$f(x) = \frac{\sum_{i=1}^n a_i x_0^{\alpha_i}(x/x_0)^{\alpha_i}}{\sum_{i=1}^n a_i x_0^{\alpha_i}} f(x_0) \ge \prod_{i=1}^n \Big(\frac{x}{x_0}\Big)^{\alpha_i a_i x_0/f(x_0)} f(x_0), \quad \text{by (GA)}$$
$$= \Big(\frac{x}{x_0}\Big)^\alpha f(x_0) = ax^\alpha = g(x).$$

There is equality if and only if $\big((x/x_0)^{\alpha_i},\ 1 \le i \le n\big)$ is constant; that is if and only if $x = x_0$. □

2.5.1 CALCULUS PROBLEMS Problems solved by elementary calculus can often be solved using (GA), usually in its simplest forms 2.2.1(2) or 2.2.2(5); [*Niven*], [*Boas & Klamkin; Frame; Lim*].

[16] The classical *isoperimetric property* is: of all convex closed curves of a given area the one with the least perimeter is the circle. It is known that the perimeter of an ellipse cannot be expressed in terms of elementary functions; [*CE p.521; EM5 pp.206–207*].

EXAMPLE (i) $\lim_{n\to\infty} \sqrt[n]{n} = 1$; [*Rooin 2001a*]. Assume that $n \ge 3$, then:

$$0 \le \sqrt[n]{n} - 1 = \frac{n-1}{1 + n^{1/n} + \cdots + n^{(n-1)/n}} \le \frac{n-1}{n\sqrt[n]{n^{1/n+2/n+\cdots+(n-1)/n}}}, \text{ by (GA)},$$
$$= \frac{n-1}{n^{(3n-1)/2n}} \le \frac{1}{\sqrt[3]{n}}.$$

Alternatively, [*Sinnadurai 1961*]:

$$1 < n^{1/2n} = (n^{n/2})^{1/n^2} < \frac{\overbrace{1+\cdots+1}^{n^2-n\text{ terms}} + \overbrace{\sqrt{n}+\cdots+\sqrt{n}}^{n\text{ terms}}}{n^2}, \text{ by (GA)},$$
$$= \frac{n^2 - n + n\sqrt{n}}{n^2} = 1 - \frac{1}{n} + \frac{1}{\sqrt{n}} < 1 + \frac{1}{\sqrt{n}}.$$

This gives: $1 < n^{1/n} = (n^{1/2n})^2 < 1 + \dfrac{3}{\sqrt{n}}$.

2.5.2 POPULATION MATHEMATICS An application to population mathematics, [*Anderson D 1979b*], is based on estimating the lower bound for the positive root of the equation

$$w_1 + w_2x + \cdots w_nx^{n-1} - x^k = 0, \tag{35}$$

where $\underline{w}$ is positive and $k > n - 1 \ge 1$. Using the results of I 1.1 it is easily seen that this equation has exactly one positive root which we will assume is not equal to 1, for in that case the discussion is trivial.

Equation (35) can be written as $0 = \dfrac{1}{W_n}\displaystyle\sum_{i=1}^{n} w_ix^{i-1} - \dfrac{x^k}{W_n}$, using (GA) we easily get that $0 \ge \left(\displaystyle\prod_{i=1}^{n} x^{(i-1)w_i}\right)^{1/W_n} - \dfrac{x^k}{W_n}$. This leads to a lower bound for x, namely $x > W_n^{\alpha}$, where $\alpha = k - \dfrac{1}{W_n}\displaystyle\sum_{i=1}^{n} w_{i+1}$.

For further discussion the reader is referred to Anderson's paper.

2.5.3 PROVING OTHER INEQUALITIES This is of course one of the main applications of (GA), and much of the rest of this book is evidence of this. Here we give a few isolated examples of this use, many more can be found in [*Herman, Kučera & Šimša pp.151–166*].

(α) It is possible to compare the geometric and arithmetic means with different weights; [*Bullen 1967; Dragomir & Goh 1997a; Iwamoto; Mitrinović & Vasić 1966a,c; Wang C L 1979d*].

THEOREM 21 *If $\underline{a}, \underline{u}, \underline{v}$ are n-tuples then:*
(a)
$$\mathfrak{G}_n(\underline{a};\underline{u}) \le \frac{V_n}{U_n}\frac{\mathfrak{G}_n(\underline{u};\underline{u})}{\mathfrak{G}_n(\underline{v};\underline{u})}\mathfrak{A}_n(\underline{a};\underline{v}),$$
with equality if and only if $\underline{a}\,\underline{v}\,\underline{u}^{-1}$ is constant;
(b)
$$\left(\frac{\mathfrak{A}_n(\underline{a};\underline{u})}{\mathfrak{G}_n(\underline{a};\underline{u})}\right)^{U_n}\left(\frac{\mathfrak{A}_n(\underline{a};\underline{v})}{\mathfrak{G}_n(\underline{a};\underline{v})}\right)^{V_n} \le \left(\frac{\mathfrak{A}_n(\underline{a};\underline{u}+\underline{v})}{\mathfrak{G}_n(\underline{a};\underline{u}+\underline{v})}\right)^{U_n+V_n}.$$

□ (a) It is easily seen that
$$\frac{\mathfrak{G}_n(\underline{a};\underline{u})}{\mathfrak{A}_n(\underline{a};\underline{v})} = \left(\frac{\mathfrak{G}_n(\underline{a}\,\underline{v}\,\underline{u}^{-1};\underline{u})}{\mathfrak{A}_n(\underline{a}\,\underline{v}\,\underline{u}^{-1};\underline{u})}\right)\left(\frac{V_n}{U_n}\right)\left(\frac{\mathfrak{G}_n(\underline{u};\underline{u})}{\mathfrak{G}_n(\underline{v};\underline{u})}\right)$$
which implies the result by (GA).
(b) This is an immediate consequence of the similar inequality associated with (J), I 4.2 Theorem 15(b), applied to the negative log function. □

(β) The following theorem is in [*Lupaş & Mitrović*].

THEOREM 22 *If $\underline{a}$ is an n-tuple then*
$$\begin{aligned} 1+\mathfrak{G}_n(\underline{a}) \le& \mathfrak{G}_n(\underline{e}+\underline{a}) \le 1+\mathfrak{A}_n(\underline{a}); \\ 1+\frac{1}{\mathfrak{G}_n(\underline{a})} \le& \mathfrak{G}_n(\underline{e}+\underline{a}^{-1}) \le 1+\frac{1}{\mathfrak{H}_n(\underline{a})}. \end{aligned} \tag{36}$$
There is equality if and only if $\underline{a}$ is constant.

□ By 1.2(7) and 1.2(10) it suffices to prove (36).
The right hand inequality is an immediate consequence of (GA) and the observation that $\mathfrak{A}_n(\underline{e}+\underline{a}) = 1+\mathfrak{A}_n(\underline{a})$.
For the left inequality consider,
$$\begin{aligned} \prod_{i=1}^n(1+a_i) =& 1+\sum_{m=1}^n\sum_m!(a_{i_1}\dots a_{i_m}) \\ \ge& 1+\sum_{m=1}^n\binom{n}{m}\prod_m!(a_{i_1}\dots a_{i_m})^{1/\binom{n}{m}}, \text{ by (GA)}, \\ =& 1+\sum_{m=1}^n\binom{n}{m}\Big(\prod_{j=1}^n a_j\Big)^{m/n} = \big(1+\mathfrak{G}_n(\underline{a})\big)^n. \end{aligned}$$
The case of equality follows from that of (GA).. □

REMARK (i) Another proof of (36) has been given by Kečkić, [*Kečkić*].

REMARK (ii) A part of this inequality has been generalized by Pečarić, see I 2.1 Remark(v), and similar but more general results can be found in some papers of Kovačeć and others; [*Alzer 1990o; Kovačeć 1981a,b; Mitrović 1973; Pećarić 1983a*]. In particular it is immediate from the above proof that the right-hand inequality in (36) holds for weighted means. See also III 3.1.3 Remark (iv).

(γ) The following inequality is in [*Kalajdžić 1970*].

THEOREM 23 *Let $\underline{a}, \underline{w}$ be n-tuples with $W_n = 1$ and let $b > 1$, then:*

$$\sum_{i=1}^{n} b^{a_i} \geq \frac{1}{(n-1)!} \sum ! b^{a^{w_{i_1}} \ldots a^{w_{i_n}}},$$

with equality if and only if $\underline{a}$ is constant.

□ Let $\underline{b} = b^{\underline{a}} = \left(b^{a_1}, \ldots, b^{a_n}\right)$, $\underline{\tilde{w}} = \left(w_{i_1}, \ldots, w_{i_n}\right)$ and consider the sum on the right-hand side of the above inequality

$$\begin{aligned}
\sum ! b^{\mathfrak{G}_n(\underline{a};\underline{\tilde{w}})} &\leq \sum ! b^{\mathfrak{A}_n(\underline{a};\underline{\tilde{w}})}, \text{ by (GA)},\\
&= \sum ! \mathfrak{G}_n(\underline{b};\underline{\tilde{w}})\\
&\leq \sum ! \mathfrak{A}_n(\underline{b};\underline{\tilde{w}}) = (n-1)! \sum_{i=1}^{n} b^{a_i}.
\end{aligned}$$

The case of equality is immediate. □

(δ) (GA) can be used to give a simple proof of a special case of I 2.2(11); the case when $p = n+1, q = n$; [*Forder; Georgakis; Goodman; Melzak; Sinnadurai 1961*].

$$\begin{aligned}
\left(1\left(1+\frac{x}{n}\right)^n\right)^{1/(n+1)} &< \frac{1}{n+1}\Big(1 + \overbrace{\left(1+\frac{x}{n}\right) + \cdots + \left(1+\frac{x}{n}\right)}^{n \text{ terms}}\Big), \text{ by (GA)},\\
&= 1 + \frac{x}{n+1},
\end{aligned}$$

that is

$$\left(1+\frac{x}{n}\right)^n < \left(1+\frac{x}{n+1}\right)^{n+1}.$$

REMARK (iii) This shows $\left(1+\dfrac{1}{n}\right)^n$ increases strictly with n, and a similar argument can be used to show that $\left(1+\dfrac{1}{n}\right)^{n+1}$ strictly decreases with n; see I 2.2(a).

(ϵ) Here we collect various results, without proofs.

THEOREM 24 *(a)* [MIJALKOVIĆ & KELLER; LYONS]

$$\mathfrak{A}_n(\underline{a};\underline{w}) \le \mathfrak{G}_n(\underline{a};\underline{a}\,\underline{w}).$$

(b) [ZACIU]

$$\left(\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{A}_n(\underline{b})}\right)^{\mathfrak{A}_n(\underline{a})} \le \frac{\mathfrak{G}_n(\underline{a}^{\underline{a}})}{\mathfrak{G}_n(\underline{b}^{\underline{b}})}$$

(c) [DAYKIN & SCHMEICHEL]

If $a_{n+k} = a_k, b_k = A_{k+n} - A_k$, $1 \le k \le n$, then

$$n\mathfrak{G}_n(\underline{a}) \le \mathfrak{G}_n(\underline{b}).$$

(d) [AI, P. 348]

If $\tilde{\Delta}^j \underline{a} > 0, 0 \le j \le k$, and $\tilde{\Delta}^{k+1}\underline{a} = \underline{0}$ then

$$1 < \frac{\mathfrak{A}_{m+n}(\underline{a})}{\mathfrak{A}_n(\underline{a})} < \frac{\binom{m+n-1}{k}}{\binom{n-1}{k}}.$$

(e) [AI PP.208–209, 345–346], [MITRINOVIĆ, PEČARIĆ & VOLENEC], [KLAMKIN 1968, 1976; PEČARIĆ, JANIĆ & KLAMKIN 1981]

If $\underline{a}$ is an n-tuple define the n-tuples $\underline{b}, \underline{c}$ by $b_k = \mathfrak{A}_n(\underline{a}) - (n-1)a_k$, and $c_k = \dfrac{\mathfrak{A}_n(\underline{a}) - a_k}{n-1}$, $1 \le k \le n$; and assume that $\underline{b}$ is non-negative then

$$\mathfrak{G}_n(\underline{b}) \le \mathfrak{G}_n(\underline{a}) \le \mathfrak{G}_n(\underline{c}).$$

(f) [AI P.344], [ÅKERBERG]

$$\frac{1}{n}\sum_{k=1}^{n} \frac{k!^{1/k}}{k+1}\mathfrak{G}_n(\underline{a}) < \mathfrak{A}_n(\underline{a}).$$

(g) [AI P.215], [SCHAUMBERGER 1975B]

If $0 < a \le b$ then

$$\frac{(b-a)^2}{8\mathfrak{A}(a,b)} \le \mathfrak{A}(a,b) - \mathfrak{G}(a,b) \le \frac{(b-a)^2}{8\mathfrak{G}(a,b)}.$$

REMARK (iv) The inequalities of (e) extend the results quoted from [*AI*].

REMARK (v) The result in (g) has been extended; see below 4.1 Theorem 2.

(ϕ) The inequality I 2.2 (24) is a generalization of I 2.2 (20); another generalization is the following; see [*Janić & Vasić*].

THEOREM 25 *If $\underline{x}$ is an n-tuple and if $1 \leq m < n$, then*

$$\frac{x_1 + \cdots + x_m}{x_{m+1} + \cdots + x_n} + \frac{x_2 + \cdots + x_{m+1}}{x_{m+2} + \cdots + x_n} + \cdots + \frac{x_n + \cdots + x_{m-1}}{x_m + \cdots + x_{n-1}} \geq \frac{nm}{n-m}$$

□ The left-hand side of this inequality is equal to

$$\begin{aligned}
&\frac{X_n - (x_{m+1} + \cdots + x_n)}{x_{m+1} + \cdots + x_n} + \frac{X_n - (x_{m+2} + \cdots + x_1)}{x_{m+2} + \cdots + x_1} + \cdots \\
&\qquad \cdots + \frac{X_n - (x_m + \cdots + x_{n-1})}{x_m + \cdots + x_{n-1}} \\
&\qquad = X_n \left(\frac{1}{x_{m+1} + \cdots + x_n} + \cdots + \frac{1}{x_m + \cdots + x_{n-1}} \right) - n \\
&\qquad \geq \frac{n^2 X_n}{(x_{m+1} + \cdots + x_n) + \cdots + (x_m + \cdots + x_{n-1})} - n, \\
&\qquad\qquad \text{by (HA)}, \\
&\qquad = \frac{n^2 X_n}{(n-m)X_n} - n = \frac{nm}{n-m}.
\end{aligned}$$

□

2.5.4 PROBABILISTIC APPLICATIONS The *mean, or expected value, of a random variable* X assuming the discrete value x_i with probability p_i, $1 \leq i \leq n$, is $E(X) = \sum_{i=1}^{n} p_i x_i$. Hence we can write

$$\mathfrak{A}_n(\underline{a}; \underline{w}) = E(X); \; \mathfrak{G}_n(\underline{a}; \underline{w}) = \exp\bigl(E(\log X)\bigr); \; \mathfrak{H}_n(\underline{a}; \underline{w}) = 1/E(1/X).$$

Then (GA) can be expressed as

$$E\bigl(\log(1/Y)\bigr) \geq 0, \quad \text{where } Y = X/E(X);$$

the inequality between the geometric and harmonic means can be written as

$$E\bigl(\log(1/Z)\bigr) \geq 0, \quad \text{where } Z = (1/X)/E(1/X)$$

and (HA) can be written as

$$E(X)E(1/X) = E(1/Y) = E(1/Z) \geq 1.$$

The further discussion of these results is outside the scope of this book and reference should be made to [*Pearce*].

3 Refinements of the Geometric Mean-Arithmetic Mean Inequality

3.1 The Inequalities of Rado and Popoviciu There are many refinements of (GA) and we first consider those that result from rewriting this inequality in one of the forms

$$\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w}) \ge 0 \tag{1}$$

$$\frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{G}_n(\underline{a};\underline{w})} \ge 1 \tag{2}$$

and then asking if the left-hand sides of (1) and (2) can be given more precise lower bounds, in general, or for n-tuples satisfying certain conditions. The consideration of this question often leads to still further proofs of (GA).

Theorem 1 *If $\underline{a}, \underline{w}$ are n-tuples, $n \ge 2$ then*

$$W_n\Big(\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})\Big) \ge W_{n-1}\Big(\mathfrak{A}_{n-1}(\underline{a};\underline{w}) - \mathfrak{G}_{n-1}(\underline{a};\underline{w})\Big), \tag{R}$$

$$\left(\frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{G}_n(\underline{a};\underline{w})}\right)^{1/W_n} \ge \left(\frac{\mathfrak{A}_{n-1}(\underline{a};\underline{w})}{\mathfrak{G}_{n-1}(\underline{a};\underline{w})}\right)^{1/W_{n-1}}, \tag{P}$$

with equality in (R) if and only if $a_n = \mathfrak{G}_{n-1}(\underline{a};\underline{w})$, and in (P) if and only if $a_n = \mathfrak{A}_{n-1}(\underline{a};\underline{w})$.

□ We give several proofs of these results.

(i) First we consider the case of equal weights and use a method due to Jacobsthal, see 2.4.3 proof (xxi) of (GA); [*Jacobsthal*].

$$\begin{aligned} n\mathfrak{A}_n(\underline{a}) &= \mathfrak{G}_{n-1}(\underline{a})\Big((n-1)\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{G}_{n-1}(\underline{a})} + \Big(\frac{\mathfrak{G}_n(\underline{a}}{\mathfrak{G}_{n-1}(\underline{a})}\Big)^n\Big) \\ &\ge \mathfrak{G}_{n-1}(\underline{a})\Big((n-1)\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{G}_{n-1}(\underline{a})} + n\frac{\mathfrak{G}_n(\underline{a})}{\mathfrak{G}_{n-1}(\underline{a})} - (n-1)\Big), \text{ by I 1.2(7)} \end{aligned}$$

This is just (R), in the case of equal weights. Further, from I 1.2(7), there is equality if and only if $\mathfrak{G}_n(\underline{a}) = \mathfrak{G}_{n-1}(\underline{a})$, or equivalently if and only if $a_n = \mathfrak{G}_{n-1}(\underline{a})$.

Similarly,

$$\begin{aligned} \left(\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{G}_n(\underline{a})}\right)^n &= \left(\frac{\mathfrak{A}_{n-1}(\underline{a})}{\mathfrak{G}_{n-1}(\underline{a})}\right)^{n-1}\left(1 + \frac{1}{n}\Big(\frac{a_n}{\mathfrak{A}_n(\underline{a})} - 1\Big)\right)^n \frac{\mathfrak{A}_{n-1}(\underline{a})}{a_n} \\ &\ge \left(\frac{\mathfrak{A}_{n-1}(\underline{a})}{\mathfrak{G}_{n-1}(\underline{a})}\right)^{n-1}, \text{ by } (\sim\text{B}). \end{aligned}$$

The case of equality follows from that of (~B).

(*ii*) Simple calculations show that (R) is equivalent to

$$\frac{w_n}{W_n}a_n + \frac{W_{n-1}}{W_n}\mathfrak{G}_{n-1}(\underline{a};\underline{w}) \geq a_n^{w_n/W_n}\mathfrak{G}_{n-1}^{W_{n-1}/W_n}(\underline{a};\underline{w}), \tag{3}$$

which is a consequence of (GA).

A similar reduction gives a proof of (P).

(*iii*) This is a modification of 2.4.1 proof (*iii*) of (GA) and is due to Liouville; [*Liouville*].

Let $x = a_n$ and $f(x) = W_n\big(\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})\big)$. Differentiation shows that for $x \geq 0$, f has a single turning point, a minimum, at $x = x_0 = \mathfrak{G}_{n-1}(\underline{a};\underline{w})$. Simple calculations show that $f(x_0) = W_{n-1}\big(\mathfrak{A}_{n-1}(\underline{a};\underline{w}) - \mathfrak{G}_{n-1}(\underline{a};\underline{w})\big)$. Hence$f(x) > f(x_0)$, $x \neq x_0$, and this completes the proof of (R).

If instead we consider $g(x) = \left(\dfrac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{G}_n(\underline{a};\underline{w})}\right)^{W_n}$ a similar discussion leads to (P).

(*iv*) (R) is an immediate consequence of I 4.2(7) in the case $f(x) = e^x$ and the a_i there are replaced by $\log a_i$, $1 \leq i \leq n$.

In a similar way we get (P) by putting $f(x) = -\log x$.

(*v*) The following ingenious proof is due to Nanjundiah who obtains (R) and (P) simultaneously by appealing twice to (~B) in the form I 2.1(3); [*Nanjundiah 1946*].

$$\begin{aligned}\left(\frac{\mathfrak{A}_n^{W_n}(\underline{a};\underline{w})}{\mathfrak{A}_{n-1}^{W_{n-1}}(\underline{a};\underline{w})}\right)^{1/w_n} &\geq \frac{W_n}{w_n}\mathfrak{A}_n(\underline{a};\underline{w}) - \frac{W_{n-1}}{w_n}\mathfrak{A}_{n-1}(\underline{a};\underline{w}) = a_n\\ &= \left(\frac{\mathfrak{G}_n^{W_n}(\underline{a};\underline{w})}{\mathfrak{G}_{n-1}^{W_{n-1}}(\underline{a};\underline{w})}\right)^{1/w_n}\\ &\geq \frac{W_n}{w_n}\mathfrak{G}_n(\underline{a};\underline{w}) - \frac{W_{n-1}}{w_n}\mathfrak{G}_{n-1}(\underline{a};\underline{w}).\end{aligned}$$

(*vi*) A simple proof based on (B) has been given recently; [*Redheffer & Voigt*].

Rewrite (3) as

$$a_n \geq \frac{W_n}{w_n}\mathfrak{G}_n(\underline{a};\underline{w}) - \frac{W_{n-1}}{w_n}\mathfrak{G}_{n-1}(\underline{a};\underline{w}), \tag{4}$$

and notice that this is just (~B) with $1 + x = \mathfrak{G}_n(\underline{a};\underline{w})/\mathfrak{G}_{n-1}(\underline{a};\underline{w})$ and $\alpha = W_n/w_n$.

(*vii*) Proofs are easily obtained from the inequalities in (a) and (b) of 2.4.5 proof (*lvi*); [*Rüthing 1982*]. □

REMARK (i) Repeated application of (R) or (P) leads to (GA) together with the case of equality; also repeated application of (R) gives a particular case of I 4.2(7);

$$\begin{aligned}W_n\big(\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})\big) &\geq W_{n-1}\big(\mathfrak{A}_{n-1}(\underline{a};\underline{w}) - \mathfrak{G}_{n-1}(\underline{a};\underline{w})\big) \geq \cdots\\ \cdots \geq W_2\big(\mathfrak{A}_2(\underline{a};\underline{w}) - \mathfrak{G}_2(\underline{a};\underline{w})\big) &\geq W_1\big(\mathfrak{A}_1(\underline{a};\underline{w}) - \mathfrak{G}_1(\underline{a};\underline{w})\big) = 0.\end{aligned}$$

REMARK (ii) (R) seems to occur for the first time in [*HLP p.61*] where it is given, in the equal weight case as an exercise and attributed to Rado. Known as *Rado's inequality* it has been rediscovered many times; see [*Bullen 1965, 1970b, 1971b; Bullen & Marcus; Čakalov 1946, 1963; Dinghas 1943/44; Ling; Popoviciu 1934b; Stubben*].

REMARK (iii) (P) was proved first in the equal weight case by Popoviciu, [*Popviciu 1934b*], and is known as *Popoviciu's inequality*. It was implied in an earlier paper but the author of that paper failed to notice what he had proved, [*Simonart*]. As in the case of (R) this inequality has been rediscovered many times; see [*Bullen 1965,1967,1968; Bullen & Marcus; Dinghas 1943/44; Kestelman; Klamkin 1968; Mitrinović 1966; Mitrinović & Vasić 1966b,c*].

REMARK (iv) By adding the inequalities (4) we get another inequality also known as *Rado's inequality*;

$$\sum_{k=m+1}^{n} w_k a_k \geq W_n \mathfrak{G}_n(\underline{a};\underline{w}) - W_m \mathfrak{G}_m(\underline{a};\underline{w}),\ 1 \leq m < n.$$

REMARK (v) For an alternative formulation of proof (*v*) see below 3.4 Remark (ii).

A point of some logical interest is that the apparently stronger inequalities (R) and (P) are equivalent to (GA) since, for instance, proof (*ii*) uses (GA) to prove both (P) and (R). Moreover this proof can be used to obtain the following extension of 2.2.3 Lemma 5.

LEMMA 2 *It is sufficient to prove* (GA), *2.1 Theorem 1, in the case of equal weights and where in addition $\underline{a}$ has all terms except one equal.*

□ By 2.2.3 Lemma 5 it is sufficient to consider the case of equal weights, and so since by Remark (i) (R) implies (GA), it is sufficient to prove (R) in the equal weight case. That is, from (3), to prove that

$$\frac{a_n}{n} + \frac{n-1}{n}\mathfrak{G}_{n-1}(\underline{a}) \geq a_n^{1/n}\mathfrak{G}_{n-1}^{(n-1)/n}(\underline{a}), \tag{5}$$

with equality if and only if $a_n = \mathfrak{G}_{n-1}(\underline{a})$.
Inequality (5) and the case of equality follows from the case of (GA) that satisfies the conditions of this lemma.

□

REMARK (vi) Since inequality (5) follows from I 1.2(7) with $x = \big(a_n/\mathfrak{G}_{n-1}(\underline{a})\big)^{1/n}$ and n replaced by $n-1$ we get yet another proof of (GA).

COROLLARY 3 *If $\underline{a}, \underline{w}$ are n-tuples, $n \geq 2$, then*

$$\mathfrak{A}_n(\underline{a};\underline{w})-\mathfrak{G}_n(\underline{a};\underline{w})\geq\frac{1}{W_n}\max_{1\leq i,j\leq n}\left\{a_iw_i+a_jw_j-(w_i+w_j)\big(a_i^{w_i}a_j^{w_j}\big)^{1/(w_i+w_j)}\right\}, \tag{6}$$

$$\frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{G}_n(\underline{a};\underline{w})}\geq\left(\max_{1\leq i,j\leq n}\left\{\frac{w_ia_i+w_ja_j}{w_i+w_j}\frac{1}{a_i^{w_i}a_j^{w_j}}\right\}\right)^{1/W_n}. \tag{7}$$

□ By Remark (i) the left-hand side of (6) is not less than

$$\frac{W_2}{W_n}\big(\mathfrak{A}_2(\underline{a};\underline{w})-\mathfrak{G}_2(\underline{a};\underline{w})\big)=\frac{1}{W_n}\big(w_1a_1+w_wa_2-W_2\big(a_1^{w_1}a_2^{w_2}\big)^{1/W_2}\big). \tag{8}$$

This implies (6) since the n-tuples $\underline{a}, \underline{w}$ can be simultaneously re-ordered to maximize the right-hand side of (8) and leaving the left-hand side unaltered.
Inequality (7) is proved in a similar manner. □

REMARK (vii) Inequalities (6), (7) give more precise right-hand sides to (1) and (2); they are special cases of I 4.2(8). In the case of equal weights they are particularly symmetric:

$$\mathfrak{A}_n(\underline{a})-\mathfrak{G}_n(\underline{a})\geq\frac{1}{n}\big(\sqrt{\max\underline{a}}-\sqrt{\min\underline{a}}\big)^2;$$

$$\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{G}_n(\underline{a})}\geq\left(\max_{1\leq i,j\leq n}\left\{\frac{1}{2}+\frac{1}{4}\Big(\frac{a_i}{a_j}+\frac{a_j}{a_i}\Big)\right\}\right)^{1/n}.$$

A Rado type inequality involving the arithmetic and harmonic means has been proved; [*Klamkin 1968*].

THEOREM 4 *If $\underline{a}$ and $\underline{w}$ are n-tuples, $n \geq 2$, then*

$$W_n\sqrt{\frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{H}_n(\underline{a};\underline{w})}}\geq w_n+W_{n-1}\sqrt{\frac{\mathfrak{A}_{n-1}(\underline{a};\underline{w})}{\mathfrak{H}_{n-1}(\underline{a};\underline{w})}}, \tag{9}$$

with equality if and only if $a_n=\sqrt{\mathfrak{A}_{n-1}(\underline{a};\underline{w})\mathfrak{H}_{n-1}(\underline{a};\underline{w})}$.

□

$$\begin{aligned}\frac{W_n^2\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{H}_n(\underline{a};\underline{w})}&=\big(W_{n-1}\mathfrak{A}_{n-1}(\underline{a};\underline{w})+a_nw_n\big)\Big(\frac{W_{n-1}}{\mathfrak{H}_{n-1}(\underline{a};\underline{w})}+\frac{w_n}{a_n}\Big)\\&=\frac{W_{n-1}^2\mathfrak{A}_{n-1}(\underline{a};\underline{w})}{\mathfrak{H}_{n-1}(\underline{a};\underline{w})}+w_n^2+w_nW_{n-1}\Big(\frac{\mathfrak{A}_{n-1}(\underline{a};\underline{w})}{a_n}+\frac{a_n}{\mathfrak{H}_{n-1}(\underline{a};\underline{w})}\Big)\\&\geq\frac{W_{n-1}^2\mathfrak{A}_{n-1}(\underline{a};\underline{w})}{\mathfrak{H}_{n-1}(\underline{a};\underline{w})}+w_n^2+2w_nW_{n-1}\sqrt{\frac{\mathfrak{A}_{n-1}(\underline{a};\underline{w})}{\mathfrak{H}_{n-1}(\underline{a};\underline{w})}},\quad\text{by (GA)},\\&=\left(w_n+W_{n-1}\sqrt{\frac{\mathfrak{A}_{n-1}(\underline{a};\underline{w})}{\mathfrak{H}_{n-1}(\underline{a};\underline{w})}}\right)^2,\end{aligned}$$

which implies (9). The case of equality follows from that for (GA). □

REMARK (viii) Repeated application of (9) gives a proof of (HA), as well as refinements of (HA) along the lines of Remark (i), inequality (7) and Remark (vii).

REMARK (ix) Rado-like inequalities between the harmonic and geometric means have been given by Alzer; [*Alzer 1989d*].

3.2 EXTENSIONS OF THE INEQUALITIES OF RADO AND POPOVICIU Various authors have given extensions to inequalities (R) and (P). Interesting though these may be they seem to have few applications. For this reason, and since most, if not all, of these extensions can be deduced from a more general result to be given later, full details will not be given in this section; [*Bullen 1967, 1968, 1969a,b, 1970b, 1971b; Gavrea & Gurzău 1986; Mitrinović & Vasić 1966a,b,c, 1967a,b, 1968a,b,c*].

3.2.1 MEANS WITH DIFFERENT WEIGHTS The most obvious extension is to allow the means in (R) and (P) to have different weights. Unlike the situation for 2.5.3 Theorem 21(a) such results are not immediate consequences of the original inequalities. A particularly fruitful method of discovering these extensions has been used by Mitrinović & Vasić, [*AI p.90*]. The proofs are based on proofs (*ii*) and (*iii*) of Theorem 1.
Some of these results are collected in the following theorem.

THEOREM 5 *(a) Let $\lambda, \mu \in \mathbb{R}$ with $\lambda\mu > 0$, and $\underline{a}, \underline{u}, \underline{v}$ be n-tuples, $n \geq 2$, with $U_n > \mu u_n$. Then*

$$\begin{aligned} V_n\mathfrak{A}_n(\underline{a};\underline{v}) - \frac{\lambda v_n}{u_n}U_n\mathfrak{G}_n^{\mu}(\underline{a};\underline{u}) \geq & \\ & V_{n-1}\mathfrak{A}_{n-1}(\underline{a};\underline{v}) \\ & - (\lambda\mu)^{U_n/(U_n-\mu u_n)}\frac{v_n}{\mu u_n}(U_n - \mu u_n)\mathfrak{G}_{n-1}^{(\mu U_{n-1})/(U_n-\mu u_n)}(\underline{a};\underline{u}). \end{aligned} \tag{10}$$

If $U_n < \mu u_n$ then (~10) holds. In both cases there is equality if and only if $a_n = (\lambda\mu)^{U_n/(U_n-\mu u_n)}\mathfrak{G}_n^{\mu U_{n-1}/(U_n-\mu u_n)}(\underline{a};\underline{u})$.
(b) Let $\lambda, \mu > 0$, and $\underline{a}, \underline{u}, \underline{v}$ n-tuples, $n \geq 2$, with $\mathfrak{A}_{n-1}(\underline{a};\underline{v}) + \dfrac{\lambda V_n}{V_{n-1}} \geq 0, V_n \geq \mu v_n$ then

$$\begin{aligned} \frac{\left(\mathfrak{A}_n(\underline{a};\underline{v}) + \lambda\right)^{V_n}}{\left(\mathfrak{G}_n^{\mu}(\underline{a};\underline{u})\right)^{(v_nU_n)/u_n}} \geq & \\ & \frac{1}{\mu^{\mu v_n}}\left(\frac{V_{n-1}}{V_n - \mu v_n}\right)^{V_n-\mu v_n}\frac{\left(\mathfrak{A}_{n-1}(\underline{a};\underline{v}) + \dfrac{\lambda V_n}{V_{n-1}}\right)^{V_n-\mu v_n}}{\left(\mathfrak{G}_{n-1}^{\mu}(\underline{a};\underline{u})\right)^{(v_nU_{n-1})/u_n}}. \end{aligned} \tag{11}$$

If $V_n < \mu v_n$ then ($\sim$11) holds. In both cases there is equality if and only if $a_n = \mu\mathfrak{A}_n(\underline{a};\underline{v}) + \lambda$.

(c) Let $\alpha, \beta, \lambda \in \mathbb{R}$ with $\lambda > 0$, and $\underline{a}, \underline{u}, \underline{v}$ be n-tuples, $n \ge 2$, with $\beta(\alpha - \beta u_n) > 0$. If $(\alpha - \beta u_n) > 0$ then

$$\frac{\Big(\mathfrak{A}_n(\underline{a};\underline{v}) + \lambda\Big)^{\alpha}}{\mathfrak{G}_n^{\beta U_n}(\underline{a};\underline{u})} \ge \tag{12}$$

$$\Big(\frac{v_n}{\beta u_n}\Big)^{\beta u_n}\Big(\frac{\alpha}{v_n}\Big)^{\alpha}\Big(\frac{V_{n-1}}{\alpha - \beta u_n}\Big)^{\alpha-\beta u_n}\frac{\Big(\mathfrak{A}_{n-1}(\underline{a};\underline{w}) + \dfrac{V_n}{V_{n-1}}\Big)^{\alpha-\beta u_n}}{\mathfrak{G}_{n-1}^{\beta U_{n-1}}(\underline{a};\underline{u})}.$$

If $\alpha - \beta u_n < 0$ then ($\sim$12) holds. In both cases there is equality if and only if $(\alpha - \beta u_n)v_n a_n = \beta u_n\big(V_{n-1}\mathfrak{A}_{n-1}(\underline{a};\underline{v}) + \lambda V_n\big)$.

□ (a) (*i*) If $\lambda = \mu = 1$ (10) reduces to

$$V_n\mathfrak{A}_n(\underline{a};\underline{v}) - \frac{v_n}{u_n}U_n\mathfrak{G}_n(\underline{a};\underline{u}) \ge V_{n-1}\mathfrak{A}_{n-1}(\underline{a};\underline{v}) - \frac{v_n}{u_n}U_{n-1}\mathfrak{G}_{n-1}(\underline{a};\underline{u}),$$

and proof (*iii*) of Theorem 1 suggests putting $x = a_n$ and $f(x)$ equal to the left-hand side of this last inequality.

The method of Mitrinović & Vasić that leads to the general case consists of introducing two extra parameters and putting $x = a_n$ and $g_{\lambda,\mu}(x) = g(x)$ equal to the left-hand side of (10). Then

$$g'(x) = v_n\Big(1 - \lambda\,\mu\, x^{(\mu u_n - U_n)/U_n}\mathfrak{G}_{n-1}^{\mu U_{n-1}/U_n}(\underline{a};\underline{u})\Big),$$

and so $g'(x) = 0$ if $x = \Big((\lambda\mu)^{U_n}\mathfrak{G}_{n-1}^{\mu U_{n-1}}(\underline{a};\underline{w})\Big)^{1/(U_n - \mu u_n)}$.

Under the first set of hypotheses this is a minimum, whilst under the second set it is a maximum.

The proof of both cases then proceeds as the proof (*iii*) of Theorem 1.

(*ii*) The method of proof (*ii*) of Theorem 1 can also be used here. However, unlike proof (*i*) above, this method gives no help in discovering the correct form of the inequality to be proved. Simple manipulations show that (10) is equivalent to

$$\frac{\mu u_n}{U_n}a_n + \frac{U_n - \mu u_n}{U_n}\Big((\lambda\mu)^{U_n}\mathfrak{G}_{n-1}^{\mu U_{n-1}}(\underline{a};\underline{u})\Big)^{1/(U_n - \mu u_n)} \ge a_n^{\frac{\mu u_n}{U_n}}\lambda\mu\mathfrak{G}_{n-1}^{\mu U_{n-1}/U_n}(\underline{a};\underline{u}), \tag{13}$$

which is an immediate consequence of (GA).

(b) The proof of (11), of which (P) is a special case, follows using either of the methods used in (a). However it is of some interest to note that (b) is the same as (a). Putting $\lambda = \dfrac{\alpha v_n U_n}{u_n V_n}\mathfrak{G}_n^{\beta}(\underline{a};\underline{u}) - \mathfrak{A}_n(\underline{a};\underline{v})$, $\mu = \dfrac{u_n V_n}{v_n U_n}\beta$, (11) reduces to (10), with λ,μ replaced by α,β respectively.

(c) As in (a) or (b) put $x = a_n$ and consider $f(x) = \dfrac{\Big(\mathfrak{A}_n(\underline{a};\underline{v}) + \lambda\Big)^{\alpha}}{\Big(\mathfrak{G}_n(\underline{a};\underline{u})\Big)^{\beta U_n}}$. Then f has a unique extremal at $x = \beta\dfrac{u_n}{v_n}\dfrac{V_{n-1}\mathfrak{A}_{n-1}(\underline{a};\underline{v}) + \lambda V_n}{\alpha - \beta u_n}$, which is a minimum if the first hypothesis holds, and a maximum if the second holds. □

Remark (i) It might be noted that (13) shows that (10) is independent of $\underline{v}$.

Example (i) We will see later, IV 3, that (R) and (P) are special cases of a more general result. Here we note that the single inequality (10) also includes them both by choosing special values of λ and μ. Take $\lambda = \mu = 1$ and $\underline{u} = \underline{v} = \underline{w}$ then (10) reduces to (R); taking $\lambda = \mathfrak{A}_n(\underline{a};\underline{w})/\mathfrak{G}_n(\underline{a};\underline{w})$, and $\mu = 1, \underline{u} = \underline{v} = \underline{w}$, (10) reduces to (P).

Example (ii) If $\alpha = V_n, \beta = v_n/u_n$ then (12) reduces to

$$\frac{\Big(\mathfrak{A}_n(\underline{a};\underline{v}) + \lambda\Big)^{V_n}}{\mathfrak{G}_n^{v_n U_n/u_n}(\underline{a};\underline{u})} \ge \frac{\Big(\mathfrak{A}_{n-1}(\underline{a};\underline{v}) + \lambda\dfrac{V_n}{V_{n-1}}\Big)^{V_{n-1}}}{\mathfrak{G}_{n-1}^{v_n U_{n-1}/u_n}(\underline{a};\underline{u})},$$

which is a multiplicative analogue of (10).

Remark (ii) Inequality (10) could be further extended by considering

$$g_{\lambda,\mu,\nu,\theta}(x) = g(x) = V_n\mathfrak{A}_n(\underline{a};\underline{w}) - \lambda\mathfrak{G}_n^{\mu}(\underline{a};\underline{u}) - \nu\mathfrak{G}_n^{\theta}(\underline{a};\underline{v})$$

where $x = a_n$ and $\theta u_n V_n = \mu v_n U_n$.
Various choices of the parameters give results obtained independently by several authors.

Example (iii) Taking $\lambda = \mu = 1$ inequality (10) reduces to a particularly simple extension of (R) due to Mitrinović, [*Mitrinović 1968c*]:

$$\frac{V_n}{v_n}\mathfrak{A}_n(\underline{a};\underline{v}) - \frac{U_n}{u_n}\mathfrak{G}_n(\underline{a};\underline{u}) \ge \frac{V_{n-1}}{v_n}\mathfrak{A}_{n-1}(\underline{a};\underline{v}) - \frac{U_{n-1}}{u_n}\mathfrak{G}_{n-1}(\underline{a};\underline{u}).$$

Example (iv) Taking $\lambda = u_n V_n/v_n U_n$, $\lambda\mu = 1$ in (10) gives the following inequality that is another generalization of (R):

$$V_n\Big(\mathfrak{A}_n(\underline{a};\underline{v}) - \mathfrak{G}_n^{v_n U_n/u_n V_n}(\underline{a};\underline{u})\Big) \ge V_{n-1}\Big(\mathfrak{A}_{n-1}(\underline{a};\underline{v}) - \mathfrak{G}_{n-1}^{v_n U_{n-1}/u_n V_{n-1}}(\underline{a};\underline{u})\Big).$$

REMARK (iii) For further extensions see [*Wang C L 1979a,1984c*].

3.2.2 INDEX SET EXTENSIONS Let $\underline{a}, \underline{w}$ be sequences and I an index set, $I \in \mathcal{I}$, see Notations 4, then extending the notations of Notations 6(xi) and I 4.2 Theorem 15, we write

$$\mathfrak{A}_I(\underline{a};\underline{w}) = \frac{1}{W_I}\sum_{i\in I} w_i a_i; \quad \mathfrak{G}_I(\underline{a};\underline{w}) = \Big(\prod_{i\in I} w_i a_i\Big)^{1/W_I}.$$

Using this notation (GA), (R) and (P) can be expressed in the following form.

THEOREM 6 *Let $\underline{a}, \underline{w}$ be sequences and define the following functions on the index sets, $I \in \mathcal{I}$,*

$$\rho(I) = W_I\Big(\mathfrak{A}_I(\underline{a};\underline{w}) - \mathfrak{G}_I(\underline{a};\underline{w})\Big), \qquad \pi(I) = \left(\frac{\mathfrak{A}_I(\underline{a};\underline{w})}{\mathfrak{G}_I(\underline{a};\underline{w})}\right)^{1/W_I};$$

then both ρ and π are non-negative increasing functions.

The results of this section are concerned with obtaining more properties of the functions ρ and π; [*Bullen 1965,1967; Everitt 1961; Mitrinović & Vasić 1966b*].

THEOREM 7 *The set function ρ is super-additive, and the set function π is log-superadditive.*

REMARK (i) The first part of this theorem is a special case of I 4.2 Theorem 15(a), and both parts follow from more precise results which we now give.

If $\underline{a}$ is an $(n+m)$-tuple, $n, m \in \mathbb{N}^*$, $\underline{a} = (a_1, \dots, a_{n+m})$, we will write

$$\widetilde{\underline{a}} = (a_{n+1}, \dots a_{n+m}); \; \widetilde{\mathfrak{A}}_m(\underline{a};\underline{w}) = \frac{1}{\widetilde{W}_m}\sum_{i=n+1}^{n+m} w_i a_i; \; \widetilde{W}_m = \sum_{i=n+1}^{n+m} w_i; \text{ etc.} \tag{14}$$

THEOREM 8 *If $\underline{a}$ and $\underline{w}$ are $(n+m)$-tuples then*

$$(a) \quad W_{n+m}\big(\mathfrak{A}_{n+m}(\underline{a};\underline{w}) - \mathfrak{G}_{n+m}(\underline{a};\underline{w})\big) \ge W_n\Big(\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})\Big) + \widetilde{W}_m\Big(\widetilde{\mathfrak{A}}_m(\underline{a};\underline{w}) - \widetilde{\mathfrak{G}}_m(\underline{a};\underline{w})\Big)$$

$$(b) \quad \left(\frac{\mathfrak{A}_{n+m}(\underline{a};\underline{w})}{\mathfrak{G}_{n+m}(\underline{a};\underline{w}}\right)^{W_{n+m}} \ge \left(\frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{G}_n(\underline{a};\underline{w}}\right)^{W_n}\left(\frac{\widetilde{\mathfrak{A}}_m(\underline{a};\underline{w})}{\widetilde{\mathfrak{G}}_m(\underline{a};\underline{w}}\right)^{\widetilde{W}_m}.$$

There is equality in (a) if and only if $\mathfrak{G}_n(\underline{a};\underline{w}) = \widetilde{\mathfrak{G}}_m(\underline{a};\underline{w})$; and there is equality in (b) if and only if $\mathfrak{A}_n(\underline{a};\underline{w}) = \widetilde{\mathfrak{A}}_m(\underline{a};\underline{w})$.

□ 3.1 Theorem 1 is a special case of this result, $m = 1$, and proof (*ii*) of that theorem can be used to prove Theorem 8. □

REMARK (ii) This result has been extended to allow for different weights in the different means; [*Bullen 1967; Mitrinović & Vasić 1966b*].

REMARK (iii) The inequality in 2.5.3 Theorem 21 (b) also has index set extensions; see [*Dragomir & Goh 1997a*].

3.3 A LIMIT THEOREM OF EVERITT Given a sequence $\underline{a}$ then the equal weight case of (R) says that the sequence $n\big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\big)$, $n \in \mathbb{N}$, is increasing. It follows that $\lim_{n\to\infty} n\big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\big)$ exists, in $\overline{\mathbb{R}}$, and it is natural to ask under what conditions this limit is finite. Everitt gave a complete answer to this question, and in this section we present his work; [*Everitt 1967,1969*].

THEOREM 9 *If $\underline{a}$ is a sequence then $\lim_{n\to\infty} n\big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\big)$ is finite if and only if: either (a) $\sum_{n=1}^{\infty} a_i$ converges, or (b) for some $\alpha > 0$, $\sum_{i=1}^{\infty}(a_i - \alpha)^2$ converges.*

□ The proof is given by considering the four possible types of behaviour of $\underline{a}$:
(α) $\lim_{n\to\infty} a_n = 0$; ($\beta$) $\lim_{n\to\infty} a_n = \alpha$, $\alpha > 0$;
(γ) $\limsup_{n\to\infty} a_n = \infty$; ($\delta$) $0 \le \liminf_{n\to\infty} a_n < \limsup_{n\to\infty} a_n < \infty$.
(α) Using 3.2.2 Theorem 6, and the notation defined there, if $1 \le k < n$,

$$\begin{aligned}\sum_{i=1}^{n} a_i \ge n\big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\big) =& \rho(\{1,\ldots,n-1,n\}) \ge \rho(\{1,\ldots,k,n\}) \\ =& a_n + \sum_{i=1}^{k} a_i - (k+1)\Big(a_n \prod_{i=1}^{k} a_i\Big)^{1/(k+1)}.\end{aligned}$$

Letting $n \to \infty$, and then letting $k \to \infty$ we get that $\lim_{n\to\infty} n\big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\big) = \sum_{i=1}^{\infty} a_i$, which completes the proof in this case.
(β) First remark that this hypothesis implies:

$$\lim_{n\to\infty} \mathfrak{A}_n(\underline{a}) = \alpha, \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(a_i - \alpha)^2 = 0.$$

Now define $\tau_{ij} = a_i - \mathfrak{A}_j(\underline{a})$; obviously $\sum_{i=1}^{j} \tau_{ij} = 0$, and we investigate the properties of the two quantities $\displaystyle\sum_{i=1}^{j} \tau_{ij}^2, \quad \sum_{i=1}^{j} \tau_{ij}^3$.
Now:

$$\begin{aligned}\sum_{i=1}^{j}(a_i - \alpha)^2 =& \sum_{i=1}^{j} \tau_{ij}^2 + j\big(\mathfrak{A}_j(\underline{a}) - \alpha\big)^2 \ge \sum_{i=1}^{j} \tau_{ij}^2 \\ \ge& \sum_{i=1}^{k} \tau_{ij}^2 = \sum_{i=1}^{k} \big((a_i - \alpha) + (\alpha - \mathfrak{A}_j(\underline{a})\big)^2, \quad i \le k \le j.\end{aligned}$$

From this, and the preliminary remark, we make two deductions. First, letting $j \to \infty$, and then $k \to \infty$, we get that

$$\lim_{j\to\infty} \sum_{i=1}^{j} \tau_{ij}^2 = \sum_{i=1}^{\infty} (a_i - \alpha)^2; \tag{15}$$

also,

$$\lim_{j\to\infty} \frac{1}{j} \sum_{i=1}^{j} \tau_{ij}^2 = 0. \tag{16}$$

Now, given $\epsilon > 0$ choose n_0 such that if $n \geq n_0$ then $|a_n - \alpha| < \epsilon$. If $j > n_0$ then

$$\begin{aligned}\sum_{i=1}^{j} \tau_{ij}^3 &\leq \sum_{i=1}^{j} \tau_{ij}^2 \left(|a_i - \alpha| + |\alpha - \mathfrak{A}_j(\underline{a})|\right) \\ &< \sum_{i=1}^{n_0} \tau_{ij}^2 |a_i - \alpha| + \epsilon \sum_{i=n_0+1}^{j} \tau_{ij}^2 + |\alpha - \mathfrak{A}_j(\underline{a})| \sum_{i=1}^{j} \tau_{ij}^2.\end{aligned} \tag{17}$$

Hence

$$\sum_{i=1}^{j} \tau_{ij}^3 = O\Big(\sum_{i=1}^{j} \tau_{ij}^2\Big), \; j \to \infty. \tag{18}$$

If we assume that $\sum_{i=1}^{\infty} (a_i - \alpha)^2 = \infty$, then using (15), (17) implies that

$$\sum_{i=1}^{j} \tau_{ij}^3 = o\Big(\sum_{i=1}^{j} \tau_{ij}^2\Big), \; j \to \infty. \tag{19}$$

We now proceed to consider the proof of the theorem in this case.

$$\begin{aligned}\mathfrak{G}_n(\underline{a}) &= \mathfrak{A}_n(\underline{a}) \prod_{i=1}^{n} \left(1 + \frac{\tau_{in}}{\mathfrak{A}_n(\underline{a})}\right)^{1/n} = \mathfrak{A}_n(\underline{a}) \exp\left(\frac{1}{n} \sum_{i=1}^{n} \log\left(1 + \frac{\tau_{in}}{\mathfrak{A}_n(\underline{a})}\right)\right) \\ &= \mathfrak{A}_n(\underline{a}) \exp\left(\frac{1}{n}\Big(\sum_{i=1}^{n} \frac{\tau_{in}}{\mathfrak{A}_n(\underline{a})} - \frac{1}{2}\frac{\tau_{in}^2}{\mathfrak{A}_n^2(\underline{a})} + O(|\tau_{in}|^3)\Big)\right) \\ &= \mathfrak{A}_n(\underline{a}) \exp\left(-\frac{1}{2n\mathfrak{A}_n^2(\underline{a})}\Big(\sum_{i=1}^{n} \tau_{in}^2 + O\big(\sum_{i=1}^{n} |\tau_{in}|^3\big)\Big)\right).\end{aligned}$$

If then $\sum_{i=1}^{\infty} (a_i - \alpha)^2 < \infty$ then from (15) and (18)

$$\mathfrak{G}_n(\underline{a}) = \mathfrak{A}_n(\underline{a}) \exp\left(\frac{-1 + O(1)}{2\mathfrak{A}_n(\underline{a})} \sum_{i=1}^{n} \tau_{in}^2\right),$$

or

$$n\big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\big) = \frac{1 + O(1)}{2\mathfrak{A}_n(\underline{a})} \sum_{i=1}^{n} \tau_{in}^2 + O(1),$$

so $\lim_{n\to\infty} n\big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\big) < \infty$.
If, on the other hand, $\sum_{i=1}^{\infty}(a_i - \alpha)^2 = \infty$, then from (15) and (19),

$$\mathfrak{G}_n(\underline{a}) = \mathfrak{A}_n(\underline{a}) \exp\Big(\frac{-1+o(1)}{2\mathfrak{A}_n(\underline{a})}\sum_{i=1}^{n}\tau_{in}^2\Big),$$

or

$$n\big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\big) = \frac{1+o(1)}{2\mathfrak{A}_n(\underline{a})}\sum_{i=1}^{n}\tau_{in}^2,$$

and so by (15) $\lim_{n\to\infty} n\big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\big) = \infty$.

(γ) Assume, without loss in generality, that $\lim_{n\to\infty} a_n = \infty$, then by 3.2.2 Theorem 6

$$\rho\big(\{1,\dots,n\}\big) \ge \rho\big(\{1,n\}\big) = a_1 + a_n - \sqrt{a_1 a_n},$$

and so $\lim_{n\to\infty} n\big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\big) = \infty$.

(δ) Let $\underline{\beta} = \liminf_{n\to\infty} a_n$, $\overline{\beta} = \limsup_{n\to\infty} a_n$ and $\epsilon = (\overline{\beta} - \underline{\beta})/3$; further suppose that $0 < a_n \le \lambda, n \in \mathbb{N}^*$.
Now choose two sequences of integers $\underline{p}$, $\underline{q}$ such that if $i \in \mathbb{N}^*$ then

$$p_i < q_i < p_{i+1}; \quad a_{p_i} < \underline{\beta} + \epsilon; \quad a_{q_i} > \overline{\beta} - \epsilon.$$

Then $a_{p_i} - a_{q_i} > \epsilon$, $i \in \mathbb{N}^*$, and so

$$\rho\big(\{p_1, q_i\}\big) = a_{p_i} + a_{q_i} - 1\sqrt{a_{p_1} a_{q_i}} > \frac{\epsilon^2}{4\lambda}. \tag{20}$$

So by 3.2.2 Theorems 6 and 7, and (20)

$$\begin{aligned}\rho\big(\{1,\dots,q_i\}\big) \ge&\rho\big(\{p_1,q_1,\dots,p_i,q_i\}\big)\\ \ge&\rho\big(\{p_1,q_1,\dots,p_{i-1},q_{i-1}\}\big) + \rho\big(\{p_i,q_i\}\big)\ \dots\dots\\ &\dots\dots \ge \sum_{k=1}^{i}\rho\big(\{p_k,q_k\}\big) \ge \frac{i\epsilon^2}{\lambda};\end{aligned}$$

and so $\lim_{n\to\infty} n\big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\big) = \infty$. □

REMARK (i) Everitt's original proof of case (γ), [*Everitt 1967*], contained an error that was pointed out by Diananda in his review of this paper, [*Diananda 1969*]. Diananda later supplied a correct proof; [*Everitt 1969*].

REMARK (ii) It might be noted that this theorem gives some information on an upper bound for $\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})$; this topic is taken up in more detail later; see 4 below.

A completely different limit is that given by the following result of Alzer, [*Alzer 1994a*]. Let $\underline{k} = \{1, 2, \ldots, k\}$, $k = 1, 2, \ldots$, then

$$\lim_{n\to\infty}\left(n\frac{\mathfrak{A}_n(\underline{n})}{\mathfrak{G}_n(\underline{n})} - (n-1)\frac{\mathfrak{A}_{n-1}(\underline{n-1})}{\mathfrak{G}_{n-1}(\underline{n-1})}\right) = \frac{e}{2}.$$

The author has given other results for the means of this special sequence; [*DI p.18*], [*Alzer 1994b,c,d, 1995b*].

3.4 NANJUNDIAH'S INEQUALITIES In this section we establish a very interesting extension of (GA), inequality (31) below.
Given two sequences $\underline{a}, \underline{w}$ let us write

$$\underline{\mathfrak{A}}(\underline{a};\underline{w}) = \underline{\mathfrak{A}} = \{\mathfrak{A}_1(\underline{a};\underline{w}), \mathfrak{A}_2(\underline{a};\underline{w}), \ldots\ldots\};$$
$$\underline{\mathfrak{G}}(\underline{a};\underline{w}) = \underline{\mathfrak{G}} = \{\mathfrak{G}_1(\underline{a};\underline{w}), \mathfrak{G}_2(\underline{a};\underline{w}), \ldots\ldots\},$$

for the sequences of arithmetic and geometric means.
Regarding $\underline{a} \mapsto \underline{\mathfrak{A}}$, $\underline{a} \mapsto \underline{\mathfrak{G}}$ as two maps of sequences into sequences Nanjundiah's ingenious idea was to define inverse mappings as follows:[17]

$$\mathfrak{A}_n^{-1}(\underline{a};\underline{w}) = \frac{W_n}{w_n}a_n - \frac{W_{n-1}}{w_n}a_{n-1}, \quad \mathfrak{G}_n^{-1}(\underline{a};\underline{w}) = \frac{a_n^{W_n/w_n}}{a_{n-1}^{W_{n-1}/w_n}}, \ n \in \mathbb{N}^*.$$

These will be called, respectively, *the sequences of inverse arithmetic and geometric means of $\underline{a}$ with weight $\underline{w}$*, and we will use the above notation to write

$$\underline{\mathfrak{A}}^{-1}(\underline{a};\underline{w}) = \underline{\mathfrak{A}}^{-1} = \{\mathfrak{A}_1^{-1}(\underline{a};\underline{w}), \mathfrak{A}_2^{-1}(\underline{a};\underline{w}) \ldots\},$$
$$\underline{\mathfrak{G}}^{-1}(\underline{a};\underline{w}) = \underline{\mathfrak{G}}^{-1} = \{\mathfrak{G}_1^{-1}(\underline{a};\underline{w}), \mathfrak{G}_2^{-1}(\underline{a};\underline{w}) \ldots\}.$$

As usual in case of equal weights reference to the weights will be omitted in these inverse means.
These inverse means have the elementary properties (Co), (Ho) and (Re) of the original means; see 1.1 Theorem 2. In addition the obvious analogue of 1.2(8) holds, the inverse arithmetic mean is (Ad) and the inverse geometric mean has the property 1.1(9).

LEMMA 10 *(a) With the above notations*

$$\mathfrak{A}_n(\underline{\mathfrak{A}}^{-1};\underline{w}) = \mathfrak{A}_n^{-1}(\underline{\mathfrak{A}};\underline{w}) = \mathfrak{G}_n(\underline{\mathfrak{G}}^{-1};\underline{w}) = \mathfrak{G}_n^{-1}(\underline{\mathfrak{G}};\underline{w}) = a_n, \ n \in \mathbb{N}^*.$$

(b) If $n > 1$,

$$\mathfrak{G}_n^{-1}(\underline{a};\underline{w}) \ge \mathfrak{A}_n^{-1}(\underline{a};\underline{w}), \tag{21}$$

[17] Remember that W_0=0; see Notations 6(xi). See earlier 2.4.3 proof (xx), 3.1 Theorem 1 proof (v).

with equality only if $a_{n-1} = a_n$.

(c) If $n > 1$ *then*

$$\mathfrak{G}_n^{-1}(\underline{a};\underline{w}) + \mathfrak{G}_n^{-1}(\underline{b};\underline{w}) \geq \mathfrak{G}_n^{-1}(\underline{a}+\underline{b};\underline{w}), \tag{22}$$

with equality only if $a_n b_{n-1} = a_{n-1} b_n$.

□ (a) Elementary.

(b) If we put $\alpha = W_n/w_n$ then $\alpha > 1$, and if further we put $a = a_n$, $b = a_{n-1}$, (21) becomes (B) in the form I 2.1(2). Alternatively we can note that $\mathfrak{A}_n^{-1}$ and $\mathfrak{G}_n^{-1}$ are arithmetic and geometric means with general weights and use 3.7 Theorem 23.

(c) By the elementary properties noted above and (b) we have

$$\begin{aligned}\frac{\mathfrak{G}_n^{-1}(\underline{a};\underline{w}) + \mathfrak{G}_n^{-1}(\underline{b};\underline{w})}{\mathfrak{G}_n^{-1}(\underline{a}+\underline{b};\underline{w})} &= \mathfrak{G}_n^{-1}(\underline{a}/(\underline{a}+\underline{b});\underline{w}) + \mathfrak{G}_n^{-1}(\underline{b}/(\underline{a}+\underline{b});\underline{w})\\ &\geq \mathfrak{A}_n^{-1}(\underline{a}/(\underline{a}+\underline{b});\underline{w}) + \mathfrak{A}_n^{-1}(\underline{b}/(\underline{a}+\underline{b});\underline{w})\\ &= \mathfrak{A}_n^{-1}(\underline{a}/(\underline{a}+\underline{b}) + \underline{b}/(\underline{a}+\underline{b});\underline{w}) = 1.\end{aligned}$$

The case of equality follows from that of (b)

□

REMARK (i) Clearly (a) justifies the name inverse mean given above.

REMARK (ii) Inequality (21) is an analogue of (GA) for these means and can be used to prove both (R) and (P). In fact proof (v) of 3.1 Theorem 1 is just two applications of (21) with $\underline{a}$ replaced by $\underline{\mathfrak{A}}$, and then by $\underline{\mathfrak{G}}$;

$$\begin{aligned}\left(\frac{\mathfrak{A}_n^{W_n}(\underline{a};\underline{w})}{\mathfrak{A}_{n-1}^{W_{n-1}}(\underline{a};\underline{w})}\right)^{1/w_n} &= \mathfrak{G}_n^{-1}(\underline{\mathfrak{A}};\underline{w}) \geq \mathfrak{A}_n^{-1}(\underline{\mathfrak{A}};\underline{w}) = a_n = \mathfrak{G}_n^{-1}(\underline{\mathfrak{G}};\underline{w})\\ &\geq \mathfrak{A}_n^{-1}(\underline{\mathfrak{G}};\underline{w}) = \frac{W_n}{w_n}\mathfrak{G}_n(\underline{a};\underline{w}) - \frac{W_{n-1}}{w_n}\mathfrak{G}_{n-1}(\underline{a};\underline{w}).\end{aligned}$$

REMARK (iii) Inequality (22) is an analogue of III 3.1.3(10), and will be used to give a proof of that result, III 3.1.1 Theorem 8.

It is natural to consider what happens when the means in Lemma 10 (a) are interchanged.

LEMMA 11 *If* $n > 1$ *and if* $W_n a_n$, $n \in \mathbb{N}^*$, *is strictly increasing, and* W_n/w_n, $n \in \mathbb{N}^*$ *is increasing then*

$$\mathfrak{G}_n^{-1}\left(\underline{\mathfrak{A}}^{-1};\underline{w}\right) \geq \mathfrak{A}_n^{-1}\left(\underline{\mathfrak{G}}^{-1};\underline{w}\right), \tag{23}$$

with equality only if $a_{n-2} = a_{n-1} = a_n$.

□ On writing out (23) we have to prove that

$$\frac{\left(\dfrac{W_n}{w_n}a_n - \dfrac{W_{n-1}}{w_n}a_{n-1}\right)^{W_n/w_n}}{\left(\dfrac{W_{n-1}}{w_{n-1}}a_{n-1} - \dfrac{W_{n-2}}{w_{n-1}}a_{n-2}\right)^{W_{n-1}/w_n}} \geq \frac{W_n}{w_n}\left(\frac{a_n^{W_n/w_n}}{a_{n-1}^{W_{n-1}/w_n}}\right) - \frac{W_{n-1}}{w_n}\left(\frac{a_{n-1}^{W_{n-1}/w_{n-1}}}{a_{n-2}^{W_{n-2}/w_{n-1}}}\right). \tag{24}$$

Rewriting (24) with a simpler notation what we have to show is that

$$\frac{(ra-(r-1)c)^r}{(qc-(q-1)b)^{r-1}} \geq r\frac{a^r}{c^{r-1}} - (r-1)\frac{c^q}{b^{q-1}}, \tag{25}$$

and the conditions of the lemma imply the restrictions $ra > (r-1)c$, $qc > (q-1)b$, $r > 1$, $q \geq 1$, and $r \geq q$.

The case $q = 1$, which occurs if $n = 2$, is an immediate consequence of ($\sim$J), I 4.2 Remark (i), and the convexity of $x^r, x > 0$, $r \geq 1$. So we now assume that $q > 1$. Let us define β by $rc - (r-1)\beta = qc - (q-1)b$, when the left hand side of (25) becomes

$$\frac{(ra-(r-1)c)^r}{(rc-(r-1)\beta)^{r-1}}, \tag{26}$$

which by (22) is greater than or equal to

$$\frac{(ra)^r}{(rc)^{r-1}} - \frac{((r-1)c)^r}{((r-1)\beta)^{r-1}}. \tag{27}$$

Now by (GA) $\beta = \dfrac{r-q}{r-1}c + \dfrac{q-1}{r-1}b \geq \left(c^{r-q}b^{q-1}\right)^{1/(r-1)}$, or equivalently

$$\frac{c^r}{\beta^{r-1}} \leq \frac{c^q}{b^{q-1}}. \tag{28}$$

Collecting (27), (26) and (25) we have proved (24).

For equality in the use of (GA) we need that $b = c$, when $\beta = c$. For equality in the application of (22) we then need $a = c$. This completes the proof of the lemma. □

Remark (iv) The reasons for the restrictions on $\underline{a}$ and $\underline{w}$ are clear from the proof: (i) the left hand side of (23), or equivalently (24), needs $W_n a_n$ to be strictly increasing; (ii) for β to be an arithmetic mean the weights must be positive; and

the condition W_n/w_n strictly increasing ensures that $r - q > 0$. While if W_n/w_n is increasing then the case $r = q$ means that $\beta = b$ and the discussion is simpler.

REMARK (v) When Lemma 11 is applied to different sequences $\underline{a}$ we must check that the first condition holds for those sequences. In particular, in the case of equal weights the first condition reduces to na_n being strictly increasing and the second is satisfied.

THEOREM 12 *If $n > 1$, and if W_n/w_n, $n \in \mathbb{N}^*$, is increasing then*

$$\left(\frac{\mathfrak{G}_n(\underline{\mathfrak{A}};\underline{w})}{\mathfrak{A}_n(\underline{\mathfrak{G}};\underline{w})}\right)^{W_n} \geq \left(\frac{\mathfrak{G}_{n-1}(\underline{\mathfrak{A}};\underline{w})}{\mathfrak{A}_{n-1}(\underline{\mathfrak{G}};\underline{w})}\right)^{W_{n-1}}, \tag{29}$$

with equality only if $a_n = \mathfrak{G}_{n-1}(\underline{a};\underline{w}) = \mathfrak{A}_{n-1}(\underline{\mathfrak{G}};\underline{w})$.

□ Since $W_n\mathfrak{A}_n(\underline{a};\underline{w})$ is strictly increasing we can apply Lemma 11 to $\underline{\mathfrak{A}}$ to get, using Lemma 10(a),

$$\begin{aligned}\mathfrak{A}_n^{-1}\big(\underline{\mathfrak{G}}^{-1}(\underline{\mathfrak{A}};\underline{w});\underline{w}\big) \leq \mathfrak{G}_n^{-1}\big(\underline{\mathfrak{A}}^{-1}(\underline{\mathfrak{A}};\underline{w});\underline{w}\big) &= \mathfrak{G}_n^{-1}(\underline{a};\underline{w})\\ &= \mathfrak{A}_n^{-1}\big(\underline{\mathfrak{A}}(\underline{\mathfrak{G}}^{-1};\underline{w});\underline{w}\big).\end{aligned}$$

In other words $W_n\big(\mathfrak{A}_n(\underline{\mathfrak{G}}^{-1};\underline{w}) - \mathfrak{G}_n^{-1}(\underline{\mathfrak{A}};\underline{w})\big)$ is an increasing sequence.
This implies that

$$\mathfrak{A}_n(\underline{\mathfrak{G}}^{-1};\underline{w}) \geq \mathfrak{G}_n^{-1}(\underline{\mathfrak{A}};\underline{w}). \tag{30}$$

In (30) replace $\underline{a}$ by $\underline{\mathfrak{G}}$ to get

$$\mathfrak{G}_n^{-1}\big(\underline{\mathfrak{A}}(\underline{\mathfrak{G}};\underline{w});\underline{w}\big) \leq \mathfrak{A}_n(\underline{a};\underline{w}) = \mathfrak{G}_n^{-1}\big(\underline{\mathfrak{G}}(\underline{\mathfrak{A}};\underline{w});\underline{w}\big),$$

which completes the proof since the cases of equality follow from those of Lemma 11. □

COROLLARY 13 *If $n \geq 1$ then*

$$\mathfrak{G}_n(\underline{\mathfrak{A}};\underline{w}) \geq \mathfrak{A}_n(\underline{\mathfrak{G}};\underline{w}), \tag{31}$$

with equality only if $\underline{a}$ is constant.

□ Rearrange , if necessary so that W_n/w_n, $n \in \mathbb{N}^*$ is increasing. This makes no difference to $\left(\frac{\mathfrak{G}_n(\underline{\mathfrak{A}};\underline{w})}{\mathfrak{A}_n(\underline{\mathfrak{G}};\underline{w})}\right)^{W_n}$, which by (29) exceeds 1. □

REMARK (vi) Theorem 12 is a Popoviciu type extension of (31) and we can also prove a Rado type extension by a similar argument. All we have to do is to start the proof of Theorem 12 by applying Lemma 11 to the sequence $\underline{\mathfrak{G}}$. For this however we need an extra lemma since it is not immediate that $W_n\mathfrak{G}_n(\underline{a};\underline{w})$ is strictly increasing.

LEMMA 14 *If $W_n a_n$ is strictly increasing then so is $W_n \mathfrak{G}_n(\underline{a};\underline{w})$ provided W_n/w_n is increasing.*

□ Note that $W_n \mathfrak{G}_n(\underline{a};\underline{w}) = \mathfrak{G}_n(\underline{\alpha};\underline{w})$, where

$$\alpha_i^{w_i} = \begin{cases} \dfrac{W_i^{W_i}}{W_{i-1}^{W_{i-1}}} a_i^{w_i} & \text{if } 1 < i \le n, \\ a_1^{w_1} & \text{if } i = 1. \end{cases}$$

Then $\alpha_i = W_i a_i \beta_i$, where

$$\beta_i = \begin{cases} \left(1 + \dfrac{w_i}{W_{i-1}}\right)^{W_{i-1}/w_i} & \text{if } 1 < i \le n, \\ 1 & \text{if } i = 1. \end{cases}$$

By hypothesis W_n/w_n is increasing and so β_n is increasing by I 1.2.2(11). Hence α_n is strictly increasing and so therefore is $W_n G_n(\underline{a};\underline{w})$. □

THEOREM 15 *If $n > 1$, $W_n a_n$, $n \in \mathbb{N}^*$, strictly increasing, and W_n/w_n, $n \in \mathbb{N}^*$, is increasing then*

$$W_n \left(\mathfrak{G}_n(\underline{\mathfrak{A}};\underline{w}) - \mathfrak{A}_n(\underline{\mathfrak{G}};\underline{w})\right) \ge W_{n-1} \left(\mathfrak{G}_{n-1}(\underline{\mathfrak{A}};\underline{w}) - \mathfrak{A}_{n-1}(\underline{\mathfrak{G}};\underline{w})\right),$$

with equality only if $a_n = \mathfrak{A}_{n-1}(\underline{a};\underline{w}) = \mathfrak{G}_{n-1}(\underline{\mathfrak{A}};\underline{w})$.

Nanjundiah used the equal weight case of (31) to prove the important classical inequality, *Carleman's inequality.* This inequality has been the object of much research; for further information see [*AI p.131; DI pp.44-45; EM2 p.25; HLP pp.249–250*], [*Pečarić & Stolarsky*]. Other proofs of this inequality are given later, see 3.6 Remark (i) and IV 4.2 Remark (ii).

COROLLARY 16 [CARLEMAN'S INEQUALITY] *If $\underline{a}$ is a non-null sequence then*

$$\mathfrak{A}_n\left(\underline{\mathfrak{G}}(\underline{a})\right) < e\,\mathfrak{A}_n(\underline{a}), \qquad \text{or} \quad \sum_{i=1}^{n} \mathfrak{G}_n(\underline{a}) < e \sum_{i=1}^{n} a_i, \; n \in \mathbb{N}^*;$$

if further $\sum_{i=1}^{\infty} a < \infty$ then

$$\sum_{i=1}^{\infty} \mathfrak{G}_n(\underline{a}) < e \sum_{i=1}^{\infty} a_i;$$

the constant is best possible.

□ The equal weight case of (31) can be written as $\sum_{i=1}^{n} \mathfrak{G}_n(\underline{a}) \le \frac{n}{\sqrt[n]{n!}} \mathfrak{G}_n(\underline{s})$, where $\mathfrak{G}_n(\underline{s})$ is the geometric mean of the first n terms of the sequence $\underline{s} = \{a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots\}$. However $n/\sqrt[n]{n!} < e$, I 2.2, and so

$$\sum_{i=1}^{n} \mathfrak{G}_n(\underline{a}) < e\,\mathfrak{G}_n(\underline{s}) < e \sum_{i=1}^{n} a_i.$$

The rest follows as in [*HLP*]. □

REMARK (vii) All the results of this section are stated in [*Nanjundiah 1952*], and proved in the author's unpublished thesis that was communicated privately to the author who then published Nanjundiah's proofs; [*Bullen 1996b*]. In the meantime, based on numerical results, the equal weights case of (31) was conjectured. A proof of this conjecture was given by Kedlaya, [*Kedlaya 1994*]. After the appearance of this paper several other papers appeared, all apparently without knowing of Nanjundiah's work. In these papers alternative proofs were given for Kedlaya's result and also of the general inequality (31); further various extensions were made; [*Holland; Kedlaya 1999; Matsuda; Mond & Pečarić 1996a,b; Pečarić 1995a*], see III 3.2.6, VI 5.

REMARK (viii) These results can be regarded as mixed mean results; this topic is taken up in III 5.3.

3.5 KOBER-DIANANDA INEQUALITIES Throughout this section $\underline{a}$ is an n-tuple that is non-constant and non-negative, $\underline{w}$ is a positive n-tuple with $W_n = 1$. Further we will use the following notations:

$$w = \min \underline{w}; \qquad W = \max \underline{w};$$

$$\Delta_n = \mathfrak{A}_n(\underline{a}; \underline{w}) - \mathfrak{G}_n(\underline{a}; \underline{w}); \quad D_n = \mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a});$$

$$\Sigma_n = \sum_{1 \le i < j \le n} w_i w_j (\sqrt{a_i} - \sqrt{a_j})^2; \quad S_n = \sum_{1 \le i < j \le n} (\sqrt{a_i} - \sqrt{a_j})^2.$$

It is worthwhile noting some other forms for Σ_n and S_n.

$$\Sigma_n = \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j (\sqrt{a_i} - \sqrt{a_j})^2 = \sum_{i=1}^{n} w_i a_i - \sum_{i,j=1}^{n} w_i w_j \sqrt{a_i a_j}; \tag{32}$$

$$S_n = \frac{1}{2} \sum_{i,j=1}^{n} (\sqrt{a_i} - \sqrt{a_j})^2 = (n-1) \sum_{i=1}^{n} a_i - 2 \sum_{1 \le i < j \le n} \sqrt{a_i a_j}. \tag{33}$$

THEOREM 17 *(a) With the above assumptions and notations*

$$\frac{w}{n-1} \le \frac{\Delta_n}{S_n} \le W; \tag{34}$$

$$\frac{1}{1-w} \le \frac{\Delta_n}{\Sigma_n} \le \frac{1}{w}. \tag{35}$$

(b) There is equality on the left-hand sides of (34) and (35) only if either $n = 2$, or $n > 2$ and the a_i with minimum weight is not zero, and the rest of the $\underline{a}_i$ are all zero.

(c) There is equality on the right-hand side of (35) only if either $n = 2$, or $n > 2$ and the a_i with minimum weight is zero, and the rest of the $\underline{a}_i$ are all equal and non-zero. The upper bound of (34) is in general not attained.

□ *(i) The left-hand side of (34).* Using (33) we see that,

$$\begin{aligned}\Delta_n - \frac{w}{n-1}S_n &= \sum_{i=1}^n w_i a_i - w\sum_{i=1}^n a_i + \frac{2w}{n-1}\sum_{1\le i<j\le n}^n \sqrt{a_i a_j} - \mathfrak{G}_n(\underline{a};\underline{w}) \\ &= \sum_{i=2}^n (w_i - w)a_i + \frac{2w}{n-1}\sum_{1\le i<j\le n}^n \sqrt{a_i a_j} - \mathfrak{G}_n(\underline{a};\underline{w}),\end{aligned}$$

where we have assumed without loss in generality that $w_1 = w$.

Now the two sums in the last line contain terms involving the a_i, $2 \le i \le n$, and the $\sqrt{a_i a_j}$, $1 \le i < j \le n$, with weights that sum to 1; that is $\sum_{i=2}^n (w_i - w) + \left(\frac{n(n-1)}{2}\right)\left(\frac{2w}{n-1}\right) = 1$. Hence applying (GA) we find that these sums exceed $\mathfrak{G}_n(\underline{a};\underline{w})$, so that $\Delta_n - \frac{w}{n-1}S_n \ge 0$, which is equivalent to the left-hand side of (34).

Further by the case of equality in (GA) this inequality is strict, the $\underline{a}$ not being constant, unless $n = 2$, or if $n > 2$, $a_1 \ne 0$ and all the other a_i, $2 \le i \le n$, are zero.

(ii) The left-hand side of (35). The proof in this case is similar to that of *(i)*, but using (32).

$$\begin{aligned}\Delta_n - \frac{1}{1-w}\Sigma_n &= \frac{1}{1-w}\Big(\sum_{i,j=1}^n w_i w_j\sqrt{a_i a_j} - w\sum_{i=1}^n w_i a_i\Big) - \mathfrak{G}_n(\underline{a};\underline{w}) \\ &= \frac{1}{1-w}\Big(\sum_{\substack{i,j=1\\ i\ne j}}^n w_i w_j\sqrt{a_i a_j} + \sum_{i=2}^n w_i(w_i - w)a_i\Big) - \mathfrak{G}_n(\underline{a};\underline{w}),\end{aligned}$$

where again, we have assumed without loss in generality that $w = w_1$.

As in (i) the sums form an arithmetic mean, and an application of (GA) shows that $\Delta_n - \dfrac{1}{1-w}\Sigma_n \geq 0$, which is equivalent to the left-hand side of (35). The cases of equality follow as in (i) .

(iii) *The right-hand side of (35).* Using (32) this is equivalent to proving

$$\Phi_n = \Phi_n(\underline{a}) = \frac{1}{2}\sum_{i,j=1}^{n} w_i w_j\big(\sqrt{a_i} - \sqrt{a_j}\big)^2 - w\big(\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})\big) \geq 0. \qquad (36)$$

The proof of (36) is by induction on n and since the case $n = 1$ is obvious assume the result for all k, $1 \leq k < n$. Assume, without loss in generality that $a_n = \min \underline{a}$, put $\tilde{w} = \min\{w_1, \ldots, w_{n-1}\}$, and $\gamma = \mathfrak{G}_{n-1}(\underline{a};\underline{w}')$, where $\underline{w}' = \big(\dfrac{w_1}{1-w_n}, \ldots, \dfrac{w_{n-1}}{1-w_n}\big)$. Then

$$\begin{aligned}\Phi_n(\gamma, \gamma, \ldots, \gamma, a_n) = (w_n - w)(1-w_n)\gamma + w_n(1 - w - w_n)a_n + w\gamma^{1-w_n}a_n^{w_n}\\ - 2w_n(1-w_n)\sqrt{\gamma a_n}\end{aligned}$$

and by (GA),

$$\Phi_n(\gamma, \gamma, \ldots, \gamma, a_n) \geq 0. \qquad (37)$$

Now if $\underline{a}' = \big(a_1, \ldots, a_{n-1}, \mathfrak{A}_{n-1}(\underline{a};\underline{w}')\big)$, and

$$\Phi_{n-1}(\underline{a}') = \frac{1}{2}\sum_{i,j=1}^{n-1} w_i' w_j'\big(\sqrt{a_i} - \sqrt{a_j}\big)^2 - \frac{\tilde{w}}{1-w_n}\Big(\mathfrak{A}_{n-1}(\underline{a};\underline{w}') - \gamma\Big),$$

then by the induction hypothesis

$$\Phi_{n-1}(\underline{a}') \geq 0. \qquad (38)$$

Further

$$\begin{aligned}\Phi_n(\underline{a}) - &\Phi_n(\gamma, \gamma, \ldots, \gamma, a_n) - (1-w_n)^2\Phi_{n-1}(\underline{a}')\\ =&(\tilde{w} - w + w_n)\Big(\sum_{i=1}^{n-1} w_i a_i - (1-w_n)\gamma\Big)\\ &\qquad - 2w_n\sqrt{a_n}\Big(\sum_{i=1}^{n-1} w_i\sqrt{a_i} - (1-w_n)\sqrt{\gamma}\Big).\end{aligned}$$

Now by (GA) both of the terms in the brackets are non-negative. Hence using (37) and (38)

$$\begin{aligned}\Phi_n(\underline{a}) \geq& w_n\Big(\sum_{i=1}^{n-1} w_i a_i - (1-w_n)\gamma\Big) - 2w_n\sqrt{\gamma}\Big(\sum_{i=1}^{n-1} w_i\sqrt{a_i} - (1-w_n)\sqrt{\gamma}\Big),\\ =& w_n\sum_{i=1}^{n-1} w_i(\sqrt{a_i} - \sqrt{\gamma})^2 \geq 0. \end{aligned} \qquad (39)$$

This completes the proof.

Further there is equality in (39) if and only if $a_i = \cdots = a_{n-1} = \gamma$. Inequalities (37) and (39) are equalities together if and only if $w = w_n$, $a_n = 0$ and $a_1 = \cdots = a_{n-1} > 0$.

(iv) The right-hand side of (34). We can write $S_n = nD_n + F_n$ where $F_n = n^2\Phi_n$, and Φ_n is as above but with equal weights.

Hence by (39)

$$\frac{D_n}{S_n} \le \frac{1}{n}. \tag{40}$$

Further, assuming without loss in generality that $w_n = W$,

$$D_n - \frac{\Delta_n}{nW} = \frac{1}{nW}\sum_{i=1}^{n-1}(W - w_i)a_i + \frac{1}{nW}\mathfrak{G}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a}) \ge 0,$$

by (GA). So

$$\frac{\Delta_n}{D_n} \le nW. \tag{41}$$

Combining (40) and (41) completes the proof of this case.

The final remark in (c) is shown by the following example. If we take $n = 3$ and $w_1 = w_2 = 1/4, w_3 = 1/2$, then the upper bound of Δ_3/S_3 occurs if either $a_1 = 0, a_2 = (\sqrt{3} - 1)^2 a_3$, or $a_2 = 0, a_1 = (\sqrt{3} - 1)^2 a_3$, when

$$\frac{\Delta_3}{S_3} \le \frac{1}{4}(1 + \frac{1}{\sqrt{3}}) < W = \frac{1}{2}.$$

If we put $r = 2(1 + \frac{1}{\sqrt{3}})$ what has to be shown is that

$$(r-2)a_3 + \big(4\sqrt[4]{a_1a_2} - r(\sqrt{a_1} + \sqrt{a_2})\big)\sqrt{a_3} + (r-1)(a_1 + a_2) - r\sqrt{a_1a_2} \ge 0. \tag{42}$$

This holds, since the coefficient of a_3 is not positive, if and only if

$$\big(4\sqrt[4]{a_1a_2} - r(\sqrt{a_1} + \sqrt{a_2})\big)^2 - 4(r-2)\big((r-1)(a_1 + a_2) - r\sqrt{a_1a_2}\big) \le 0.$$

Since $3r^2 - 12r + 8 = 0$ this is equivalent to

$$\sqrt[4]{a_1a_2}(\sqrt{a_1} + \sqrt{a_2}) \ge 2\sqrt{a_1a_2}, \tag{43}$$

which is obviously true. For simultaneous equality in (42) and (43) we must have that $2(r-2)\sqrt{a_3} = -4\sqrt{a_1a_2} + r(\sqrt{a_1} + \sqrt{a_2})$ and $a_1 = 0$, or $a_2 = 0$, or $a_1 = a_2$, and since not all of a_1, a_2, a_3 are equal this completes the proof. □

REMARK (i) The result (34) is due to Kober; Diananda proved (35) and gave the above simplified proof of Kober's result; [*Diananda 1963a,b; Kober*]. A multiplicative analogue of Theorem 16 has been given, where the differences Δ_n, D_n are replaced by ratios; [*Bullen 1967*].

REMARK (ii) Both Kober and Diananda gave consideration to the behaviour of the ratios Δ_n/S_n, and Δ_n/Σ_n as $\underline{a}$ tends to a constant n-tuple; in particular if the limit is not the zero n-tuple then Δ_n/Σ_n tends to 2.

REMARK (iii) The following inequality can be found in [*Beck 1969*a]; see also [*Motzkin 1965b*].

$$\frac{(\min \underline{a})^3}{2} \sum_{1\le i\le j\le n} w_i w_j \Big(\frac{1}{a_i} - \frac{1}{a_j}\Big)^2 \le \Delta_n \le \frac{\mathfrak{G}_n(\underline{a};\underline{w})}{2(\min \underline{a})^2} \sum_{1\le i,j\le n} w_i w_j (a_i - a_j)^2.$$

REMARK (iv) Some of these inequalities are related to the mixed mean inequalities of III 5.3; see [*Mitrinović & Pečarić 1988b*]. In particular the inequality $F_n \ge 0$, see the equal weight case of (39), is equivalent to

$$(n-2)\mathfrak{M}_n(1,0;1;\underline{a}) + \mathfrak{M}_n(1,0;n;\underline{a}) \ge \mathfrak{M}_n(1,0;2;\underline{a}),$$

using the notation of III 5.3

REMARK (v) The inequality (39), $\Phi_n \ge 0$, or equivalently

$$\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w}) \le \frac{1}{2w}\sum_{i,j=1}^{n} w_i w_j \big(\sqrt{a_i} - \sqrt{a_j}\big)^2$$

and the above inequality of Beck, are examples of converse inequalities, a topic discussed in more detail later, see 4 below.

REMARK (vi) Alzer has refined the right-hand side of (34); see [*Alzer 1992*a].

REMARK (vii) Aczél, in his review of the above paper by Alzer, [*Zbl., 0776. 26013*], pointed out that the Kober-Diananda inequalities are helpful in situations where the geometric mean is needed but the arithmetic mean is easier to calculate; [*Aczél 1991*].

3.6 REDHEFFER'S RECURRENT INEQUALITIES A very interesting general method for obtaining inequalities is in [*Redheffer 1967, 1981*]. The method leads to certain refinements of (GA) but has much wider applications.

Definition 18 *Let I_k, $1 \le k \le n$, be intervals, and $f_k, g_k : \prod_{i=1}^{k} I_i \mapsto \mathbb{R}$, $\mu_k \in \mathbb{R}$, $1 \le k \le n$, then*

$$\sum_{i=1}^{n} \mu_i f_i \le \sum_{i=1}^{n} g_i \tag{44}$$

is called a recursible or recurrent inequality *if there exist functions $F_k : \mathbb{R} \mapsto \mathbb{R}$, $1 \le k \le n$, such that*

$$F_k(\mu) f_{k-1}(a_i, \ldots, a_{k-1}) = \sup_{a_k \in I_k} \{\mu f_k(a_i, \ldots, a_k) - g_k(a_i, \ldots, a_k)\},$$

where f_0 is defined to be 1.

A good motivation for this definition is given in the first reference.

Theorem 19 *The recurrent inequality (44) holds for all $\underline{a} \in \prod_{i=1}^{n} I_i$ if and only if there exist $\delta_k \in \mathbb{R}$, $1 \le k \le n+1$, $\delta_{n+1} = 0$, $\delta_1 \le 0$, such that*

$$\mu_k = F_k^{-1}(\delta_k) - \delta_{k+1}, \quad 1 \le k \le n;$$

where for each k, $1 \le k \le n$, $F_k^{-1}(\delta)$ being any one of the solutions of $F_k(\mu) = \delta$.

□ The proof is by induction on n; since the case $n = 1$ is obvious let us assume the result for all k, $1 \le k < n$.
The inequality will hold for $k = n$ if and only if it holds for the unfavourable choices of a_n, assuming $a_1, \ldots, a_{n-1}$ fixed.
Since the inequality is recurrent we get, using the induction hypothesis, the relations of the theorem for k, $1 \le k \le n-2$ and also

$$F_n(\mu_n) + \mu_{n-} = F_{n-1}^{-1}(\delta_{n-1});$$

and defining $\delta_n = F_n(\mu_n)$, completes the proof. □

As an application we give the following result.

Corollary 20 *If β_k, $1 \le k \le n+1$, are non-negative, $\beta_1 \le 1, \beta_{n+1} = 0$, and λ_k, $1 \le k \le n$, positive then*

$$\sum_{k=1}^{n} k\Big((\lambda_k\beta_k)^{1/k} - \beta_{k+1}^{1/k}\Big)\mathfrak{G}_k(\underline{a}) \le \sum_{k=1}^{n} \lambda_k a_k. \tag{45}$$

□ The inequality $\sum_{k=1}^{n} \mu_k \mathfrak{G}_k(\underline{a}) \le \sum_{k=1}^{n} \lambda_k a_k$ is recurrent.
To see this consider

$$\mu\mathfrak{G}_k(\underline{a}) - \lambda_k a_k = (\mu t - \lambda_k t^k)\mathfrak{G}_{k-1}(\underline{a}),$$

where $t^k = a_k/\mathfrak{G}_{k-1}(\underline{a})$, $k > 1$. We then obtain that, $k > 1$,

$$F_k(\mu) = \begin{cases} (k-1)\lambda_k^{-1/(k-1)}\left(\dfrac{\mu}{k}\right)^{k/(k-1)}, & \text{if } \mu \geq 0, \\ 0, & \text{if } \mu < 0; \end{cases}$$

and $F_1(\mu) = 0$, $\mu \leq \lambda_1$, $F_1(\mu) = \infty$, $\mu > \lambda_1$.
Since $F_k(\mu) \geq 0$ implies that $\delta_k \geq 0$, put $\delta_k = (k-1)\beta_k^{1/(k-1)}$ to get that

$$\mu_k = k\Big((\lambda_k\beta_k)^{1/k} - \beta_{k+1}^{1/k}\Big),$$

where the $\beta_k, 1 \leq k \leq n-1$, are as stated. □

Corollary 21 *If* $\mathfrak{A}_n(\underline{\mathfrak{G}}) = \mathfrak{A}_n(\mathfrak{G}_1(\underline{a}), \mathfrak{G}_2(\underline{a}), \ldots, \mathfrak{G}_n(\underline{a}))$ *then*

$$\mathfrak{A}_n(\underline{\mathfrak{G}}) \exp\left(\frac{\mathfrak{G}_n(\underline{a})}{\mathfrak{A}_n(\underline{\mathfrak{G}})}\right) \leq e\mathfrak{A}_n(\underline{a}), \tag{46}$$

and

$$1 \leq \frac{1}{2}\left(\frac{\mathfrak{A}_n(\underline{\mathfrak{G}})}{\mathfrak{G}_n(\underline{a})} + \frac{\mathfrak{G}_n(\underline{a})}{\mathfrak{A}_n(\underline{\mathfrak{G}})}\right) \leq \frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{G}_n(\underline{a})}, \tag{47}$$

with equality if and only if $\underline{a}$ *is constant.*

□ Apply Corollary 20 with $\beta_k = e^{t(k-1)}$, $t > 0$, $1 \leq k \leq n$. By the mean-value theorem of differentiation, see I 2.1 Footnote 1,

$$k(\beta_k^{1/k} - \beta_{k+1}^{1/k}) > -te^t,$$

so, taking $\lambda_k = 1$, $1 \leq k \leq n$, (45) leads to

$$\mathfrak{G}_n(\underline{a}) < \mathfrak{A}_n(\underline{a})e^{-t} + tA_n(\underline{\mathfrak{G}}). \tag{48}$$

The best choice of t is $\big(\mathfrak{G}_n(\underline{a}/A_n(\underline{\mathfrak{G}})\big) - 1$, which gives (46).
Now assume, without loss in generality that, $\underline{a}$ is increasing, and is not constant; then $0 \leq \mathfrak{G}_1(\underline{a}) \leq \mathfrak{G}_2(\underline{a}) \leq \cdots \leq \mathfrak{G}_n(\underline{a})$, and $A_n(\underline{\mathfrak{G}}) \leq \mathfrak{G}_n(\underline{a})$. So (47) follows from (46) using the inequality in I 2.2 Remark (ii) and (20). □

Remark (i) Both of the inequalities (46) and (47) sharpen (GA). Since the exponential term on the left-hand side of (46) is bigger than 1, (46) implies Carleman's inequality, 3.4 Corollary 16; see also [*Alzer 1993c*].

Remark (ii) A direct proof of (48) is given in [*Bullen 1969c*] using methods employed earlier in this chapter. Unfortunately the method requires that $0 \leq t \leq 2$ and so does not give (46).

3.7 The Geometric Mean-Arithmetic Mean Inequality with General Weights Clearly if all the weights are negative nothing new occurs since we only use the positive ratios $w_i/W_n, 1 \le i \le n$. Further if we allow non-negative, or non-positive, weights, with of course not all being zero, the effect is to reduce the value of n, and to change the condition of equality from "$\underline{a}$ is constant" to "$\underline{a}$ is essentially constant", see I 4.3. We will further talk of the arithmetic mean being essentially internal meaning that in 1.1 (2) the minimum and maximum is taken over the essential elements of $\underline{a}$, with analogous usages for other properties. In the case of (R), 3.1 Theorem 1, the extra generality means that either $w_n = 0$ when the inequality is trivial, or $w_n \neq 0$ when as remarked above the result includes all the cases $2 \le k \le n$ and the case of equality is unaltered.

However if both positive and negative weights occur (GA) can still hold; [*Ayoub*]. This is a consequence of 2.4.2 proof (*xvi*) which shows (GA) is just a special case of (J). Hence applying the Jensen-Steffensen theorem, I 4.3 Theorem 20, instead of (J) we get the following result.

Theorem 22 *If $n \ge 3$, $\underline{a}$ and increasing n-tuple, and $\underline{w}$ a real n-tuple satisfying*

$$W_n \neq 0, \text{ and } 0 \le \frac{W_i}{W_n} \le 1, \; 1 \le i \le n \tag{49}$$

then

$$\min \underline{a} \le \mathfrak{A}_n(\underline{a}; \underline{w}) \le \max \underline{a}, \quad \min \underline{a} \le \mathfrak{G}_n(\underline{a}; \underline{w}) \le \max \underline{a}. \tag{50}$$

and (GA) *holds with equality if and only if $\underline{a}$ is essentially constant.*

□ The proof of the left-hand inequalities in (50) is given in the preliminary remarks in the proof of I 4.3 Theorem 20; the right-hand inequalities can then be deduced in a similar manner, or by using 1.2 (8).

The proof of (GA) follows 2.4.2 proof (*xvi*), but now using the Jensen-Steffensen theorem. □

Remark (i) All the methods used to prove I 4.3 Theorem 20 can be used to obtain this result; the proofs are now simpler as this is a very special case of that theorem.

When $n = 2$ the situation is different and very elementary but is worth stating for reference; see also 2.2.2 Lemma 4(b).

Theorem 23 *If $n = 2$, $w_1w_2 < 0$, $W_2 = 1$, $\underline{a}$ a 2-tuple then (~GA) holds. More precisely, if $a_1 < a_2$ and $w_2 < 0$ then*

$$\mathfrak{A}(\underline{a}; \underline{w}) < \mathfrak{G}(\underline{a}; \underline{w}) < \min \underline{a} \tag{51}$$

while if $a_1 < a_2$ and $w_1 < 0$ then

$$\mathfrak{G}(\underline{a};\underline{w}) > \mathfrak{A}(\underline{a};\underline{w}) > \max \underline{a}. \tag{52}$$

□ This follows as a particular case of I 4.2 Remark(i) but we will give a simple direct proof. Assume, without loss in generality, that $0 < a_1 = x < a_2 = y$, and $w_1 = 1-t$, $w_2 = t$, $t \in \mathbb{R}$. Define the three functions:

$$A(t) = \mathfrak{A}(\underline{a};\underline{w}) = (1-t)x + ty, \; G(t) = \mathfrak{G}(\underline{a};\underline{w}) = x^{1-t}y^t, \quad D(t) = A(t) - G(t).$$

Now if $t < 0$ then $G(t) = x(y/x)^t < x$, which give the right-hand side of (51), while if the $1-t<0$, equivalently $t>1$ then $A(t) = (1-t)(x-y)+y > y$, giving the right-hand side of (52).
Now we easily see that $D(0) = D(1) = 0$ and $D''(t) < 0$. So D is strictly concave and hence is negative outside the interval $[0,1]$. This completes the proof. □

Remark (ii) A particular case of arithmetic and geometric means with general weights that do not satisfy (49) is discussed below in 5.8.

Example (i) A simple use of Theorem 23 is to prove inequality 2.4.6 (34); let $\underline{x}$ be an increasing n-tuple then

$$x_1 \prod_{k=2}^{n} (kx_k - \overline{k-1}x_{k-1}) \le x_1 \prod_{k=2}^{n} x_k^k x_{k-1}^{-(k-1)} = x_n^n.$$

3.8 Other Refinements of the Geometric Mean-Arithmetic Mean Inequality There are many other refinements of (GA) of which we will mention a few.

Theorem 24 [Siegel; Hunter J] *If $n \ge 2$ and $\underline{a}$ a non-constant n-tuple then:*
(a)

$$\mathfrak{G}_n(\underline{b};\underline{w})\, \mathfrak{G}_n(\underline{a};\underline{w}) \le \mathfrak{A}_n(\underline{a};\underline{w}), \tag{53}$$

where $\underline{b}$ is the n-tuple defined by $b_k = 1 + (n-k)/(t+k-1)$, $1 \le k \le n$, and t is the unique positive root of

$$\frac{1}{n!}\prod_{k=1}^{n-2}\left(\frac{t+k}{t-k}\right)^{n+k-1} \prod_{1\le i<j\le n}(a_i - a_j)^2 = \left(\prod_{i=1}^{n} a_i\right)^{n-1}; \tag{54}$$

(b)

$$\mathfrak{G}_n(\underline{a};\underline{w}) \le \sqrt[n]{\bigl(1+(n-1)t\bigr)(1-t)^{n-1}}\, \mathfrak{A}_n(\underline{a};\underline{w}), \tag{55}$$

where $0 < t < 1$, and t is a root of the equation

$$\sum_{1 \le i < j \le n} (a_i - a_j)^2 = t(n-1)\Big(\sum_{i=1}^{n} a_i\Big)^2.$$

REMARK (i) Since for all k, $1 \le k \le n-1$, $b_k > 1$ we have that $\mathfrak{G}_n(\underline{b}) > 1$, (53) is a refinement of (GA).

REMARK (ii) Eliminating $\prod_{i=1}^{n} a_i$ from (53) and (54) leads to the inequality

$$\mathfrak{A}_n^{n(n-1)}(\underline{a}) \ge \prod_{1 \le i < j \le n} (a_i - a_j)^2 \prod_{k=1}^{n} \frac{(t+n-1)^{n-1}}{(t+k-1)^{k-1}},$$

which in the case $t = 0$ is a refinement of a result of Schur; see [*Schur 1918*].

REMARK (iii) The results in [*Dinghas 1953*] follow from (55).

THEOREM 25 [FINK & JODHEIT 1976] *If $\underline{a}$ is an increasing n-tuple then*

$$\mathfrak{G}_n(\underline{a}) \le \mathfrak{A}_n(\underline{a}; \underline{w}), \tag{56}$$

where $w_i = \dfrac{W_n}{n} \displaystyle\prod_{k<i} \Big(1 - \big(1 - (\frac{a_k}{a_i})^{1/n}\big)^n\Big)$, $1 \le i \le n$, with equality if and only if $\underline{a}$ is constant.

REMARK (iv) The essence of this result is that the geometric mean with equal weights is less than the arithmetic mean with certain smaller weights.

REMARK (v) A general class of weights $\underline{w}$ for which (56) holds has been studied in [*Fink 1981*].

THEOREM 26 [SIERPIŃSKI'S INEQUALITY] *If $\underline{a}$ is an n-tuple then*

$$\frac{\mathfrak{H}_n(\underline{a})}{\mathfrak{A}_n(\underline{a})} \ge \left(\frac{\mathfrak{G}_n(\underline{a})}{\mathfrak{A}_n(\underline{a})}\right)^n \ge \left(\frac{\mathfrak{H}_n(\underline{a})}{\mathfrak{A}_n(\underline{a})}\right)^{n-1}, \tag{57}$$

with equality if $n = 1, 2$, or $n \ge 3$ and $\underline{a}$ constant.

□ The case $n = 1$ is trivial and for $n = 2$ see 2.2.1 Remark (i); so assume that $n > 2$. A simple application of 1.2(7) shows that the left-hand inequality in (57) implies the right-hand inequality, and so it is sufficient to prove that

$$\frac{\mathfrak{A}_n^{n-1}(\underline{a})\mathfrak{H}_n(\underline{a})}{\mathfrak{G}_n^n(\underline{a})} \ge 1. \tag{58}$$

The proof is by induction, and let $x = a_n$, and then put the left-hand side of (58) equal to $g(x)$, and $f = \log \circ g$

Simple calculations show that f has a unique minimum at the point $x = x'$ where $x' = \dfrac{(n-1)\mathfrak{A}_n(\underline{a}) - \mathfrak{H}_n(\underline{a})}{n-1}$; since $n > 2$ we have by (HA) that $x' > 0$. So for all $x \neq x', g(x) > g(x')$. Simple calculations, and a further application of (HA) show that $g(x') \geq \dfrac{\mathfrak{A}_{n-1}^{n-2}(\underline{a})\mathfrak{H}_{n-1}(\underline{a})}{\mathfrak{G}_{n-1}^{n-1}(\underline{a})}$ and so the result follows from the induction hypothesis. □

Remark (vi) The result is in [*Sierpiński*]; this proof is from [*Mitrinović & Vasić 1976*], and clearly gives the following Rado type extension of (58):

$$\frac{\mathfrak{A}_n^{n-1}(\underline{a})\mathfrak{H}_n(\underline{a})}{\mathfrak{G}_n^n(\underline{a})} \geq \frac{\mathfrak{A}_{n-1}^{n-2}(\underline{a})\mathfrak{H}_{n-1}(\underline{a})}{\mathfrak{G}_{n-1}^{n-1}(\underline{a})}$$

with equality if and only if $\underline{a}$ is constant.

Remark (vii) A simple proof giving the cases of equality can be found in [*Alzer 1989*a]; the same author gives a generalization of (57), [*Alzer 1989b, 1991b*].

An extension to weighted means, when the weights are decreasing, can be found in [*Pečarić & Wang; Wang C L 1979e*]. The following result generalizes these and inequality (57); [*Alzer, Ando & Nakamura*].

Theorem 27 *If $\underline{a}$ is a non-constant n-tuple, $n \geq 2$, and $\underline{w}$ is a decreasing n-tuple, strictly if $n = 2$, define*

$$f(x) = \big(\mathfrak{G}_n(\underline{a}^{1-x};\underline{w})\big)^{W_n}\big(\mathfrak{A}_n(\underline{a}^x;\underline{w})\big)^{W_{n-1}}\big(\mathfrak{H}_n(\underline{a}^x;\underline{w})\big)^{w_n}, \quad x \in \mathbb{R}.$$

If $\underline{a}$ is increasing then f is strictly increasing on $]-\infty, 0]$, while if $\underline{a}$ is decreasing f is strictly increasing on $[0, \infty[$.

In particular we get the following generalization of (58).

Corollary 28 *If $\underline{a}, \underline{w}$ are decreasing n-tuples, $n \geq 3$, then*

$$\frac{\mathfrak{A}_n^{W_{n-1}}(\underline{a};\underline{w})\mathfrak{H}_n^{w_n}(\underline{a};\underline{w})}{\mathfrak{G}_n^{W_n}(\underline{a};\underline{w})} \geq 1,$$

with equality if and only if $\underline{a}$ is constant.

□ Take $x = 0, 1$ in Theorem 27. □

Remark (viii) It is clear that it is sufficient to require that the n-tuples $\underline{a}, \underline{w}$ be similarly ordered; see I 3.3 Definition 15.

REMARK (ix) The case, $x = r, r > 0$, of Theorem 27 is given in III 6.4 Theorem 9.

Daykin & Eliezer have given a convex function whose value increases from one side of (GA) to the other, thus providing many refinements of this inequality; [*Daykin & Eliezer 1967*]. As these results follow from I 4.2 Theorem 18, we only give a later result of Chong K M.

THEOREM 29 [CHONG K M 1977] *If $\underline{a}$ is a non-constant n-tuple and*

$$A(x) = \mathfrak{A}_n\Big(\underline{a}^x \mathfrak{G}_n^{1-x}(\underline{a}; \underline{w}); \underline{w}\Big),\ 0 \le x \le 1,$$

then A is strictly increasing.

REMARK (x) This result generalizes (GA) since $A(0) = \mathfrak{G}_n(\underline{a}; \underline{w})$ and $A(1) = \mathfrak{A}_n(\underline{a}; \underline{w})$.

This result has been extended in [*Pečarić 1983–1984*] where it is shown that

$$\frac{\mathfrak{G}_n\big(x\,\mathfrak{A}_n(\underline{a}; \underline{w}) + (1-x)\underline{a}; \underline{w}\big)}{\mathfrak{A}_n\big(x\,\mathfrak{A}_n(\underline{a}; \underline{w}) + (1-x)\underline{a}; \underline{w}\big)}$$

is an increasing function: and Chong K M has given another simple function with a similar property:

$$\prod_{i=1}^{n}\Big(x\,\mathfrak{A}_n(\underline{a}; \underline{w}) + (1-x)a_i\Big)^{w_i/W_n}.$$

REMARK (xi) For further results of this type see III 2.5.4.

THEOREM 30 [WANG C L 1980D] *If $\alpha > 0, \beta \ge 0, \gamma > 0$, $\underline{a}$ and $\underline{w}$ n-tuples with $\underline{a} \le \sqrt{\gamma/\beta}$, and if $f(x) = \left(\beta x + \gamma x^{-1}\right)^\alpha$ then*

$$f\Big(\mathfrak{A}_n(\underline{a}; \underline{w})\Big) \le f\Big(\mathfrak{G}_n(\underline{a}; \underline{w})\Big) \le \mathfrak{A}_n\big(f(\underline{a}); \underline{w}\big). \tag{59}$$

□ Since f is decreasing on $[0, \sqrt{\gamma/\beta}]$, see I 2.2(f), the first inequality in (59) follows from (GA). Further $g = f \circ \exp$ is convex so the other inequality follows from (J) and 1.2(7). □

REMARK (xii) The original proof of Wang used the ideas from linear programming as well as Lagrange multipliers, see 2.4 Footnote 10; the above simple proof is due to Pečarić.

REMARK (xiii) Inequality (59) generalizes a result of Mitrinović & Djoković, see [*AI p. 282*], and reduces to (GA) if $\alpha = \gamma = 1$, $\beta = 0$.

THEOREM 31 [ROOIN 2001B] *If*

$$\alpha_{ij},\, \beta_{ij} \ge 0,\ 1 \le i,j \le n,\ \text{and}\ \sum_{i=1}^{n}\alpha_{ij} = \sum_{j=1}^{n}\alpha_{ij} = \sum_{i=1}^{n}\beta_{ij} = \sum_{j=1}^{n}\beta_{ij} = 1,$$

then

$$\mathfrak{G}_n(\underline{a}) \le \sqrt[n]{\prod_{i=1}^{n}\sum_{j=1}^{n}\big((1-t)\alpha_{ij} + t\beta_{ij}\big)a_j} \le \mathfrak{A}_n(\underline{a}),\ 1 \le t \le 1.$$

In particular

$$\mathfrak{G}_n(\underline{a}) \le \sqrt[n]{\prod_{i=1}^{n}\big((1-t)a_i + ta_{n+1} - i\big)} \le \mathfrak{A}_n(\underline{a}),\ 1 \le t \le 1.$$

□ These follow from I 4.2 Theorem 18 on taking $f(x) = -\log x$. □

A simple question that arises from (GA) is whether or not it is always true that the geometric mean is nearer to the arithmetic mean than it is to the harmonic mean. In the case of $n = 2$ and equal weights the answer is immediate since

$$\mathfrak{H}(a,b) < \mathfrak{G}(a,b) < \mathfrak{A}(a,b),\ \text{and}\ \mathfrak{H}(a,b)\mathfrak{A}(a,b) = \mathfrak{G}^2(a,b)$$

easily imply that

$$\big(\mathfrak{A}(a,b) - \mathfrak{G}(a,b)\big) - \big(\mathfrak{G}(a,b) - \mathfrak{H}(a,b)\big) = \mathfrak{A}(a,b) + \mathfrak{H}(a,b) - 2\sqrt{(\mathfrak{A}(a,b)(\mathfrak{H}(a,b)} > 0,$$

or

$$1 < \frac{\mathfrak{A}(\underline{a}) - \mathfrak{G}(\underline{a})}{\mathfrak{G}(\underline{a}) - \mathfrak{H}(\underline{a})} < \frac{\mathfrak{A}(\underline{a})}{\mathfrak{H}(\underline{a})}.$$

However if $n > 2$ it is possible for both $\mathfrak{G}_n(\underline{a})$ to be nearer to $\mathfrak{A}_n(\underline{a})$ and for $\mathfrak{G}_n(\underline{a})$ to be nearer to $\mathfrak{H}_n(\underline{a})$; see [*Scott*]. The general situation is elucidated by the following theorem of Pearce & Pečarić; [*Pearce & Pečarić 2001*], see also [*Lord*].

THEOREM 32 *If $\alpha > 0$ and $n \ge 2$ and $\underline{a}$ is a non-constant n-tuple then*

$$\frac{1}{n-1} < \frac{\mathfrak{A}_n^\alpha(\underline{a}) - \mathfrak{G}_n^\alpha(\underline{a})}{\mathfrak{G}_n^\alpha(\underline{a}) - \mathfrak{H}_n^\alpha(\underline{a})} < (n-1)\left(\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{H}_n(\underline{a})}\right)^\alpha \tag{60}$$

In particular

$$\frac{1}{n-1} < \frac{\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})}{\mathfrak{G}_n(\underline{a}) - \mathfrak{H}_n(\underline{a})} < (n-1)\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{H}_n(\underline{a})}$$

□ The case $n = 2$ can be handled as above so assume that $n \ge 3$.

$$\begin{aligned}\mathfrak{G}_n^\alpha(\underline{a}) &< \big(\mathfrak{A}_n^\alpha(\underline{a})\big)^{(n-1)/n}\big(\mathfrak{H}_n^\alpha(\underline{a})\big)^{1/n}, && \text{by Sierpiński's inequality, (58),}\\ &< \frac{n-1}{n}\mathfrak{A}_n^\alpha(\underline{a}) + \frac{1}{n}\mathfrak{H}_n^\alpha(\underline{a}), && \text{by (GA).}\end{aligned}$$

This, on rearranging, gives the first inequality in (60). The second inequality in (60) follows by applying the first inequality to the n-tuple $\underline{a}^{-1}$. □

Further generalizations of (GA) can be found in many places in this book, and of course turn up in the literature almost daily; see also [*Alzer 1985, 1987b, 1989c, 1990c, 1991d, 1992b; Chuan; Dragomir 1993, 1998; Hao 1990, 1993; Kritikos 1928; Sándor 1990a; Wang W L 1994a,b*].

4 Converse Inequalities

(GA) in either of the forms 3.1(1) or (2) can be regarded as giving lower bounds to certain expressions. In general there are no interesting upper bounds unless the n-tuples $\underline{a}$ are restricted to a compact, see Notations 9.

The topic of converse inequalities will be discussed fully for more general means in a later section; see IV 6. Here a few simple results are given as their proofs differ from those of the more general results.

4.1 Bounds for the Differences of the Means The basic result here is found in [*Mond & Shisha 1967a,b; Shisha & Mond*]; see also [*Alzer 1990b; Bullen 1979, 1980 1994b; Dostor; Tung*]

Theorem 1 *Let $\underline{a}$ be a non-constant n-tuple with $0 < m \le \underline{a} \le M$, and $\underline{w}$ and n-tuple with $W_n = 1$; then*

$$0 < \mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w}) \le \theta M + (1-\theta)m - M^\theta m^{1-\theta}, \tag{1}$$

where θ, $0 < \theta < 1$, is determined by

$$\mathfrak{G}(m, M; 1-\theta, \theta) = \mathfrak{L}(m, M). \tag{2}$$

There is equality on the right-hand inequality of (1) if and only if for some index set I, a subset of $\{1,\ldots,n\}$, $W_I = \theta, a_i = M, i \in I, a_i = m, i \notin I$.

□ We give two proofs of this result.

(*i*) This theorem follows easily from proof (*iii*) of 3.1 Theorem 1. The function f defined there has a unique turning point, a minimum, and so if as here, its domain is $[m, M]$ the maximum of the function occurs at an end point, that is either at m or at M.

Since this argument can be applied to an f that uses $x = a_i$ for any i, $1 \le i \le n$, it follows that the maximum of the difference in (1) must occur when for all i, $a_i = m$ or $a_i = M$. For such an n-tuple $\underline{a}$, and some y, $0 \le y \le 1$,

$$\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w}) = yM + (1-y)m - M^y m^{1-y} = g(y), \text{ say.}$$

Simple calculations show that g has its maximum at $y = \theta$, where θ is given by (2), where $\mathfrak{L}(m, M) = (M - m)/(\log M - \log m)$, the logarithmic mean; see 2.4.5 Footnote 15, and 5.5 below.

(*ii*) The method used in proof (*iii*) of (J), I 4.2 Theorem 12 can be adapted to give an inductive proof.

In the case $n = 2$ define the difference function as in the proof of 3.7 Theorem 23, $D_2(x, y; t) = D_2(t) = \mathfrak{A}(x, y; 1 - t, t) - \mathfrak{G}(x, y; 1 - t, t)$, $0 \le t \le 1$, $0 < x < y$; then $D_2(0) = D_2(1) = 0$ and there is no loss in generality in the assumptions on x, y. Simple calculations give $D_2'(t) = (y - x) - \big(\log y - \log x\big)\mathfrak{G}(x, y; 1 - t, t)$ and $D_2''(t) = -\big(\log y - \log x\big)^2 \mathfrak{G}_2(x, y; 1 - t, t)$. Hence D_2 is strictly concave and so positive unless $t = 0$ or $t = 1$.

Further D_2 has a unique maximum at $t = \theta$ where $D_2'(\theta) = 0$, that is when $\mathfrak{G}_2(x, y; 1 - \theta, \theta) = \mathfrak{L}(x, y)$, which is just (2) in this case. That is $0 \le D_2(t) \le D_2(\theta)$, with equality on the left if and only if $t = 1$ or 0, and on the right if and only if $t = \theta$. This gives both a proof of (GA), and of the above result for $n = 2$.

If we regard D_2 as a function of x then simple calculus show that it is a strictly decreasing function, while as a function of y it is strictly increasing.

Hence if $u \le x < y \le v$, with not both $x = u$ and $y = v$ then $D_2(x, y; t) < D_2(u, v; t)$. In particular as a function of t the maximum of $D_2(x, y; t)$ is strictly less than the maximum of $D_2(u, v; t)$. This remark is used in the sequel.

We now consider the case $n = 3$. Define, for $0 < x < y < z$,

$$D_3(x, y, z; s, t) = D_3(s, t) = \mathfrak{A}_3(x, y, z; 1 - s - t, s, t) - \mathfrak{G}_3(x, y, z; 1 - s - t, s, t),$$

and $0 \le s \le 1$, $0 \le t \le 1, 0\ \ 0 \le s + t \le 1$; that is (s, t) lies in the triangle T of I 3.2 Theorem 12. Note that if $s = 0$, $t = 0$ or $s + t = 1$ then D_3 reduces to a case of $n = 2$.

There is no loss of generality in the restriction on the numbers x, y, z since if all are equal $D_3(s, t) = 0$, and if two are equal D_3 reduces to a case of $n = 2$.

Since D is continuous it attains its maximum and minimum on T, and if this is an interior point we must have that

$$\begin{aligned}\frac{\partial D}{\partial s} =&(y - x) - \mathfrak{G}_2\big(x, y, z; 1 - s - t, s,\big)(\log y - \log x) = 0,\\ \frac{\partial D}{\partial t} =&(z - x) - \mathfrak{G}_2\big(x, y, z; 1 - s - t, s, t\big)\big(\log z - \log x\big) = 0.\end{aligned}$$

So at such a turning points $\mathfrak{L}(x, y) = \mathfrak{L}(x, z)\Big(= \mathfrak{G}_2\big(x, y, z; 1 - s - t, s,\big)\Big)$. However $\mathfrak{L}(x, y) < \mathfrak{L}(x, z)$, see below 5.5 Theorem 10.

Hence the maximum and minimum of D_3 occur on the boundary of T; but on the boundary of T, as we have noted, D_3 reduces to a D_2, and we know that D_2 is positive except at the end points where it is zero. Further the discussion of the $n = 2$ case shows that on each edge D_3 has a local maximum and that the largest of the three of these occurs on the edge $s = 0$, since on that side the numbers x, z are used. Further the maximum occurs when $t = \theta$, given by $\mathfrak{G}_2(x, y; 1-\theta, \theta) = \mathfrak{L}(x, z)$. This proves that

$$0 \le D_3(x, y, z; s, t) \le D_2(x, z; 1-\theta, \theta),$$

proving (GA) in this case, and the above result.
The method clearly extends by induction to any value of n. □

The following results are of a different type.

THEOREM 2 *Let $\underline{a}, \underline{w}$ be as in the previous theorem, then:*
(a) [CARTWRIGHT & FIELD]

$$\frac{1}{2M}\sum_{k=1}^{n} w_k\big(a_k - \mathfrak{G}_n(\underline{a}; \underline{w})\big)^2 \le \mathfrak{A}_n(\underline{a}; \underline{w}) - \mathfrak{G}_n(\underline{a}; \underline{w}) \le \frac{1}{2m}\sum_{k=1}^{n} w_k\big(a_k - \mathfrak{A}_n(\underline{a}; \underline{w})\big)^2;$$

(b) [MERCER A]

$$\frac{1}{4M}\sum_{k=1}^{n}\big(w_k a_k^2 - \mathfrak{G}_n^2(\underline{a}; \underline{w})\big) \le \mathfrak{A}_n(\underline{a}; \underline{w}) - \mathfrak{G}_n(\underline{a}; \underline{w}) \le \frac{1}{4m}\sum_{k=1}^{n}\big(w_k a_k^2 - \mathfrak{G}_n^2(\underline{a}; \underline{w})\big).$$

REMARK (i) The left-hand side in (a) with $\mathfrak{A}_n$ instead of $\mathfrak{G}_n$ is the original result; the stronger form, that has a similar proof, is due to Alzer; [*Alzer 1997a*].

REMARK (ii) The equal weight case of (a) has been improved; [*Raşa*].

REMARK (iii) The result in (b) is not comparable with that in (a) and is found together with several other similar inequalities, in [*Mercer A 1999*]; see also [*Mercer A 2000, 2001, 2002*].
The proof of (b) depends on the following interesting mean-value theorem due to Mercer.

LEMMA 3 *If $\underline{a}, \underline{w}$ are as in the previous theorem, $n \ge 2$, and if $f, g \in \mathcal{C}^2(m, M)$ then for some y, $m \le y \le M$,*

$$\frac{\mathfrak{A}_n\big(f(\underline{a}); \underline{w}\big) - f\big(\mathfrak{A}_n(\underline{a}; \underline{w})\big)}{\mathfrak{A}_n\big(g(\underline{a}); \underline{w}\big) - g\big(\mathfrak{A}_n(\underline{a}; \underline{w})\big)} = \frac{f''(y)}{g''(y)},$$

provided the denominator on the left-hand side is never zero.

REMARK (iv) For an extension of this result and of Theorem 2(b) see III 6.4 Theorem 8.

4.2 BOUNDS FOR THE RATIOS OF THE MEANS In the equal weight case a very simple converse inequality has been given by Dočev; the proof is based on the centroid method, see I 4.4 Theorem 24; [*Mitrinović & Vasić pp.43–44*], [*Dočev*].

THEOREM 4 *Let $\underline{a}$, M and m be as in Theorem 1, $\mu = M/m$, then*

$$\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{G}_n(\underline{a})} < \frac{(\mu-1)\mu^{1/(\mu-1)}}{e\log\mu}. \tag{3}$$

Further the right-hand side of (3) increases as a function of M, and decreases as a function of m.

□ Since $f(x) = -\log x$, $m \le x \le M$ is strictly convex the centroid of the points $(a_i, f(a_i))$, $1 \le i \le n$, the point $\big(\mathfrak{A}_n(\underline{a};\underline{w}), -\log\mathfrak{G}_n(\underline{a};\underline{w})\big)$, lies strictly above the graph of f and below the chord ℓ joining $(m, f(m))$ to $(M, f(M))$.
If then γ is chosen so that the graph of $g(x) = \gamma + f(x)$ is tangent to ℓ it is immediate that the centroid lies below the graph of g, that is $-\log\mathfrak{G}_n(\underline{a};\underline{w}) < \gamma - \log\mathfrak{A}_n(\underline{a};\underline{w})$; or

$$\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{G}_n(\underline{a})} < e^{\gamma}.$$

It remains then to calculate γ. Since ℓ is a tangent to the graph of g, at (x_0, y_0) say,

$$\frac{1}{x_0} = \frac{\log\mu}{m(\mu-1)}, \quad \gamma = y_0 + \log x_0, \quad \frac{y_0 + \log m}{x_0 - m} = \frac{-\log\mu}{m(\mu-1)}.$$

These easily show that e^{γ} is the right-hand side of (3).
Further note that since f is strictly convex increasing M increases the slope of ℓ, while decreasing m decreases the slope of ℓ; see I 4.1 (3). This observation readily implies the last part of the theorem. □

REMARK (i) Inequality (3) is referred to as *Dočev's inequality*.

The following result also only considers the equal weight case; see [*Loewner & Mann*].

THEOREM 5 *If* $0 < \alpha \le \dfrac{a_i}{\mathfrak{A}_n(\underline{a})} \le \beta \le 2$, $1 \le i \le n$, *then*

$$1 \le \left(\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{G}_n(\underline{a})}\right)^n \le \frac{(\beta/\alpha)^{\left[n(\beta-1)/(\beta-\alpha)\right]}\beta^{-n+1}}{1 + (\beta-\alpha)\left[n(\beta-1)/(\beta-\alpha)\right] - (n-1)(\beta-1)}.$$

□ We first find the minimum of $f(\underline{x}) = \prod_{i=1}^{n}(a+x_i)$ in the domain defined by

$$-a \le -p \le x_i \le q,\ 1 \le i \le n;\quad \sum_{i=1}^{n} x_i = c,\ -np \le c \le nq.$$

If $n = 2$ the minimum is at $(-p, c+p)$ if $cp \le q$, or at $(q, c-q)$ if $c - q \ge -p$. In either case, at most one of x_1, x_2 is different from $-p$ or q.

We now show that this is true in general.

Using Lagrange multipliers we se that the minimum occurs on the boundary and so one of the elements of $\underline{x}$, x_n say, is equal to $-p$ or q. Assume that $x_n = q$. Now we have to find a point at which $\prod_{i=1}^{n-1}(a+x_i)$ has its minimum in the domain

$$-a \le -p \le \underline{x}_i \le q,\ 1 \le i \le n-1;\quad \sum_{i=1}^{n-1} x_i = c - q,\ -(n-1)p \le c \le (n-1)q.$$

The result now follows by an induction argument.

Now apply this to the case $a = \mathfrak{A}_n(\underline{a}), x_i = a_i - \mathfrak{A}_n(\underline{a})$, when $c = 0$. From the above $\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})$ is maximized if f is minimized, and that occurs if at least $(n-1)$ of the elements of $\underline{x}$ are equal to either $-p\mathfrak{A}_n(\underline{a})$ or to $q\mathfrak{A}_n(\underline{a})$. For simplicity put $\mathfrak{A}_n(\underline{a}) = 1$ and let $x_i = -p, 1 \le i \le r$, and $x_i = q$, $r+1 \le i \le r+s$, $r+s = n-1$. Then $-p \le x_n = rp - sq < q$, and so $r \le nq/(p+q) \le r+1$. Now either $nq/(p+q)$ is not an integer when $r = [nq/(p+q)]$, or it is an integer when it is equal to either r or $r+1$. In the latter case $s = nq/(p+q)$, $x_n - -p$ so we could take $r = nq/(p+q)$.

In both cases therefore:

$r = [nq/(p+q)]$, $s = n - 1 - [nq/(p+q)]$, $x_n = (p+q)[nq/(p+q)] - nq + q$,

and putting $\alpha = 1 - p, \beta = 1L + q$ simple rearrangements lead to the theorem. □

THEOREM 6 [DRAGOMIR] *If $\underline{a}, \underline{w}$ are two n-tuples then*

$$\max\left\{\frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{G}_n(\underline{a};\underline{w})}, \frac{\mathfrak{G}_n(\underline{a};\underline{w})}{\mathfrak{H}_n(\underline{a};\underline{w})}\right\} \le \exp\left(\frac{1}{W_n^2}\sum_{1\le i<j\le n} w_i w_j \frac{(a_i - a_j)^2}{a_i a_j}\right).$$

□ The proof uses the following inequality proved in [*Dragomir, Dragomir & Pranesh; Dragomir & Goh 1996*]; if $W_n = 1$,

$$0 \le \sum_{i=1}^{n} w_i \log a_i - \log\Big(\sum_{i=1}^{n} w_i a_i\Big) \le \sum_{1\le i<j\le n} w_i w_j \frac{(a_i - a_j)^2}{a_i a_j}.$$

□

A very simple converse inequality for the $n = 2$ case of (GA), in the form 2.2.2 (5), has been given by Zhuang, [*Zhuang 1991*].

THEOREM 7 *If $0 < \alpha \le a \le A$, $0 < \beta \le b \le B$, $p > 1$ and p' the conjugate index then*

$$\frac{a^p}{p} + \frac{b^{p'}}{p'} \le Kab, \tag{4}$$

where

$$K = \max\left\{\frac{\alpha^p/p + B^{p'}/p'}{\alpha B}, \frac{A^p/p + \beta^{p'}/p'}{A\beta}\right\}.$$

□ The function $f(a,b) = (\frac{a^p}{p} + \frac{b^{p'}}{p'})\big/ab$, $a > 0, b > 0$, attains its minimum, namely 1, inside the rectangle in the hypotheses, when $a^p = b^{p'}$; this is just the case $n = 2$ of (GA). Further this is the only turning point of f. So the maximum on the given rectangle is on the boundary. It is then clear that this is at either the point (α, B), or (A, β). □

A completely different result has recently been given by Alzer, [*Alzer 2001a*]. To state the theorem we need the following definitions. The *digamma or psi function* is

$$\mathcal{F}(x) = \psi(x+1) = (\log x!)'.$$

The derivatives of this function are called the the *multigamma, or polygamma, functions*,

$$\psi^{(k)}(x) = (-1)^{k+1}k!\sum_{n=1}^{\infty}\frac{1}{(x+n)^{k+1}},\ x > 0, k = 1, 2, \ldots;$$

see [*DI pp. 71–72*], where there are further references for these functions.

THEOREM 8 *If $\underline{a}$ and $\underline{w}$ are n-tuples, $n \ge 2$, $W_n = 1$, and if $k \in \mathbb{N}^*$, then*

$$\left(\frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{G}_n(\underline{a};\underline{w})}\right)^k \le \frac{\mathfrak{G}_n\left(\psi^{(k)}(\underline{a});\underline{w}\right)}{\psi^{(k)}\left(\mathfrak{A}_n(\underline{a};\underline{w})\right)}, \tag{5}$$

with equality if and only if $\underline{a}$ is constant.
The exponent on the left-hand side, k, cannot be replaced by any larger number.

□ The proof of (5) depends on showing that the function

$$f(x) = \log\left(x^k|\psi^{(k)}(x)|\right),\ x > 0,$$

is strictly convex, when the result follows from (J).
For the rest if $\underline{a}$ is not constant and if (5) holds with exponent k replaced by α then,

$$\alpha \le \frac{\sum_{i=1}^n w_i \log|\psi^{(k)}(a_i)| - \log|\psi^{(k)}\left(\mathfrak{A}_n(\underline{a};\underline{w})\right)|}{\log \mathfrak{A}_n(\underline{a};\underline{w}) - \log \mathfrak{G}_n(\underline{a};\underline{w})},$$

which, on letting $a_1 \to \infty$ and using an asymptotic property of the multigamma function, $\log|\psi^{(k)}(x)| \sim -k\log x$, $x \to \infty$, implies that $\alpha \le k$. □

REMARK (ii) For another converse inequality due to Alzer see below, 5.7 Theorem 18.

5 Some Miscellaneous Results

In this section various properties of the elementary means of this chapter are discussed. The various results are not related to one another nor, in general, to the inequalities given earlier.

5.1 AN INDUCTIVE DEFINITION OF THE ARITHMETIC MEAN We prove that the equal weighted arithmetic mean of n-tuples can be defined from equal weighted arithmetic means of $(n-1)$-tuples. This fact has been used extensively by Aumann in his study of the axiomatics of means; see [*Aumann 1933a,b*].

THEOREM 1 *Let $\underline{a}$ be an n-tuple and define the n-tuples $\underline{a}^{(r)}$, $r \in \mathbb{N}^*$, recursively as follows:*
$\underline{a}^{(1)}$ = the arithmetic means of the n possible $(n-1)$-tuples from $\underline{a}$;
$\underline{a}^{(r)}$ = the arithmetic means of the n possible $(n-1)$-tuples from $\underline{a}^{(r-1)}$, $r \ge 2$.
Then $\lim_{r\to\infty} a_i^{(r)} = \mathfrak{A}_n(\underline{a})$, $1 \le i \le n$.

□ Assume for simplicity that for all r, $a_1^{(r)} \le a_n^{(r)}$. Then the result is immediate from the following identities.

$$\sum_{i=1}^n a_i^{(r)} = \sum_{i=1}^n a_i^{(r-1)} = \sum_{i=1}^n a_i,\ r \ge 2; \quad a_n^{(r)} - a_1^{(r)} = \frac{a_n^{(r-1)} - a_1^{(r-1)}}{n-1},\ r \ge 2.$$

□

REMARK (i) This result has been extended; see [*Kritikos 1949*].

5.2 AN INVARIANCE PROPERTY Given a sequence $\underline{a}$ it is natural to ask which of its properties are also properties of the sequences of means, $\underline{\mathfrak{A}}$, $\underline{\mathfrak{G}}$, defined in 3.4. We will only consider the sequence $\underline{\mathfrak{A}}$ in the equal weight case, and in several instances the answers are immediate:
(a) if $m \le \underline{a} \le M$ then $m \le \underline{\mathfrak{A}} \le M$ since the arithmetic mean is internal, 1.1 Theorem 2 (I);
(b) if $\lim_{n\to\infty} a_n = \alpha$ then, by a well known result of Cauchy, $\lim_{n\to\infty} \mathfrak{A}_n(\underline{a}) = \alpha$, [*Hardy p.10*];

(c) if $\underline{a}$ is increasing, strictly, so is $\mathfrak{A}$, using internality of both the arithmetic mean and the weighted arithmetic mean and the formula

$$\mathfrak{A}_{n+1}(\underline{a}) = \frac{n}{n+1}\mathfrak{A}_n(\underline{a}) + \frac{1}{n+1}a_{n+1}.$$

This last case is generalized in the following theorem.

THEOREM 2 *(a) If $\underline{a}$ is k-convex so is $\mathfrak{A}$.*
(b) if $\underline{a}$ is bounded and of bounded k-variation so is $\mathfrak{A}$.

□ (a) Since $\underline{a}$ is k-convex then $\Delta^k a_n \geq 0$, $n \in \mathbb{N}^*$. Hence

$$\sum_{i=1}^{n} \frac{(i+k-1)!}{(i-1)!}\Delta^k a_i \geq 0.$$

or equivalently

$$\frac{(n+k)!}{(n-1)!}\Delta^k \mathfrak{A}_n(\underline{a}) \geq 0,$$

which implies the result.
(b) By I 3.1 Theorem 3, $\underline{a}$ being bounded and of bounded k-variation implies that $\underline{a} = \underline{b} - \underline{c}$, where $\underline{b}, \underline{c}$ are k-convex. So by (a) $\mathfrak{A}_n(\underline{b})$ and $\mathfrak{A}_n(\underline{c})$ are bounded and k-convex. The result then follows since $\mathfrak{A}_n(\underline{a}) = \mathfrak{A}_n(\underline{b}) - \mathfrak{A}_n(\underline{c})$. □

REMARK (i) Part (a) of Theorem is due to Ozeki. Later Vasić, Kečkić, Lacković & Mitrović extended the case $k = 2$ to weighted means, finding necessary and sufficient conditions on the weights for the result to hold; see also [*Andrica, Raşa & Toader; Lacković & Simić; Mitrinović, Lacković & Stanković; Ozeki 1965; Popoviciu 1968; Toader 1983; Vasić, Kečkić, Lacković & Mitrović*].

The following slightly different result is due to Lupaş and Sivakumar; [*Lupaş 1988*].

THEOREM 3 *If $\underline{a}$ is a sequence such that $m_p \leq \Delta^p a_n \leq M_p$, $n = 0, 1, 2, \ldots$ then*

$$\frac{m_p}{p+1} \leq \Delta^p \mathfrak{A}_n(\underline{a}) \leq \frac{M_p}{p+1}\ n = 0, 1, 2, \ldots.$$

5.3 ČEBIŠEV'S INEQUALITY We now consider the problem mentioned in 1.1 Remark (vi): to obtain a relation between $\mathfrak{A}_n(\underline{a}), \mathfrak{A}_n(\underline{b})$ and $\mathfrak{A}_n(\underline{a}\,\underline{b})$.

THEOREM 4 *(a) If $\underline{a}$, $\underline{b}$ and $\underline{w}$ are n-tuples with $\underline{a}$ and $\underline{b}$ similarly ordered then,*

$$\mathfrak{A}_n(\underline{a};\underline{w})\mathfrak{A}_n(\underline{b};\underline{w}) \leq \mathfrak{A}_n(\underline{a}\,\underline{b};\underline{w}), \tag{1}$$

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are oppositely ordered then (~1) holds.
In both cases there is equality if and only if either $\underset{\sim}{a}$, or $\underset{\sim}{b}$, is constant..

□ (*i*) Suppose then that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are similarly ordered[18].

$$\begin{aligned} W_n\sum_{i=1}^n w_ia_ib_i - \Big(\sum_{i=1}^n w_ia_i\Big)\Big(\sum_{i=1}^n w_ib_i\Big) &= \sum_{i,j=1}^n (w_iw_ja_jb_j - w_iw_ja_ib_j) = \sum_{i,j=1}^n (w_iw_ja_ib_i - w_iw_ja_jb_i) \\ &= \frac{1}{2}\sum_{i,j=1}^n (w_iw_ja_jb_j - w_iw_ja_ib_j + w_iw_ja_ib_i - w_iw_ja_jb_i) \\ &= \frac{1}{2}\sum_{i,j=1}^n w_iw_j(a_i-a_j)(b_i-b_j) \ge 0, \end{aligned} \tag{2}$$

which is equivalent to (1).

Following I 3.3 Remark (viii) we can assume that both $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are decreasing. Then since (2) contains the term $w_1w_n(a_1 - a_n)(b_1 - b_n)$ we see that the sum can be zero if and only if either $\underset{\sim}{a}$ or $\underset{\sim}{b}$ is constant.
(*ii*) A simple proof of the equal weight case of (1) can be based on I 3.3 Theorem 16. Let $\underset{\sim}{b}^{(j)} = (b_1^{(j)}\ldots, b_n^{(j)}) = (b_j, b_{j+1},\ldots, b_n, b_1, \ldots b_{j-1})$, $1 \le j \le n$, putting $b_0 = b_n$. Since $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are similarly ordered, $\sum_{i=1}^n a_ib_i \ge \sum_{i=1}^n a_ib_i^{(j)}$, $1 \le j \le n$. Adding over j of these leads immediately to the required result.
A similar proof can be given for (~ 1), when $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are oppositely ordered. □

Inequality (1), in the equal weight case, is due to Čebišev, and is called *Čebišev's inequality*; see [*AI pp.36–37; DI pp.50–5; Hermite; HLP pp.43–44; PPT pp.197–198*], [*Herman, Kučera & Šimša pp.148–150, 159*], [*Čebišev 1883; Daykin; Djoković 1964; Jensen 1888*]. A history of this inequality can be found in the excellent expository paper [*Mitrinović & Vasić 1974*].
Generalizations are given in many places; see for instance [*Pearce, Pečarić & Šunde; Pečarić 1985b; Pečarić & Dragomir 1990; Popoviciu 1959a; Toader 1996*].

Remark (i) The requirement that the n-tuples be similarly ordered is sufficient for (1) but it is not necessary; other sufficient conditions can be found in [*Labutin 1947*]. The problem of giving necessary and sufficient conditions for the validity of Čebišev's inequality has been solved; see [*Sasser & Slater*].

Nanjundiah has given a proof of (1) that yields a Rado type extension; [*Bullen 1996b*]. It is based on the use of Nanjundiah's inverse arithmetic means, see 3.4.

[18] See I 3.3 Definition 15.

Lemma 5 *With the notation of 3.4 Lemma 10 and assuming that $n > 1$ and $(a_{n-1}, a_n), (b_{n-1}, b_n)$ are similarly ordered*

$$\mathfrak{A}_n^{-1}(\underline{a};\underline{w})\mathfrak{A}_n^{-1}(\underline{b};\underline{w}) \geq \mathfrak{A}_n^{-1}(\underline{a}\,\underline{b};\underline{w}),$$

with equality only if $a_n = a_{n-1}$, or $b_n = b_{n-1}$.

□ This is an immediate consequence of the elementary computation

$$\mathfrak{A}_n^{-1}(\underline{a};\underline{w})\mathfrak{A}_n^{-1}(\underline{b};\underline{w}) - \mathfrak{A}_n^{-1}(\underline{ab};\underline{w}) = W_{n-1}W_n(a_n - a_{n-1})(b_n - b_{n-1}).$$

□

Theorem 6 *If $n > 1$ and if $\underline{a}$ and $\underline{b}$ are both decreasing then*

$$\begin{aligned} W_n\big(\mathfrak{A}_n(\underline{a}\,\underline{b};\underline{w}) &- \mathfrak{A}_n(\underline{a};\underline{w})\mathfrak{A}_n(\underline{b};\underline{w})\big) \\ &\geq W_{n-1}\left(\mathfrak{A}_{n-1}(\underline{a}\,\underline{b};\underline{w}) - \mathfrak{A}_{n-1}(\underline{a};\underline{w})\mathfrak{A}_{n-1}(\underline{b};\underline{w})\right), \end{aligned} \tag{3}$$

with equality only if $a_n = \mathfrak{A}_{n-1}(\underline{a};\underline{w})$ or $b_n = \mathfrak{A}_{n-1}(\underline{b};\underline{w})$.

□ Since $\underline{a}$ and $\underline{b}$ are decreasing so are $\underline{\mathfrak{A}}(\underline{a};\underline{w})$ and $\underline{\mathfrak{A}}(\underline{b};\underline{w})$, 5.2 (c). So we can apply Lemma 5 using these sequences to get

$$\begin{aligned} \mathfrak{A}_n^{-1}(\underline{\mathfrak{A}}(\underline{a};\underline{w})\underline{\mathfrak{A}}(\underline{b};\underline{w});\underline{w}) &\leq \mathfrak{A}_n^{-1}(\underline{\mathfrak{A}}(\underline{a};\underline{w});\underline{w})\mathfrak{A}_n^{-1}(\underline{\mathfrak{A}}(\underline{b};\underline{w});\underline{w}) \\ &= a_n b_n \\ &= \mathfrak{A}_n^{-1}(\underline{\mathfrak{A}}(\underline{a}\,\underline{b};\underline{w});\underline{w}) \qquad \text{by 3.4 Lemma 10 (a);} \end{aligned}$$

which is just (3). The case of equality follows from that of Lemma 5. □

A weaker result in the equal weight case, due to Janić, is that $n^2\big(\mathfrak{A}_n(\underline{a}\,\underline{b}) - \mathfrak{A}_n(\underline{a})\mathfrak{A}_n(\underline{b})\big)$ increases with n; [*AI p.206*], [*Djoković 1964*]. The best result in this direction is the following, due to Alzer; [*Alzer1989e*].

Theorem 7 *If $\underline{a}, \underline{b}$ are increasing n-tuples and $\underline{w}$ another n-tuple and k an integer with $2 \leq k < n$, and $a_1 < a_k$, $b_1 < b_k$ then*

$$\frac{W_n^2}{W_n - w_1}\big(\mathfrak{A}_n(\underline{a}\,\underline{b};\underline{w}) - \mathfrak{A}_n(\underline{a};\underline{w})\mathfrak{A}_n(\underline{b};\underline{w})\big) \geq \frac{W_k^2}{W_k - w_1}\big(\mathfrak{A}_k(\underline{a}\,\underline{b};\underline{w}) - \mathfrak{A}_k(\underline{a};\underline{w})\mathfrak{A}_k(\underline{b};\underline{w})\big).$$

Remark (ii) The equal weight case of this result improves the quoted result of Janić and the result of Nanjunduiah as it says that $\big(n^2/(n-1)\big(\mathfrak{A}_n(\underline{a})\mathfrak{A}_n(\underline{b}) - \mathfrak{A}_n(\underline{ab})\big)$ decreases with n.

EXAMPLE (i) Suppose that $a_2 = \cdots = a_n = b_2 = \cdots = a_n = 1$ then

$$\frac{\mathfrak{A}_n(\underline{a}\,\underline{b};\underline{w}) - \mathfrak{A}_n(\underline{a};\underline{w})\mathfrak{A}_n(\underline{b};\underline{w})}{\mathfrak{A}_k(\underline{a}\,\underline{b};\underline{w}) - \mathfrak{A}_k(\underline{a};\underline{w})\mathfrak{A}_k(\underline{b};\underline{w})}$$
$$= \frac{W_n W_k^2(w_1a_1b_1 + W_n - w_1) - W_k^2(w_1a_1 + W_n - w_1)(w_1b_1 + W_n - w_1)}{W_k W_n^2(w_1a_1b_1 + W_k - w_1) - W_n^2(w_1a_1 + W_k - w_1)(w_1b_1 + W_k - w_1)}.$$

If now we let first $a_1 \to 1$, and then let $b_1 \to 1$ the right-hand side has limit $\dfrac{W_k^2(W_n - w_1)}{W_n^2(W_k - w_1)}$. This shows that the constant in the inequality in Theorem 7 cannot be improved.

REMARK (iii) McLaughlin & Metcalf have studied inequality (1) from the point of view of functions of an index set; see 3.2.2. They showed that under suitable conditions the difference between the right-and left-hand sides, considered as a function of index sets, is super-additive; [*McLaughlin & Metcalf 1968b*].

Another result due to Alzer is the following lower bound for the difference between the two sides in (1); [*Alzer 1992c*]

THEOREM 8 *If $\underline{a}$ and $\underline{b}$ are strictly increasing n-tuples and $\underline{w}$ another n-tuple, $n \ge 2$, then*

$$\mathfrak{A}_n(\underline{a}\,\underline{b};\underline{w}) - \mathfrak{A}_n(\underline{a};\underline{w})\mathfrak{A}_n(\underline{b};\underline{w}) \ge \left(\mathfrak{A}_n(\underline{w}^2) - \mathfrak{A}_n^2(\underline{w})\right) \min_{2\le i,j\le n} \left\{(a_i - a_{i-1})((b_j - b_{j-1})\right\}.$$

This inequality is strict unless for some positive α, β, $a_i = a_1 + (i-1)\alpha$, and $b_i = b_1 + (i-1)\beta$, $1 \le i \le n$.

□ Let $c_n = \min_{\le i,j \le n} \left\{(a_i - a_{i-1})((b_j - b_{j-1})\right\}$. Then if $i > j$,

$$\begin{aligned}(a_i - a_j)(b_i - b_j) &= \sum_{k=j+1}^{i} (a_k - a_{k-1}) \sum_{m=j+1}^{i} (b_m - b_{m-1}) \\ &= \sum_{k,m=j+1}^{i} (a_k - a_{k-1})(b_m - b_{m-1}) \\ &\ge \sum_{k,m=j+1}^{i} c_n = (i-j)^2 c_n.\end{aligned}$$

Now if S_n denotes the left-hand side of the identity (2), the above calculation shows that

$$S_n \ge \frac{c_n}{2} \sum_{i,j=1}^{n} w_i w_j (i-j)^2.$$

Simple manipulations show that this is just the inequality to be proved. □

REMARK (iv) A related result can be found in [*AI pp.340–341*].

Writing (1) in the form

$$\Big(\sum_{i=1}^{n} w_i a_i\Big)\Big(\sum_{i=1}^{n} w_i b_i\Big) \le W_n \Big(\sum_{i=1}^{n} w_i a_i b_i\Big)$$

and applying this to infinite sums leads to various interesting elementary inequalities. For instance, [*DI p.251*]:

$$\tan x \tan y \le \tan 1 \tan xy \quad \text{if } 0 \le x, y \le 1, \text{ or } 1 \le x, y \le \pi/2;$$
$$\arcsin x \arcsin y \le \frac{1}{2} \arcsin xy, \quad \text{if } 0 \le x, y \le 1.$$

REMARK (v) For an interesting application of Čebišev's inequality see below 5.5 Remark (vii).

The following definition allows us to generalize inequality (1).

DEFINITION 9 *Give two sequences $\underline{a}$ and $\underline{w}$ we say that $\underline{a}$ is* monotonic in the mean, increasing in the mean, *respectively* decreasing in the mean *if the sequence $\mathfrak{A}_1(\underline{a};\underline{w}), \mathfrak{A}_2(\underline{a};\underline{w}) \ldots$ is monotonic, increasing, respectively decreasing.*

THEOREM 10 *If the m sequences $\underline{a}^{(k)}$, $1 \le k \le m$ are monotonic in the mean in the same sense then*

$$\prod_{k=1}^{m} \mathfrak{A}_n(\underline{a}^{(k)};\underline{w}) \le \mathfrak{A}_n\Big(\prod_{k=1}^{m} \underline{a}^{(k)};\underline{w}\Big)$$

REMARK (vi) In the case $m = 2$ this is due to Burkill & Mirsky, and a simple proof of the general case has been given by Vasić & Pečarić; see [*Burkill & Mirsky; Vasić & Pečarić 1982a*]. For further generalizations see [*Daykin; Pečarić 1984c; Sun X H; Vasić & Pečarić 1982a*].

REMARK (vii) The concept of monotonic in the mean has been extended to *sequences k-convex in the mean*; see [*Toader 1983, 1985*].

5.4 A RESULT OF DIANANDA

THEOREM 11 *If $\underline{a}$ and $\underline{w}$ are n-tuples, $k \in \mathbb{N}^*$ and $\underline{a} \ge k$, then*

$$1 \le R = \frac{\mathfrak{A}_n([\underline{a}];\underline{w})}{\mathfrak{A}_n(\underline{a};\underline{w})} \prod_{i=1}^{n} \frac{a_i}{[a_i]} < \Big(1 + \frac{1}{k}\Big)^{n-1}.$$

In particular if $\underline{a} \geq n-1$ *then* $1 \leq R < e$. *Further :* (a) $R = 1$ *if and only if all the elements of* $\underline{a}$ *are integers, and* (b) R *tends to* $\left(1+\dfrac{1}{k}\right)^{n-1}$ *if and only if each element of* $\underline{a}$ *tends to* $(k+1)$ *from below.*

REMARK (i) This result of Diananda has been generalized by Kalajdić; [*Dianada 1975, Kalajdžić 1973*].

5.5 INTERCALATED MEANS Let $a < b$ and as in proof (*ii*) of 2.2.2 Lemma 4 let us insert n arithmetic, n geometric and n harmonic means between a and b; that is define the n-tuples $\underline{a}, \underline{g}$ and $\underline{h}$ as follows:

$$a_i = a + \frac{i}{n+1}(b-a); \quad g_i = a\left(\frac{b}{a}\right)^{i/(n+1)}; \quad h_i = \frac{ab}{b - \dfrac{k}{n+1}(b-a)}; \quad 1 \leq i \leq n.$$

The following facts are easily verified:

$$\begin{gathered}
\mathfrak{A}_n(\underline{a}) = \mathfrak{A}(a,b); \quad \mathfrak{G}_n(\underline{g}) = \mathfrak{G}(a,b); \quad \mathfrak{H}_n(\underline{h}) = \mathfrak{H}(a,b);\\
\lim_{n\to\infty} \mathfrak{H}_n(\underline{a}) = \lim_{n\to\infty} \mathfrak{A}_n(\underline{g}) = \frac{b-a}{\log b - \log a};\\
\lim_{n\to\infty} \mathfrak{H}_n(\underline{g}) = \lim_{n\to\infty} \mathfrak{A}_n(\underline{h}) = \frac{\log b - \log a}{b-a};\\
\lim_{n\to\infty} \mathfrak{G}_n(\underline{a}) = e^{-1}\left(\frac{b^a}{a^b}\right)^{1/(b-a)}; \quad \lim_{n\to\infty} \mathfrak{G}_n(\underline{h}) = e\left(\frac{a^b}{b^a}\right)^{1/(b-a)}.
\end{gathered}$$

The nine possible relations between these means are given in the following theorem.

THEOREM 12

$$\mathfrak{A}_n(\underline{a}) > \mathfrak{G}_n(\underline{a}) > \mathfrak{H}_n(\underline{a}) > \mathfrak{A}_n(\underline{g}) > \mathfrak{G}_n(\underline{g}) > \mathfrak{H}_n(\underline{g}) > \mathfrak{A}_n(\underline{h}) > \mathfrak{G}_n(\underline{h}) > \mathfrak{H}_n(\underline{h}).$$

□ Because of (GA) we need only prove that $\mathfrak{H}_n(\underline{a}) > \mathfrak{A}_n(\underline{g})$ and $\mathfrak{H}_n(\underline{g}) > \mathfrak{A}_n(\underline{h})$, and since the discussions are similar we just give a proof of the first. Consider then

$$\begin{aligned}
A_n(\underline{g}) =& \frac{1}{n}\sum_{i=1}^n g_i = \frac{1}{n}\sum_{i=1}^n \exp\left(\log a + \frac{i}{n+1}(\log b - \log a)\right)\\
<& \frac{1}{\log b - \log a}\int_{\log a}^{\log b} e^x\,dx, \text{ by I 4.1 (5)},\\
=& \frac{b-a}{\log b - \log a}.
\end{aligned}$$

Similarly, since $\mathfrak{H}_n(\underline{a}) = \left(\mathfrak{A}_n(\underline{a}^{-1})\right)^{-1}$, 1.2 (7), consider

$$\mathfrak{A}_n(\underline{a}^{-1}) = \frac{1}{n}\sum_{i=1}^{n} a_i^{-1} < \frac{1}{b-a}\int_a^b x^{-1}\,\mathrm{d}x = \frac{\log b - \log a}{b-a},$$

again using I 4.1 (5). This completes the proof of the first inequality above and in fact proves more, namely

$$\mathfrak{H}_n(\underline{a}) > \frac{b-a}{\log b - \log a} > A_n(\underline{g}).$$

The similar proof of the second inequality also proves more, namely

$$\mathfrak{H}_n(\underline{g}) > \frac{\log b - \log a}{b-a} > A_n(\underline{h}).$$

□

REMARK (i) In fact I 4.1 Remark (xiv) shows that $\mathfrak{H}_n(\underline{a})$ and $\mathfrak{H}_n(\underline{g})$ decrease with n while $\mathfrak{A}_n(\underline{g})$ and $\mathfrak{A}_n(\underline{h})$ increase, to the limits given above.

COROLLARY 13

$$\frac{a+b}{2} > e^{-1}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} > \frac{b-a}{\log b - \log a} > \sqrt{ab} \tag{4}$$

$$> \frac{\log b - \log a}{a^{-1} - b^{-1}} > e\left(\frac{a^b}{b^a}\right)^{1/(b-a)} > \frac{2ab}{a+b}.$$

REMARK (ii) Nanjundiah has pointed out that this last inequality is a considerable improvement on the standard estimates for e and the logarithmic function, in particular I 2.2(9). This is an implicit in a later proof of the same inequality by Králik which consists of substituting $1 + x = b/a$ in I 2.2(10). [*Nanjundiah 1946; Králik*].

The quantities $e^{-1}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$ and $\dfrac{b-a}{\log b - \log a}$ are called respectively the *identric* and *logarithmic means*[19] *of* a, b, written $\mathfrak{I}(a,b)$ and $\mathfrak{L}(a,b)$; the definitions are completed by putting $\mathfrak{I}(a,a) = \mathfrak{L}(a,a) = a$.

THEOREM 14 *The means $\mathfrak{L}(a,b)$ and $\mathfrak{I}(a,b)$ are strictly increasing as functions of both a and b.*

□ This property of the logarithmic mean is a consequence of the strict concavity of the logarithmic function and I 4.1 Remark(v). The same result and the strict convexity of the function $x\log x$, see I 4.1 Example(i), gives the property for the identric mean. □

[19] The logarithmic mean has occurred earlier in 2.4.5 Footnote 15 and 4.1.

THEOREM 15 *If a, b are positive numbers then*

$$\min\{a,b\} \le \mathfrak{L}(a,b) \le \mathfrak{I}(a,b) \le \max\{a,b\}; \tag{5}$$

more precisely

$$\mathfrak{G}(a,b) \le \mathfrak{L}(a,b) \le \mathfrak{I}(a,b) \le \mathfrak{A}(a,b). \tag{6}$$

All these inequalities are strict unless $a = b$.

□ (*i*) The outer inequalities in (5), the strict internality of the identric and logarithmic means, follow from the proof of Theorem 14. Alternatively the results are immediate from the mean-value theorem of differentiation, I 2.1 Footnote 1. Assume that $a < b$ then:

$$\mathfrak{L}(a,b) = \Big(\frac{\log b - \log a}{b-a}\Big)^{-1} = c, \text{for some } c,\ a < c < b;$$

and

$$\log \mathfrak{I}(a,b) = -1 + \frac{b\log b - a\log a}{b-a} = \log c, \text{for some } c,\ a < c < b.$$

(*ii*) The rest of (5) and (6) is proved in (4). A different proof of the inner inequality of (5) can be found in VI 2.1.1 Theorem 3; see also [*Zaiming—Perez P*].

We now give several proofs of the inequality for the logarithmic mean in (6),

$$\mathfrak{G}(a,b) \le \mathfrak{L}(a,b) \le \mathfrak{A}(a,b), \tag{7}$$

and assume without loss in generality that $a < b$.

(*iii*) (7) follows from the left inequalities in I 2.2(14) by putting $x = \frac{1}{2}\log(b/a)$, or by I 2.2(10) putting $x = b/a$.

(*ii*) Consider the graph of $f(x) = e^x$, $\log a \le x \le \log b$. Then, using the convexity of the exponential function and the Hadamard-Hermite inequality, I 4.1(4), we easily obtain the following

$$\exp\Big(\frac{\log a + \log b}{2}\Big)(\log b - \log a) < \int_{\log a}^{\log b} e^x\, dx < \frac{\exp\log b + \exp\log a}{2}(\log b - \log a),$$

which is just (7).

(*iii*) Comparing the area under the curve $y = x^{-1}$, $a \le x \le b$, with the area bounded by the lines $x = a, x = b$, the tangent to this curve at $x = (a+b)/2$, $y = 2/(a+b)$ and the X-axis gives the right-hand inequality in (7). Comparing the area under the curve $y = x^{-1}$, $\sqrt{a} \le x \le \sqrt{b}$ with the area bounded by the lines $x = a, x = b$, the chord to this and the X-axis gives the left-hand inequality in (7).

(*iv*) If $t > 0$ then by (GA),

$$t^2 + (a+b)t + \left(\frac{a+b}{2}\right)^2 > t^2 + (a+b)t + ab > t^2 + 2\sqrt{ab}\,t + ab,$$

and so

$$\int_0^\infty \frac{1}{(t+(a+b)/2)^2}\,\mathrm{d}t < \int_0^\infty \frac{1}{(t+a)(t+b))}\,\mathrm{d}t < \int_0^\infty \frac{1}{(t+\sqrt{ab})^2}\,\mathrm{d}t,$$

which is just (7).

□

REMARK (iii) Proofs (*iii*) and (*iv*) are by Burk, [*Burk 1985, 1987*]; proof (*iv*)) is due to Carlson, [*Carlson 1972b*]. Other proofs of parts of (6) have been given, see for instance [*College Math. J., 14 (1983), 353–356*], [*Pérez Marco; Yang Y 1987*].

REMARK (iv) In particular the above results show that these means are internal, monotonic and it is easily seen that they are also homogeneous.

REMARK (v) The result $\mathfrak{L}(a,b) < \mathfrak{A}(a,b), a \neq b$, occurs, heavily disguised, in [*Pólya & Szegö 1951 p.9 (5)*]. In the same reference an even stronger inequality is given

$$\mathfrak{L}(a,b) < \frac{2}{3}\mathfrak{G}(a,b) + \frac{1}{3}\mathfrak{A}(a,b) < \mathfrak{A}(a,b),\ a \neq b;$$

it is also heavily disguised; [*Polya & Szegö 1951 p.9 (6)*]. A proof is given in VI 2.1.3 Remark (v).
Sándor, [*Sándor1995a*], has shown that if $a \neq b$ then:

$$\frac{2}{e}\mathfrak{A}(a,b) < \mathfrak{I}(a,b) < \mathfrak{A}(a,b).$$

REMARK (vi) The inequality used in proof (*i*), I 2.2(14), can be extended to

$$\frac{\sinh x}{\sqrt{\sinh^2 x + \cosh^2 x}} < \tanh x < x < \sinh x < \frac{\sinh 2x}{2},\ x > 0,$$

when the same substitution, $x = \dfrac{1}{2}\log(b/a)$, leads to

$$\mathfrak{H}(a,b) \le \mathfrak{G}(a,b) \le \mathfrak{L}(a,b) \le \mathfrak{A}(a,b) \le \mathfrak{Q}(a,b),$$

where $\mathfrak{Q}(a,b) = \sqrt{(a^2+b^2)/2}$, the *quadratic mean of* a, b; see III 1. If in 1.3.1 Figure 1 the trapezium is divided into two equal areas by a line parallel to AB then its length is $\mathfrak{Q}(a,b)$.

Remark (vii) Proof (*ii*) of (7) gives another proof of (GA).

Alzer has used the Čebišev inequality, 5.3, together with the logarithmic mean to give another proof of (GA); [*Alzer 1991*a].

□ Let $\underline{a}, \underline{w}$ be n-tuples with $\underline{a}$ increasing and $W_n = 1$, then, putting $\mathfrak{A} = \mathfrak{A}_n(\underline{a}; \underline{w})$, $\mathfrak{G} = \mathfrak{G}_n(\underline{a}; \underline{w})$,

$$\log \left(a_i/\mathfrak{A}\right)^{w_i} = \frac{w_i(a_i - \mathfrak{A})}{\mathfrak{L}(a_i, \mathfrak{A})}, \ 1 \le i \le n.$$

Adding these leads to

$$\begin{aligned} \log \mathfrak{G}/\mathfrak{A} &= \log \prod_{i=1}^{n} \left(a_i/\mathfrak{A}\right)^{w_i} = \sum_{i=1}^{n} \frac{w_i(a_i - \mathfrak{A})}{\mathfrak{L}(a_i, \mathfrak{A})} \\ &= \sum_{i=1}^{n} \frac{w_i a_i}{\mathfrak{L}(a_i, \mathfrak{A})} - \Big(\sum_{j=1}^{n} w_j a_j\Big)\Big(\sum_{i=1}^{n} \frac{w_i}{\mathfrak{L}(a_i, \mathfrak{A})}\Big). \end{aligned} \tag{8}$$

By hypothesis $\underline{a}$ is increasing and so by Theorem 14 the n-tuple $\big(\mathfrak{L}(a_i, \mathfrak{A}),\ 1 \le i \le n\big)$ is also increasing. Hence by 5.3(1) the right-hand side of (8) is non-positive. This gives (GA), and the case of equality follows from that of 5.3(1). □

Remark (viii) The logarithmic mean occurs in problems of heat flow; see [*Walker, Lewis & McAdams*]. Dodd has pointed out that if a collection of incomes has a distribution between a and b that is proportional to their reciprocals then the mean income is $\mathfrak{L}(a, b)$; [*Dodd 1941b*]

Remark (ix) The topic of the identric and logarithmic means is taken up in more detail later, see VI 2.1.1

5.6 Zeros of a Polynomial and Its Derivative

Theorem 16 *Let $\underline{x}$ be the n-tuple of the real distinct zeros of a real polynomial of degree n arranged in increasing order, and let $\underline{y}$ be the $(n-1)$-tuple of the real distinct zeros of its derivative, also in increasing order, then*

$$\mathfrak{A}_j(\underline{x}) \ge \mathfrak{A}_{j-1}(\underline{y}); \quad \mathfrak{A}_j(x_{n-j+1}, \ldots, x_n) \ge \mathfrak{A}_{j-1}(y_{n-j+1}, \ldots, y_{n-1}), \quad 2 \le j \le n,$$

Further if f is a convex function on $[x_1, x_n]$ and if $\underline{A}, \underline{B}$ are defined by $A_i = \mathfrak{A}_{n-1}(\underline{x}'_i)$, $B_j = \mathfrak{A}_{n-2}(\underline{y}'_j)$ $1 \le i \le n, 1 \le j \le n-1$, then

$$\mathfrak{A}_n(f(\underline{x})) \ge \mathfrak{A}_{n-1}(f(\underline{y})); \quad \mathfrak{A}_n(f(\underline{A})) \le \mathfrak{A}_{n-1}(f(B)).$$

Remark (i) See [*AI pp.233–234*], [*Bray; Popoviciu 1944; Toda*]; see also V 2 Remark (x), and for another amusing result involving polynomials see [*Klamkin & Grosswald*].

5.7 Nanson's Inequality

THEOREM 17 *If $\underline{a}$ is a convex $(2n+1)$-tuple and if $\underline{b} = \{a_2, a_4, \ldots, a_{2n}\}$, $\underline{c} = \{a_1, a_3, \ldots, a_{2n+1}\}$ then*

$$\mathfrak{A}_n(\underline{b}) \le \mathfrak{A}_{n+1}(\underline{c}) \tag{9}$$

with equality if and only if $\underline{a}$ is an arithmetic progression.

□ Since $\underline{a}$ is convex we have that $\Delta^2 a_n \ge 0$, I 3.1. So in particular for $k = 1, 2, \ldots, n$

$$k(n-k+1)(a_{2k-1} - 2a_{2k} + a_{2k+1}) \ge 0,$$
$$k(n-k)(a_{2k} - 2a_{2k+1} + a_{2k+2}) \ge 0.$$

Adding these inequalities gives (9). The case of equality follows since $\Delta^2 a_n = \underline{0}$ implies that $\underline{a}$ is an arithmetic sequence. □

REMARK (i) An equivalent result has been generalized by Adamović & Pečarić; see [*AI pp.205–206; PPT pp.247–251*], [*Adamović & Pečarić; Andrica, Raşa & Toader; Milovanović, Pečarić & Toader; Nanson; Steinig*].
Another result involving convex n-tuples has been given by Alzer, [*Alzer 1990d*]

THEOREM 18 *If $\underline{a}$ is an n-tuple with the $(n+1)$-tuple $0, a_1, \ldots, a_n$ concave then*

$$\mathfrak{A}_n(\underline{a}) < \frac{e}{2}\mathfrak{G}_n(\underline{a}),$$

and the constant is best possible.

5.8 THE PSEUDO ARITHMETIC MEANS AND PSEUDO GEOMETRIC MEANS If $\underline{a}, \underline{w}$ are n-tuples then

$$\mathfrak{a}_n(\underline{a};\underline{w}) = \frac{W_n}{w_1}a_1 - \frac{1}{w_1}\sum_{i=2}^{n} w_i a_i, \quad \mathfrak{g}_n(\underline{a};\underline{w}) = \frac{a_1^{W_n/w_1}}{\prod_{i=2}^{n} a_i^{w_i/w_1}}; \tag{10}$$

are called the *pseudo arithmetic mean, and pseudo geometric mean of $\underline{a}$ with weight $\underline{w}$*, respectively.
Other forms of (10) are worth noting:

$$\begin{aligned} \mathfrak{a}_n(\underline{a};\underline{w}) &= \frac{W_n}{w_1}a_1 - \frac{W_{n-1}}{w_1}\mathfrak{A}_{n-1}(\underline{a}'_1;\underline{w}'_1), \\ \mathfrak{g}_n(\underline{a};\underline{w}) &= \frac{a_1^{W_n/w_1}}{\left(\mathfrak{G}_{n-1}(\underline{a}'_1;\underline{w}'_1)\right)^{(W_{n-1})/w_1}}; \end{aligned} \tag{11}$$

$$a_1 = \mathfrak{A}_n(\underline{a}^\sharp;\underline{w}) = \mathfrak{G}_n(\underline{a}^\flat;\underline{w}); \tag{12}$$

where $\underline{a}^\sharp$ and $\underline{a}^\flat$ are defined as:

$$a_i^\sharp = \begin{cases} \mathfrak{a}_n(\underline{a};\underline{w}), & \text{if } i=1,\\ a_i, & \text{if } 2\le i\le n,\end{cases} \qquad a_i^\flat = \begin{cases} \mathfrak{g}_n(\underline{a};\underline{w}), & \text{if } i=1,\\ a_i, & \text{if } 2\le i\le n.\end{cases}$$

REMARK (i) Simple examples show that in general the quantities defined in (10) do not satisfy internality, see 1.1. This explains the term pseudo mean. Indeed if we assume that $a_1 = \min \underline{a}$, then $\mathfrak{a}_n(\underline{a};\underline{w}) \ge \min \underline{a} = a_1$ implies, using (11), that $a_1 \ge \mathfrak{A}_{n-1}(\underline{a}_1';\underline{w}_1')$, which is false.

REMARK (ii) This lack of internality is not surprising since these pseudo means are cases of the arithmetic and geometric means with general weights that do not satisfy 3.7 (49), a condition that is necessary and sufficient for internality, see I 4.3 Remark(v); precisely, $\mathfrak{a}_n(\underline{a};\underline{w}) = \mathfrak{A}_n(\underline{a};\underline{w}')$, $\mathfrak{g}_n(\underline{a};\underline{w}) = \mathfrak{G}_n(\underline{a};\underline{w}')$, where

$$w_i' = \begin{cases} \dfrac{W_n}{w_1} & \text{if } i=1,\\[2mm] -\dfrac{w_i}{w_1} & \text{if } 2\le i\le n.\end{cases}$$

THEOREM 19 *If $\underline{a}, \underline{w}$ are n-tuples then*

$$\mathfrak{a}_n(\underline{a};\underline{w}) \le \mathfrak{g}_n(\underline{a};\underline{w}), \tag{13}$$

with equality if and only if $\underline{a}$ is constant.

□ We give several proofs of this result.

(*i*) From (11) we have:

$$\begin{aligned} \mathfrak{a}_n(\underline{a};\underline{w}) &\le a_i^{W_n/w_1}\big(\mathfrak{A}_{n-1}(\underline{a}_1';\underline{w}_1')\big)^{-(W_{n-1})/w_1}, && \text{by } (\sim\text{GA}),\ 3.7 \text{ Theorem 23},\\ &\le a_i^{W_n/w_1}\big(\mathfrak{G}_{n-1}(\underline{a}_1';\underline{w}_1')\big)^{-(W_{n-1})/w_1}, && \text{by (GA)},\\ &= \mathfrak{g}_n(\underline{a};\underline{w}). \end{aligned}$$

(*ii*) By (12) and (GA),

$$\begin{aligned} a_1 = \mathfrak{A}_n(\underline{a}^\sharp;\underline{w}) &\ge \mathfrak{G}_n(\underline{a}^\sharp;\underline{w})\\ &= \big(\mathfrak{a}_n(\underline{a};\underline{w})\big)^{w_1/W_n}\prod_{i=2}^n a_i^{w_i/W_n}; \end{aligned}$$

this, on simplification is just the required inequality.

(*iii*) Simple calculations, from (12), give

$$W_n\big(\mathfrak{A}_n(\underline{a}^\flat;\underline{w}) - \mathfrak{G}_n(\underline{a}^\flat;\underline{w})\big) = w_1\big(\mathfrak{g}_n(\underline{a};\underline{w}) - \mathfrak{a}_n(\underline{a};\underline{w})\big). \tag{14}$$

This implies the result by (GA).
(*iv*) Since (13) is trivial if $\mathfrak{a}_n(\underline{a};\underline{w}) \le 0$ we can assume it to be positive and apply the reverse Jensen inequality, I 4.4 Theorem 21, in a manner similar to the standard use of (J) to prove (GA), see 2.4.2 proof(*xvi*).
In all proofs the case of equality is immediate. □

REMARK (iii) The equal weight case of (13) is due to Iwamoto, Tomkins & Wang but a full discussion of these pseudo means was given later by Alzer; [*DI p.226*], [*Alzer 1990q; Iwamoto, Tomkins & Wang 1986a*].

REMARK (iv) Inequality (13) as proof (*iv*) makes obvious is an example of a reverse inequality, another similar example of which is the Aczél-Lorentz inequality, see III 2.5.7.

REMARK (v) In the case $n = 2$ (13) is just a case of 3.7 Theorem 23. Further when $n = 3$ the result follows from the discussion in the proof (*ii*) of I 4.3 Theorem 20. It is shown there that the 0-level curve of $D_3(s,t)$ at the origin lies in the second and fourth quadrants. Hence the region $\{(s,t) : s < 0, \text{ and } t < 0\}$ lies outside this level curve which implies that for such choices of s and t, ($\sim$J) holds.

COROLLARY 20 *With the same conditions as in Theorem 19,*

$$\mathfrak{g}_n(\underline{a};\underline{w}) - \mathfrak{a}_n(\underline{a};\underline{w}) \ge \mathfrak{g}_{n-1}(\underline{a};\underline{w}) - \mathfrak{a}_{n-1}(\underline{a};\underline{w});$$
$$\frac{\mathfrak{g}_n(\underline{a};\underline{w})}{\mathfrak{a}_n(\underline{a};\underline{w})} \ge \frac{\mathfrak{g}_{n-1}(\underline{a};\underline{w})}{\mathfrak{a}_{n-1}(\underline{a};\underline{w})},$$

with equality, in both cases, if and only if $a_1 = a_n$.

□ The first inequality is an immediate consequence of (14) and (R).
The second inequality follows in a similar manner from (P) and the following analogue of (14):

$$\left(\frac{\mathfrak{A}_n(\underline{a}^\sharp;\underline{w})}{\mathfrak{G}_n(\underline{a}^\sharp;\underline{w})}\right)^{W_n} = \left(\frac{\mathfrak{g}_n(\underline{a};\underline{w})}{\mathfrak{a}_n(\underline{a};\underline{w})}\right)^{w_1}.$$

The cases of equality follow from those of 3.1 Theorem 1. □

REMARK (vi) Clearly Corollary 20 gives Rado-Popoviciu type inequalities for these pseudo means that are in a certain sense sharper than the original (R) and (P).

REMARK (vii) As for (R) and (P) Corollary 20 implies the original inequality (13) and in fact can give better lower bounds, using the same idea as in 3.1 Corollary 3. In particular in the case of equal weights we get:

$$\mathfrak{g}_n(\underline{a}) - \mathfrak{a}_n(\underline{a}) \ge \max_{2\le i\le n} \frac{(a_1 - a_i)^2}{a_1}; \quad \frac{\mathfrak{g}_n(\underline{a})}{\mathfrak{a}_n(\underline{a})} \ge \max_{2\le i\le n} \frac{a_1^2}{a_1^2 - (a_1 - a_i)^2}$$

5.9 An Inequality Due to Mercer

Theorem 21 *If $\underline{a}$ is a non-constant n-tuple such that $m \le \underline{a} \le M$ then*

$$M + m - \mathfrak{A}_n(\underline{a};\underline{w}) > \frac{Mm}{\mathfrak{G}_n(\underline{a};\underline{w})}.$$

□ If $0 < m \le t \le M$ then obviously $(t-m)(M-t) \ge 0$ with equality if and only if either $t = m$ or $t = M$. Equivalently, see I 2.2(23):

$$M + m - t \ge \frac{Mm}{t}, \quad \text{with equality if and only if either } t = m \text{ or } t = M.$$

Put $t = a_i$, $1 \le i \le n$, and then take the arithmetic mean of the left-hand sides and the geometric means of the right-hand sides and the result is immediate using (GA). □

This result can be found in [*Mercer A 2002*] and the substitutions $m \mapsto m^{-1}$, $M \mapsto M^{-1}, \underline{a} \mapsto \underline{a}^{-1}$, lead to

$$M + m - \mathfrak{A}_n(\underline{a};\underline{w}) > \frac{Mm}{\mathfrak{G}_n(\underline{a};\underline{w})} > \frac{1}{M^{-1} + m^{-1} - \big(\mathfrak{H}_n(\underline{a};\underline{w})\big)^{-1}}. \tag{15}$$

For a generalization see III 6.4 Remark (x).

III THE POWER MEANS

This chapter is devoted to the properties and inequalities of the classical generalization of the arithmetic, geometric and harmonic means, the power means. The inequalities obtained in the previous chapter are extended to this scale of means. In addition some results for sums of powers are obtained, the classical inequalities of Minkowski, Cauchy and Hölder, and some generalization of these results. Various generalizations of the power mean family are also discussed.

1 Definitions and Simple Properties

DEFINITION 1 *Let $\underline{a}, \underline{w}$ be two n-tuples, $r \in \overline{\mathbb{R}}$, then the* r-th power mean of $\underline{a}$ with weight $\underline{w}$ *is*

$$\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) = \begin{cases} \left(\dfrac{1}{W_n}\displaystyle\sum_{i=1}^{n} w_i a_i^r\right)^{1/r}, & \text{if } r \in \mathbb{R}^*, \\ \mathfrak{G}_n(\underline{a};\underline{w}), & \text{if } r = 0, \\ \max \underline{a}, & \text{if } r = \infty, \\ \min \underline{a}, & \text{if } r = -\infty. \end{cases} \tag{1}$$

As in the previous chapter we will just write $\mathfrak{M}_n^{[r]}$ if the references are unambiguous, $\mathfrak{M}_n^{[r]}(\underline{a})$ will denote the equal weight case, $\mathfrak{M}^{[r]}$ will be used if $n = 2$; see II 1.1 Conventions 1, 2, 3; and if I is an index set the notation $\mathfrak{M}_I^{[r]}(\underline{a};\underline{w})$ is used in the manner of Notations 6 (xi), I 4.2, II 3.2.2.
Since $\mathfrak{M}_n^{[1]} = \mathfrak{A}_n, \mathfrak{M}_n^{[0]} = \mathfrak{G}_n, \mathfrak{M}_n^{[-1]} = \mathfrak{H}_n$ the power means form a natural extension of these elementary means. The case $r = 2$ is often called the *quadratic mean, of $\underline{a}$ with weight $\underline{w}$*, and written $\mathfrak{Q}_n(\underline{a};\underline{w})$; the case $n = 2$ was introduced in II 5.5. In addition the power means are sometimes called *Hölder means*.
The following result shows that the r-th power mean, $r \in \mathbb{R}$, is a mean in the sense of II 1.1 Theorem 2 and II 12, Theorem 6, and that the special definitions given for the cases $r = 0, \pm\infty$ are reasonable.

THEOREM 2 *(a) If $r \in \mathbb{R}$ then $\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})$ has the properties (Co), (Ho), (Mo), (Re), (Sy*), (Sy) in the case of equal weights, and is strictly internal,*

$$\min \underline{a} \le \mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) \le \max \underline{a}, \tag{2}$$

with equality if and only if $\underline{a}$ is constant.

(b)

$$\lim_{r\to 0}\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})=\mathfrak{G}_n(\underline{a};\underline{w}).$$

(c)

$$\lim_{r\to\infty}\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})=\max\underline{a};\qquad \lim_{r\to-\infty}\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})=\min\underline{a}.$$

□ (a) Immediate

(b) We give several proofs of this result, and in all of them we assume that $r\in\mathbb{R}^*$, and without loss in generality that $W_n=1$.

(i)[*Paasche*]

$$\begin{aligned}\lim_{r\to 0}\log\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})&=\lim_{r\to 0}\frac{\log\big(\sum_{i=1}^n w_ia_i^r\big)}{r}\\&=\lim_{r\to 0}\frac{\sum_{i=1}^n(w_ia_i^r\log a_i)}{\sum_{i=1}^n w_ia_i^r},\quad\text{by l'Hôpital's Rule,}\\&=\sum_{i=1}^n(w_i\log a_i)=\mathfrak{A}_n\big(\log(\underline{a});\underline{w}\big),\end{aligned}$$

which gives the result by II 1.2 (8). The use of l'Hôpital's Rule[1] is easily justified.

(ii)

$$\begin{aligned}\log\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})=&\frac{1}{r}\log\big(\sum_{i=1}^n w_ia_i^r\big)\\=&\frac{1}{r}\log\big(1+r\sum_{i=1}^n(w_i\log a_i)+O(r^2)\big),\end{aligned}$$

using the Taylor expansion of the exponential function. The result is now immediate by I 2.2 (9), and II 1.2 (8).

(iii) By the note in the proof of I 2.1 Corollary 4 $a_i^r=1+b_i$, where $b_i=O(r)$ as $r\to 0$, $1\le i\le n$; and by I 2.1(5) $a^{w_ir}=(1+b_i)^{w_i}=1+w_ib_i+O(r^2)$, as $r\to 0$. Hence,

$$\begin{aligned}\frac{\mathfrak{G}_n(\underline{a};\underline{w})}{\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})}=&\left(\frac{\prod_{i=1}^n(1+b_i)^{w_i}}{\sum_{i=1}^n w_i(1+b_i)}\right)^{1/r}\\=&\left(\frac{1+\sum_{i=1}^n w_ib_i+O(r^2)}{1+\sum_{i=1}^n w_ib_i}\right)^{1/r}\\=&\big(1+O(r^2)\big)^{1/r}=1+O(r),\ \text{as } r\to 0,\end{aligned}$$

---

[1] L'Hôpital's Rule is:*if f,g are real differentiable functions defined on the interval* $]a,b[$*, bounded or unbounded, with g' never zero and if, with $c=a$ or $c=b$, either* $\lim_{x\to c}f(x)=\lim_{x\to c}g(x)=0$ *or* $\lim_{x\to c}|g(x)|=\infty$*, then* $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$*, provided the right-hand side limits exists, finite or infinite*; [*EM5 pp.407–408*].

from which (b) is immediate.
(*iv*) Assume $r > 0$, then as in (*ii*),

$$\begin{aligned}\log \mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) =& \frac{1}{r}\log\big(\sum_{i=1}^n w_i a_i^r\big)\\ \le& \frac{1}{r}\big((\sum_{i=1}^n w_i a_i^r) - 1\big), \quad \text{by I 2.2(9)},\\ =& \sum_{i=1}^n w_i \frac{a_i^r - 1}{r} \to \sum_{i=1}^n w_i \log a_i, \ \text{ as } r \to 0,\\ =& \log \mathfrak{G}_n(\underline{a};\underline{w}).\end{aligned}$$

The opposite inequality follows from inequality (r;s) given below, 3.1.1.
The case $r < 0$ is handled similarly.
(c) Assume $r \in \mathbb{R}_+^*$ and without loss in generality that $\max \underline{a} = a_n$, then

$$\Big(\frac{w_n}{W_n}\Big)^{1/r} a_n \le \mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) \le a_n,$$

which implies the first part of (c).
A similar proof can be given for the other part of (c).
A slightly different proof of the equal weight case is given below in 2.3. □

REMARK (i) It follows from the internality that if $\underline{a}$ is a strictly increasing sequence then so is $\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}), n \in \mathbb{N}^*$; see II 5.2 (c).

REMARK (ii) More details of the behaviour of $\mathfrak{M}_n^{[r]}$ as $r \to 0, \pm\infty$ can be found in [*Gustin 1950; Shniad*].

REMARK (iii) Another proof of the equal weight case of (b) is given below, 3.1.1 Remark (xi).

The following simple identity is often useful.
If $r, s \in \mathbb{R}^*$, $t = s/r$, $\underline{b} = \underline{a}^r, \underline{c} = \underline{a}^t$ then

$$\mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) = \big(\mathfrak{M}_n^{[t]}(\underline{b};\underline{w})\big)^{t/s} = \big(\mathfrak{M}_n^{[r]}(\underline{c};\underline{w})\big)^{1/t} \tag{3}$$

in particular of course, taking $r = s$ in the first identity,

$$\mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) = \big(\mathfrak{A}_n(\underline{b};\underline{w})\big)^{1/s}. \tag{4}$$

REMARK (iv) These extend the identities II 1.2 (7), (8).

THEOREM 3 *If $\underset{\sim}{a}, \underset{\sim}{u}, \underset{\sim}{v}$ are n-tuples with $\underset{\sim}{a}$ increasing and $U_n = V_n$, and if for $2 \le k \le n$, $0 \le V_n - V_{k-1} \le U_n - U_{k-1} \le U_n = V_n$, then for $r \in \mathbb{R}^*$,*

$$\mathfrak{M}_n^{[r]}(\underset{\sim}{a}; \underset{\sim}{v}) \le \mathfrak{M}_n^{[r]}(\underset{\sim}{a}; \underset{\sim}{u}).$$

□ Assume $0 < r < \infty$ then

$$\begin{aligned}\sum_{i=1}^{n} u_i a_i^r =& a_1 U_n + \sum_{i=2}^{n} (U_{i-1} - U_n) \Delta a_{i-1}^r \\ \ge& a_1 V_n + \sum_{i=2}^{n} (V_{i-1} - V_n) \Delta a_{i-1}^r = \sum_{i=1}^{n} v_i a_i^r.\end{aligned}$$

□

REMARK (v) In particular if $0 \le \underset{\sim}{v} \le \underset{\sim}{u}$ with $U_n = V_n$ this reduces to a result of Castellano; [*Castellano 1948*]. The general result is due to Pečarić & Janić.

2 Sums of Powers

Before obtaining some deeper properties of the power means we study sums of powers.

2.1 HÖLDER'S INEQUALITY The following theorem, *Hölder's inequality*, is basic to all studies of power means and has many other applications.[2]

THEOREM 1 [HÖLDER'S INEQUALITY] *If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are two n-tuples and $p > 1$ then*

$$\sum_{i=1}^{n} a_i b_i \le \Big(\sum_{i=1}^{n} a_i^p\Big)^{1/p} \Big(\sum_{i=1}^{n} b_i^{p'}\Big)^{1/p'}. \qquad (H)$$

If $p < 1$, $p \ne 0$, then ($\sim$H) holds. In both cases there is equality if and only if $\underset{\sim}{a}^p \sim \underset{\sim}{b}^{p'}$

□ `[Case 1;` $p > 1$`]` We give seven proofs of this basic result.

(i) Inequality (H) can be written as

$$\sum_{i=1}^{n} \left(\frac{a_i^p}{\sum_{j=1}^{n} a_j^p}\right)^{1/p} \left(\frac{b_i^{p'}}{\sum_{j=1}^{n} b_j^{p'}}\right)^{1/p'} \le 1.$$

By (GA), in the form of II 2.2.2 (5), the left-hand side of this inequality is not greater than

$$\sum_{i=1}^{n} \left(\frac{1}{p}\frac{a_i^p}{\sum_{j=1}^{n} a_j^p} + \frac{1}{p'}\frac{b_i^{p'}}{\sum_{j=1}^{n} b_j^{p'}}\right) = \frac{1}{p} + \frac{1}{p'} = 1.$$

[2] p' used in Theorem 1 and elsewhere is the index conjugate to p; see Notations 4.

This completes the proof of (H) and the case of equality follows from that for (GA).

(*ii*) Put $\alpha = 1/p, \beta = 1/p', a = a_i\Big/\Big(\sum_{j=1}^n a_j^p\Big)^{1/p}, b = b_i\Big/\Big(\sum_{j=1}^n b_j^{p'}\Big)^{1/p'}$ in II 2.2.2(4). Then summing over i gives (H) as in (*i*).

(*iii*) [*Matkowski & Rätz 1993*] If $h\ :]0,\infty[\mapsto \mathbb{R}$ is concave and if $\underline{c}, \underline{d}$ are two n-tuples then ($\sim$J) is equivalent to

$$h(C_n/D_n) \geq \frac{1}{D_n}\sum_{i=1}^n d_i h(c_i/d_i);$$

further if h is strictly concave then there is equality if and only if $\underline{c} \sim \underline{d}$; I 4.2 Theorem 12.

Taking $h(t) = t^{1/p}$, $p > 1$, this inequality is

$$C_n^{1/p} D_n^{1/p'} \geq \sum_{i=1}^n c_i^{1/p} d_i^{1/p'},$$

which after a simple change of variable is (H). The case of equality is immediate.

(*iv*) [*Soloviov*] Let $\gamma(\underline{a}) = \Big(\sum_{i=1}^n a_i^p\Big)^{1/p}$ then γ is homogeneous and strictly convex by I 4.6 Example (vii). Further $\gamma_i'(\underline{v}) = v_i^{p-1}\Big(\sum_{i=1}^n v_i^p\Big)^{-1/p'}$. Taking $\underline{v} = \underline{b}^{1/(p'-1)}$, and $\underline{u} = \underline{a}$ the support inequality, I 4.6 (23), is just (H).

(*v*) A proof using II 2.4.6 Lemma 20 can be given; [*Sándor & Szabó; Ušakov*]. Using the notation of that lemma take $f_i(x) = a_i^p x^{1/p'} + b_i^{p'} x^{-1/p}$, $x \in M = \mathbb{R}_+^*$, $1 \leq i \leq n$. So $\sum_{i=1}^n \inf_{x\in M} f_i(x) = K\sum_{i=1}^n a_i b_i$, where $K = (p'/p)^{1/p'} + (p/p')^{1/p}$. Then

$F(x) = \sum_{i=1}^n a_i^p x^{1/p'} + b_i^{p'} x^{-1/p}$, and $\inf_{x\in M} F(x) = (\sum_{i=1}^n a_i)^{1/p}(\sum_{i=1}^n b_i)^{1/p'}$. This by the quoted lemma completes the proof together with the case of equality.

(*vi*) There is an inductive proof in proof(*ii*) of Corollary 2 below.

(*vii*) See also 3.1.1 Remark (viii).

[Case 2; $p < 0$] Then $0 < p' < 1$, see Notations 4; put $r = -p/p'$ then $r' = 1/p'$, and $r > 1$.

Now put $\underline{c} = \underline{a}^{-p'}, \underline{d} = \underline{a}^{p'}\underline{b}^{p'}$ then,
by (H), $\sum_{i=1}^n c_i d_i \leq \big(\sum_{i=1}^n c_i^r\big)^{1/r}\big(\sum_{i=1}^n d_i^{r'}\big)^{1/r'}$, which reduces to ($\sim$H).

[Case 3; $0 < p < 1$] Apply the above argument to p' since now $p' < 0$. □

REMARK (i) If $\underline{w}$ is some n-tuple it can be written as $\underline{w} = \underline{w}^{1/p}\underline{w}^{1/p'}$ so (H) can be given a weighted form

$$\sum_{i=1}^n w_i a_i b_i \leq \Big(\sum_{i=1}^n w_i a_i^p\Big)^{1/}\Big(\sum_{i=1}^n w_i b_i^{p'}\Big)^{1/p'}. \tag{1}$$

Corollary 2 *Let $r_i > 0$, $1 \le i \le m$, and put $1/\rho_m = \sum_{i=1}^m 1/r_i$ and let $\underline{a}_i = (a_{i1}, \dots, a_{in})$, $1 \le i \le m$, then*

$$\Big(\sum_{j=1}^n \Big(\prod_{i=1}^m a_{ij}\Big)^{\rho_m}\Big)^{1/\rho_m} \le \prod_{i=1}^m \Big(\sum_{j=1}^n a_{ij}^{r_i}\Big)^{1/r_i}, \tag{2}$$

with equality if and only if the n-tuples $\underline{a}_i^{r_i}$, $1 \le i \le m$, are pairwise dependent.

□ We give two proofs of this generalization of (H).

(*i*) Proof (*i*) of Case 1 of Theorem 1 is easily extended to this more general situation.

(*ii*) A proof of (2), and incidentally of (H), can be given by an induction argument. If $n = 2$ then (H) reduces to

$$a_1b_1 + a_2b_2 \le (a_1^p + a_2^p)^{1/p}(b_1^{p'} + b_2^{p'})^{1/p'}, \tag{3}$$

which can be proved by one of the methods used to prove (H) in Theorem 1.

We can now proceed in one of two ways.

(*i*) Assume that (H) has been proved for all integers less than n, and that $p > 0, q > 0$ and $1/p + 1/q = 1/r$. Then

$$\begin{aligned}\sum_{j=1}^n a_j^r b_j^r &\le a_n^r b_n^r + \Big(\sum_{j=1}^{n-1} a_j^p\Big)^{r/p}\Big(\sum_{j=1}^{n-1} b_j^q\Big)^{r/q}, \text{ by the induction hypothesis,}\\ &\le \Big(\sum_{j=1}^{n} a_j^p\Big)^{r/p}\Big(\sum_{j=1}^{n} b_j^q\Big)^{r/q}, \text{ by (3).}\end{aligned}$$

This proves (H), which is (2) for general n and $m = 2$.

So now assume that (2) has been proved for all n and all m, $2 \le m < k$. Then,

$$\begin{aligned}\Big(\sum_{j=1}^n \Big(\prod_{i=1}^k a_{ij}\Big)^{\rho_k}\Big)^{1/\rho_k} &= \Big(\sum_{j=1}^n a_{kj}^{\rho_k}\Big(\prod_{i=1}^{k-1} a_{ij}\Big)^{\rho_k}\Big)^{1/\rho_k}\\ &\le \Big(\sum_{j=1}^n a_{kj}^{r_k}\Big)^{1/r_k}\Big(\sum_{j=1}^n \Big(\prod_{i=1}^{k-1} a_{ij}\Big)^{\rho_{k-1}}\Big)^{1/\rho_{k-1}}, \text{ by case } m = 2 \text{ of (2),}\end{aligned}$$

and (2) follows by the induction hypothesis.

(*ii*) The order of the induction can be reversed. Inequality (3) gives (2) in the case $m = n = 2$; keeping $n = 2$ assume the result for all integers less than m. Then by an induction on m similar to that above and with $\sum_{i=1}^m 1/r_i = 1$

$$\prod_{i=1}^m a_i + \prod_{i=1}^m b_i \le \prod_{i=1}^m (a_i^{r_i} + b_i^{r_i})^{1/r_i}. \tag{4}$$

The rest of the induction follows easily, as also does the case of equality. □

REMARK (ii) If $m = 2$ (2) can be written as

$$\Big(\sum_{i=1}^{n} a_i^r b_i^r\Big)^{1/r} \le \Big(\sum_{i=1}^{n} a_i^p\Big)^{1/p}\Big(\sum_{i=1}^{n} b_i^q\Big)^{1/q}, \tag{5}$$

where $p, q > 0$ and $1/p + 1/q = 1/r$. It is easily seen that if either $p < 0$, or $q < 0$ and if $r > 0$ then ($\sim$5) holds, while if $r > 0$ (5) holds. Finally if all three of these parameters are negative then ($\sim$5) holds. This can be put in a symmetric form as follows. Let $\underline{a}, \underline{b}, \underline{c}$ be three n-tuples such that $\underline{a}\,\underline{b}\,\underline{c} = \underline{e}$ and suppose that $1/p + 1/q + 1/r = 0$, with all but one of p, q, r are positive, then

$$\Big(\sum_{i=1}^{n} a_i^p\Big)^{1/p}\Big(\sum_{i=1}^{n} b_i^q\Big)^{1/q}\Big(\sum_{i=1}^{n} c_i^r\Big)^{1/r} \ge 1; \tag{6}$$

if all but one of p, q, r are negative then ($\sim$6) holds.
This result can be extended to m, $m > 3$, n-tuples with all but one of the exponents having the same sign; see [*Aczél & Beckenbach*].

REMARK (iii) Inequality (4) is of some independent interest. In particular it shows, using I 4.6 Theorem 38, that if $p : \mathbb{R}^n \mapsto \mathbb{R}$ is a homogeneous polynomial of degree m, and if $f(\underline{a}) = p(\underline{a}^{1/m})$ then f is concave, and of course homogeneous of degree 1; see V 6 Theorem 3.

The extreme generality of (H) leads to many inequalities turning out to be special cases in impenetrable disguises, as the next result illustrates; also see below, 2.5.4 Remark (iii).

THEOREM 3 *(a) If $0 < s < 1$ then (H) is equivalent to*

$$\sum_{i=1}^{n} a_i^s b_i^{1-s} \le \Big(\sum_{i=1}^{n} a_i\Big)^{s}\Big(\sum_{i=1}^{n} b_i\Big)^{1-s}, \tag{7}$$

with equality if and only if $\underline{a} \sim \underline{b}$.
(b) [LIAPUNOV'S INEQUALITY] *If $\underline{a}$ and $\underline{w}$ are n-tuples and $r > s > t > 0$ then*

$$\Big(\sum_{i=1}^{n} w_i x_i^s\Big)^{r-t} \le \Big(\sum_{i=1}^{n} w_i x_i^t\Big)^{r-s}\Big(\sum_{i=1}^{n} w_i x_i^r\Big)^{s-t}. \tag{8}$$

(c) [RADON'S INEQUALITY] *If $p > 1$ then*

$$\frac{\Big(\sum_{i=1}^{n} a_i\Big)^p}{\Big(\sum_{i=1}^{n} b_i\Big)^{p-1}} \le \sum_{i=1}^{n} \frac{a_i^p}{b_i^{p-1}}; \tag{9}$$

with equality if and only if $\underline{a} \sim \underline{b}$. If $p < 1$ then (~9) holds.

□ (a) This is seen to be equivalent to (H) by simple change of variables.
(b) In (1) substitute $\underline{a}^p = \underline{x}^t, \underline{b}^{p'} = \underline{x}^r, p = (r-t)/(r-s)$,when $p' = (r-t)/(s-t)$.
(c) (H) implies (9) by a simple change of variable; [*HLP p.61*], [*Radon*]. □

Remark (iv) Liapunov's inequality is a particular case of an inequality between the Gini means, see below 5.2.1 Remark(iv); see also [*AI p.54; DI p.157; MPF p.101; PPT p.117*], [*Allasia 1974–1975; Giaccardi 1956; Liapunov*].

Remark (v) It has been observed by Maligranda that (7) was first proved by Rogers by a use of (GA), [*Maligranda 1998; Rogers*]. As this gives Rogers a priority he has suggested that (H) should be called the *Rogers' inequality*, or at least the *Rogers-Hölder inequality*.

Other forms of (H) are the following.

Theorem 4 *(a) If $p > 1$ then*

$$\Big(\sum_{i=1}^n a_i^p\Big)^{1/p} = \sup \sum_{i=1}^n a_i b_i,$$

where the sup is over all $\underline{b}$ such that $\sum_{i=1}^n b_i^{p'} = 1$; the sup is attained if and only if $\underline{a}^p \sim \underline{b}^{p'}$.
(b) If $p > 1$ and $\underline{a}, \underline{b}$ are n-tuples with $A_n = B_n = 1$ then

$$\sum_{i=1}^n a_i^{1/p} b_i^{1/p'} \le 1,$$

with equality if and only if $\underline{a} \sim \underline{b}$.

Remark (vi) The first part of this theorem is an extremely useful way of considering Theorem 1 and is the basis of a method of proof called *quasi-linearization*, see [*BB p.23; MPF pp.669–679*].

Remark (vii) Many other inequalities can be considered this way; for instance (GA): assuming without loss in generality that $W_n = 1$,

$$\mathfrak{G}_n(\underline{a}; \underline{w}) = \inf_{\underline{c} \in C} \sum_{i=1}^n w_i c_i a_i, \text{ where } C = \{\underline{c};\ \mathfrak{G}_n(\underline{c}; \underline{w}) = 1\}.$$

Remark (viii) Proofs of (H) can be found in many places; see [*AI p.50; BB p.19; HLP p.21*], [*Abou-Tair & Sulaiman 2000; Avram & Brown; Hölder; Iwamoto; Liu*

& Wang; Lou; Matkowski 1991; Redheffer 1981; Vasić & Kečkić 1972; Wang C L 1977; You 1989a,b; Zorio].

2.2 CAUCHY'S INEQUALITY If $p = 2$, when of course $p' = 2$, (H) reduces to

$$\sum_{i=1}^{n} a_i b_i \le \Big(\sum_{i=1}^{n} a_i^2\Big)^{1/2} \Big(\sum_{i=1}^{n} b_i^2\Big)^{1/2}, \tag{C}$$

known variously as *Cauchy's inequality*, the *Cauchy-Schwarz inequality*[3], or the *Cauchy-Schwarz-Bunyakovskiĭ inequality*, see [*Bunyakovskiĭ*]; we will refer to it as (C). A discussion of the history of this inequality can be found in [*Schreiber; Zhang, Bao & Fu*]; an exhaustive survey of (C) and related inequalities can be found in [*Dragomir 2003*].

Obviously any proof of (H) provides a proof of (C). However, the simple nature of (C) allows for various direct proofs that show that 2.1 Theorem 1 holds for all real n-tuples $\underline{a}, \underline{b}$ when $p = 2$. For instance, either of the following identities implies (C):

$$\sum_{i=1}^{n} (a_i x + b_i)^2 = x^2 \sum_{i=1}^{n} a_i^2 + 2x \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2$$

$$\Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big) - \Big(\sum_{i=1}^{n} a_i b_i\Big) = \frac{1}{2} \sum_{i,j=1}^{n} (a_i b_j - a_j b_i)^2.$$

The second identity is known as the *Lagrange identity*. For other proofs see [*1938a,b; Cannon; Dubeau 1990a,1991b; Eames; Sinnadurai 1963*].

REMARK (i) Of course (C) arises from (H) by taking $p = p'$. By taking $r_1 = \cdots = r_m = m$ in (2) we get a simple extension of (C); [*Kim S*].

$$\sum_{j=1}^{n} a_{1j} \ldots a_{mj} \le \prod_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij}^m\Big)^{1/m}.$$

THEOREM 5 *(H) and (C) are equivalent.*

□ That (H) implies (C) is trivial and for the converse we show that (C) implies the extension of (H) in 2.1 Corollary 2, (2).

The proof is in five steps and we assume in Corollary 2, without loss in generality, that $\rho_m = 1$.

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[3] This is K.Schwarz.

(i) $m = 2$ and $r_1 = r_2 = 2$: then Corollary 2 is just (C).

(ii) $m = 2^\mu$, $\mu \in \mathbb{N}^*$, $r_1 = \cdots = r_{2^\mu} = 2^\mu$: the proof of this case of (2) is by induction on μ, $\mu = 1$ being (i). So suppose the result is known for integers k, $1 \le k < \mu$. Then

$$\begin{aligned}\sum_{j=1}^{n}\prod_{i=1}^{2^\mu} a_{ij} &= \sum_{j=1}^{n}\Big(\prod_{i=1}^{2^{\mu-1}} a_{ij}\Big)\Big(\prod_{i=2^{\mu-1}+1}^{2^\mu} a_{ij}\Big)\\ &\le \Big(\sum_{j=1}^{n}\Big(\prod_{i=1}^{2^{\mu-1}} a_{ij}^2\Big)\Big)^{1/2}\Big(\sum_{j=1}^{n}\Big(\prod_{i=2^{\mu-1}+1}^{2^\mu} a_{ij}^2\Big)\Big)^{1/2}, \text{ by } (i),\\ &\le \prod_{i=1}^{2^\mu}\Big(\sum_{j=1}^{n} a_{ij}^{2^\mu}\Big)^{1/2^\mu}, \quad \text{by the induction hypothesis.}\end{aligned}$$

(iii) Now let m be any integer, $r_1 = \cdots = r_m = m$, and suppose $2^\mu > m$. Define α_{ij}, $1 \le i \le 2^\mu$, $1 \le j \le n$, as follows:

$$\alpha_{ij} = \begin{cases} a_{ij}^{m/2^\mu}, & \text{if } 1 \le i \le m,\ 1 \le j \le n,\\ \Big(\prod_{k=1}^{m} a_{kj}\Big)^{1/2^\mu}, & \text{if } m \le i \le 2^\mu,\ 1 \le j \le n.\end{cases}$$

Then by (ii)

$$\begin{aligned}\sum_{j=1}^{n}\prod_{i=1}^{m} a_{ij} &= \sum_{j=1}^{n}\prod_{i=1}^{2^\mu} \alpha_{ij}\\ &\le \prod_{i=1}^{2^\mu}\Big(\sum_{j=1}^{n} \alpha_{ij}^{2^\mu}\Big)^{1/2^\mu} = \prod_{i=1}^{m}\Big(\sum_{j=1}^{n} a_{ij}^{m}\Big)^{1/m}.\end{aligned}$$

(iv) Let m be any positive integer, $r_i \in \mathbb{Q}$, $1 \le i \le m$. Then for some integers μ, μ_i, $1 \le i \le m$, we have $r_i = \mu_i/\mu$, $1 \le i \le m$. Define $\alpha_{ij} = a_{ij}^{\mu_i}$, $1 \le i \le m$, $1 \le j \le n$, then applying (iii) to the $\{\alpha_{ij}\}$ gives the result in this case.

(v) The general case of real r_i, $1 \le i \le m$, follows by taking limits.

It remains to consider the case of equality. Let $1/r_i = q_i + p_i, q_i \in \mathbb{Q}$, $1 \le i \le m$,

then

$$\begin{aligned}
\sum_{j=1}^{n}\prod_{i=1}^{m} a_{ij} &=\sum_{j=1}^{n}\prod_{i=1}^{m} a_{ij}^{r_i(q_i+p_i)}\\
&=\sum_{j=1}^{n}\Big(\prod_{i=1}^{m} a_{ij}^{r_iq_i/Q_n}\Big)^{Q_n}\Big(\prod_{i=1}^{m} a_{ij}^{r_ip_i/P_n}\Big)^{P_n}\\
&\le\Big(\sum_{j=1}^{n}\prod_{i=1}^{m} a_{ij}^{r_iq_i/Q_n}\Big)^{Q_n}\Big(\sum_{j=1}^{n}\prod_{i=1}^{m} a_{ij}^{r_ip_i/P_n}\Big)^{P_n}, \text{ by the above,}\\
&\le\prod_{i=1}^{m}\Big(\sum_{j=1}^{n} a_{ij}^{r_i}\Big)^{q_i}\Big(\sum_{j=1}^{n}\prod_{i=1}^{m} a_{ij}^{r_ip_i/P_n}\Big)^{P_n}, \text{ by the above up to } (iv),\\
&\le\prod_{i=1}^{m}\Big(\sum_{j=1}^{n} a_{ij}^{r_i}\Big)^{q_i}\prod_{i=1}^{m}\Big(\sum_{j=1}^{n} a_{ij}^{r_i}\Big)^{p_i}, \text{ by the above,}\\
&=\prod_{i=1}^{m}\Big(\sum_{j=1}^{n} a_{ij}^{r_i}\Big)^{1/r_i}.
\end{aligned}$$

In using the above arguments up to (iv) we have strict inequality unless the n-tuples $\underline{a}_i^{r_i}$, $1 \le i \le m$, are pairwise dependent. This gives the case of strict inequality in the general case, and Theorem 3 is proved. □

EXAMPLE (i) Consider the part of an ellipse $\underline{x} = (a\cos\theta, b\sin\theta), 0 \le \theta \le \pi/2$, where $0 < b < a$. The distance, d, of the normal at the point $\underline{x}$ from the origin is given by $d = |\underline{x}.\underline{t}|/|\underline{t}|$, where $\underline{t} = (-a\sin\theta, b\cos\theta)$. Simple calculations give that $d = (a^2 - b^2)/\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$. Now by (C)

$$\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}=\sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)(\cos^2\theta + \sin^2\theta)} \ge (a+b)\sin\theta\cos\theta.$$

with equality if and only if $\theta = \theta_0$ where $\tan\theta_0 = \sqrt{b/a}$. Hence the minimum distance of a normal to the ellipse from the origin is $(a - b)$, and it occurs when $\theta = \theta_0$; [*Klamkin & McLenaghan*].

EXAMPLE (ii) (C) can be used to give a very simple proof of (HA) in the equivalent form II 2.4.3(12); see [*Batinetu*].

$$\Big(\sum_{i=1}^{n} w_ia_i\Big)\Big(\sum_{i=1}^{n}\frac{w_i}{a_i}\Big) \ge \Big(\sum_{i=1}^{n}(w_ia_i)^{1/2}(\frac{w_i}{a_i})^{1/2}\Big)^2 = W_n^2;$$

2.3 POWER SUMS For an n-tuple $\underline{a}$ write

$$\mathcal{S}_n(r,\underline{a}) = \sum_{i=1}^{n} a_i^r,\ r\in\mathbb{R},\quad \mathcal{R}_n(r,\underline{a}) = \mathcal{S}_n^{1/r}(r,\underline{a}),\ r\in\mathbb{R}^*;$$

as usual reference to n and, or $\underline{a}$, will be omitted if there is no resulting confusion. We use (H) to obtain some properties of these quantities.
Since $\mathcal{R}(r) = n^{1/r}\mathfrak{M}_n^{[r]}(\underline{a})$ we have by 1 Theorem 2(c) that $\lim_{r\to\infty}\mathcal{R}(r) = \max\underline{a}$ and $\lim_{r\to-\infty}\mathcal{R}(r) = \min\underline{a}$; or just note that,

$$\max\underline{a} \le \mathcal{R}(r) \le n^{1/r}\max\underline{a},\ r > 0;\ n^{1/r}\min\underline{a} \le \mathcal{R}(r) \le \min\underline{a},\ r < 0.$$

This also shows that the equal weight case of 1 Theorem 2(c) can be deduced from the limit values of $\mathcal{R}(r)$.
From the remark in the proof of I 2.1 Corollary 4 we have $\mathcal{S}(r) = n + O(r)$, $r \to 0$. Hence, $\lim_{r\to 0}\mathcal{R}_n(r, \underline{a}) = \infty$, if $n > 1$. The same result follows from the simple observations,

$$n^{1/r}\min\underline{a} \le \mathcal{R}(r) \le n^{1/r}\max\underline{a},\ r > 0;\ n^{1/r}\max\underline{a} \le \mathcal{R}(r) \le n^{1/r}\min\underline{a},\ r < 0.$$

LEMMA 6 *(a) $\mathcal{R}$ is strictly decreasing; that is if $r < s$ then*

$$\mathcal{R}(s) = \Big(\sum_{i=1}^n a_i^s\Big)^{1/s} < \Big(\sum_{i=1}^n a_i^r\Big)^{1/r} = \mathcal{R}(r). \tag{10}$$

(b) Both of $\mathcal{S}, \mathcal{R}$ are log-convex.
More precisely if $0 \le \lambda \le 1$ and $r, s \in \mathbb{R}$ then

$$\mathcal{S}\big((1-\lambda)r + \lambda s\big) \le \mathcal{S}^{(1-\lambda)}(r)\mathcal{S}^{\lambda}(s); \tag{11}$$

with, if $\lambda \ne 0, 1$, equality in (11) if and only if $\underline{a}$ is constant;
if $r, s > 0$ then

$$\mathcal{R}\big((1-\lambda)r + \lambda s\big) \le \mathcal{R}^{(1-\lambda)}(r)\mathcal{R}^{\lambda}(s). \tag{11$'$}$$

□ (a) Assume that $r > s$ and that $\mathcal{S}(s) = 1$. Then for all i, $1 \le i \le n$, we have $a_i^s < 1$. So if $s > 0$, $a_i^r > a_i^s$, showing that $1 < \mathcal{S}(r)$, and so $1 < \mathcal{R}(r)$; and if $s < 0$, $a_i^r < a_i^s$, showing that $1 > \mathcal{S}(r)$, and so $1 < \mathcal{R}(r)$.
This completes the proof in this special case so now assume that $\mathcal{S}(s) = \sigma$, and put, for all i, $b_i = a_i/\sigma^{1/s} = b_i/\mathcal{R}(s)$, when $\mathcal{S}(s; \underline{b}) = 1$.
From the special case we then have that $\mathcal{R}(r; \underline{b}) > 1$, which is just (10).
(b) First we consider $\mathcal{S}$

$$\sum_{i=1}^n a_i^{(1-\lambda)r+\lambda s} = \sum_{i=1}^n (a_i^r)^{(1-\lambda)}(a_i^s)^{\lambda} \le \Big(\sum_{i=1}^n a_i^r\Big)^{(1-\lambda)}\Big(\sum_{i=1}^n a_i^s\Big)^{\lambda}, \text{ by 2.1 (7).}$$

The case of equality follows from that for 2.1 (7).

Now we consider $\mathcal{R}$.
If $\mathcal{R}(r) > 1$ then $\mathcal{S}(r) > 1$ and so since $\log \mathcal{R}(r) = r^{-1} \log \mathcal{S}(r)$ the result follows from the convexity of $\log \mathcal{S}$ and $f(r) = r^{-1}$, using I 4.1 Theorem 4(e).
If $\mathcal{R}(r) = \rho < 1$ choose ρ', $0 < \rho' < \rho$ and let $a_i' = a_i/\rho'$, $1 \le i \le n$, when $\mathcal{R}(\underline{a}', r) = \rho/\rho' > 1$ and so $\mathcal{R}(\underline{a}', r)$ is log-convex.
However $\log \mathcal{R}(\underline{a}', r) = \log \mathcal{R}(r) - \log \rho'$, so $\log \mathcal{R}(r)$ is convex.

□

REMARK (i) The proof of (b) can be found in [*Beckenbach 1946*].

REMARK (ii) If $\underline{w}$ is an m-tuple with $W_m = 1$ a simple induction extends (11) and (11′) to

$$\mathcal{S}\big(\mathfrak{A}_m(\underline{r};\underline{w}),\underline{a}\big) \le \mathfrak{G}_m\big(\mathcal{S}(r_1,\underline{a}),\ldots,\mathcal{S}(r_m,\underline{a});\underline{w}\big), \tag{12}$$

$$\mathcal{R}\big(\mathfrak{A}_m(\underline{r};\underline{w}),\underline{a}\big) \le \mathfrak{G}_m\big(\mathcal{R}(r_1,\underline{a}),\ldots,\mathcal{R}(r_m,\underline{a});\underline{w}\big). \tag{12′}$$

Ursell has made some interesting observations about (11) which we will now discuss; see [*Reznick; Ursell*]. Consider (11) in the special case obtained by putting $r = 0$ and $\lambda s = p$,

$$\mathcal{S}(p) \le n^{1-p/s}\mathcal{S}^{p/s}(s). \tag{13}$$

or equivalently $\mathcal{R}(p) \le n^{(\frac{1}{p}-\frac{1}{s})}\mathcal{R}(s)$.
Inequalities (10)—(13) are best possible in the sense that given n, r, s, λ, p the constants cannot be improved. However (10) and (13) are best possible in a stronger sense. Given $n, r, s, p, \mathcal{S}(s)$ the values of $\mathcal{S}(r), \mathcal{S}(p)$ are given by (10) and (13). In the case of (13) equality implies that $(\mathcal{R}(r)/\mathcal{R}(s))^{rs/(1-r)}$ is equal to n, in particular it is an integer. In other words, given n, r, s, λ and appropriate $\mathcal{S}(r)$ and $\mathcal{S}(s)$, we can still have strict inequality in (13) unless $\mathcal{R}(r)/\mathcal{R}(s)$ has a special value. The determination of the exact range of this left-hand side given the sums on the right-hand side is completely determined by Ursell; see also [*Páles 1990b*] and IV 7.2.2.

REMARK (iii) The sums, $\mathcal{S}$ and $\mathcal{R}$, can be considered with weights; that is we define $\mathcal{S}(r, \underline{a}; \underline{w}) = \sum_{i=1}^n w_i a_i^r$ and $\mathcal{R}(r, \underline{a}; \underline{w}) = \left(\sum_{i=1}^n w_i a_i^r\right)^{1/r}$. It is immediate that Lemma 6(b) remains valid for these more general functions. Lemma 6(a) is considered in [*HLP p.29*] and [*Vasić & Pečarić 1980a*]; it is valid if $\underline{w} \ge \underline{e}$, while if $W_n \le 1$ the opposite inequality holds; see 3.1.1 Remark (ii). See also below IV 2 Theorem 13.

REMARK (iv) If $r \ge 1$ then $\left|\mathcal{R}(r,\underline{a}) - \mathcal{R}(r,\underline{b})\right| \le \min\left\{\sum_{k=1}^n |a_k - b_{i_k}|\right\}$, where the minimum is taken over all permutations $\{i_1, \ldots, i_n\}$ of $\{1, \ldots, n\}$. This answers

a question in [*Mitrinović & Adamović*]; some generalizations can be found in [*Milovanović & Milovanović 1978; Pečarić & Beesack 1986*].

Because of (HA) and Lemma 6(a) the following strengthens (12′).

LEMMA 7 *If* $W_n = 1$ *then*

$$\mathcal{R}(\mathfrak{H}_m(\underline{r};\underline{w}),\underline{a}) \le \mathfrak{G}_m(\mathcal{R}(r_1,\underline{a}),\ldots\mathcal{R}(r_m,\underline{a});\underline{w}), \tag{14}$$

with equality if and only if either $\underline{r}$ *or* $\underline{a}$ *is constant.*

□ Raise the left-hand side of (14) to the power $\mathfrak{H} = \mathfrak{H}_n(\underline{a};\underline{w})$ when we obtain,

$$\sum_{i=1}^n a_i^{\mathfrak{H}} = \sum_{i=1}^n \Big(\prod_{j=1}^n a_i^{w_j}\Big)^{\mathfrak{H}} \le \prod_{j=1}^n \Big(\sum_{i=1}^n (a_i^{w_j^{\mathfrak{H}}})^{r_j/w_j^{\mathfrak{H}}}\Big)^{w_j\mathfrak{H}/r_j},$$

by (2), since $\sum_{j=1}^n w_j\mathfrak{H}/r_j = 1$, and this implies (14). The cases of equality follow from those of (2). □

REMARK (v) By considering the case of constant $\underline{a}$ when (14) reduces to equality, it is easily seen from (10) that (14) is best possible in the sense that $\mathfrak{H}_m(\underline{a};\underline{w})$ cannot be replaced by a smaller number.

REMARK (vi) Inequality (14) is in [*McLaughlin & Metcalf 1968a*], where it is also pointed out that this inequality is best possible in another sense. The right-hand side of (12) cannot be replaced by any function of the $\mathcal{R}(r_i)$, $1 \le i \le n$, that is less than or equal to the geometric mean but which is not the geometric mean itself.

Lemma 6(a) can be used to extend (H) and 2.1 (5) as follows.

COROLLARY 8 *(a) If* $p, q > 0$ *and* $1/p + 1/q > 1$ *then*

$$\sum_{i=1}^n a_ib_i < \Big(\sum_{i=1}^n a_i^p\Big)^{1/p}\Big(\sum_{i=1}^n b_i^q\Big)^{1/q}.$$

(b) If $p, q, r > 0$ *and* $1/p + 1/q \ge 1/r$ *then*

$$\Big(\sum_{i=1}^n (a_ib_i)^r\Big)^{1/r} \le \Big(\sum_{i=1}^n a_i^p\Big)^{1/p}\Big(\sum_{i=1}^n b_i^q\Big)^{1/q},$$

with equality if and only if $1/p + 1/q = 1/r$ *and* $\underline{a}^p \sim \underline{b}^q$.

□ (a) Put $\lambda = 1/p + 1/q > 1$ and $r = \lambda p > p$; then $r' = \lambda q > q$ so

$$\begin{aligned}\sum_{i=1}^n a_ib_i &\le \Big(\sum_{i=1}^n a_i^r\Big)^{1/r}\Big(\sum_{i=1}^n b_i^{r'}\Big)^{1/r'}, \text{ by (H)},\\ &< \Big(\sum_{i=1}^n a_i^p\Big)^{1/p}\Big(\sum_{i=1}^n b_i^q\Big)^{1/q}, \text{ by (10).}\end{aligned}$$

(b) This part is either (5), or reduces to (a) by a simple change of variable. □

REMARK (vii) The properties of $\mathcal{R}(r)$, $r \le 0$, and their applications to the various results given above are easily obtained; see [*HLP p. 28*], [*Vasić & Pečarić 1979b*].

REMARK (viii) In his very interesting paper Reznick, [*Reznick*], has considered precise bounds for various products of power sums. Some of his simpler results are that for real n-tuples $\underline{a}$

$$-\frac{n}{8} \le \frac{\mathcal{S}(\underline{a};1)\mathcal{S}(\underline{a};3)}{\mathcal{S}(\underline{a};4)} \le n; \qquad \left|\frac{\mathcal{S}(\underline{a};1)\mathcal{S}(\underline{a};3)}{\mathcal{S}(\underline{a};2)}\right| \le \frac{3\sqrt{3}}{16}\sqrt{n} + \frac{5}{8} + O(n^{-1/2}).$$

REMARK (ix) Given an n-tuple $\underline{a}$ inequalities of the form

$$\sum_{i=1}^{n} a_i^{\alpha} A_i^{\beta} \le K(\alpha,\beta,n) A_n^{\alpha+\beta};$$

are considered in [*Beesack 1969*] ; in particular if $\alpha \ge 1, \alpha+\beta \ge 1$ and $\underline{a}$ is positive then $K = 1$; see [*AI p.282*].

2.4 MINKOWSKI'S INEQUALITY A very important consequence of Hölder's inequality, 2.1 Theorem 1, is *Minkowski's inequality.*

THEOREM 9 [MINKOWSKI'S INEQUALITY] *If $p > 1$ then*

$$\Big(\sum_{i=1}^{n}(a_i+b_i)^p\Big)^{1/p} \le \Big(\sum_{i=1}^{n} a_i^p\Big)^{1/p} + \Big(\sum_{i=1}^{n} b_i^p\Big)^{1/p}; \tag{M}$$

if $p < 1$, $p \ne 0$, then (~M) holds. In both cases there is equality if and only if $\underline{a} \sim \underline{b}$.

□ We give five proofs of this result.

(i) Consider the left-hand side of (M), and assume $p > 1$:

$$\begin{aligned}\sum_{i=1}^{n}(a_i+b_i)^p &= \sum_{i=1}^{n} a_i(a_i+b_i)^{p-1} + \sum_{i=1}^{n} b_i(a_i+b_i)^{p-1}\\ &\le \Big(\sum_{i=1}^{n} a_i^p\Big)^{1/p}\Big(\sum_{i=1}^{n}(a_i+b_i)^p\Big)^{(p-1)/p}\\ &\qquad + \Big(\sum_{i=1}^{n} b_i^p\Big)^{1/p}\Big(\sum_{i=1}^{n}(a_i+b_i)^p\Big)^{(p-1)/p}, \quad \text{by (H);}\end{aligned}$$

from which (M) follows.

If $p < 1$, $p \ne 0$ then (~M) follows by a similar argument from (~H).

The case of equality follows from that for 2.1 Theorem 1.

(*ii*) This proof uses quasi-linearization, see 2.1 Remark (vi) and [*BB p.26*]. Assume that $p > 1$; the other case can be dealt with in a similar manner.

By the quasi-linearization form of (H), 2.1 Theorem 4(a), if $L = \{\underline{c};\ \sum_{i=1}^n c_i^{p'} = 1\}$, then $\left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/p} = \sup_{\underline{c}\in L} \sum_{i=1}^n (a_i + b_i)c_i$. Hence

$$\begin{aligned}\left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/p} &\le \sup_{\underline{c}\in L} \sum_{i=1}^n a_i c_i + \sup_{\underline{c}\in L} \sum_{i=1}^n b_i c_i, \quad \text{by I 2.2 (15)}\\ &= \left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p}, \quad \text{by 2.1 Theorem 4(a).}\end{aligned}$$

(*iii*) [*König; Maligranda 1995*] With $p > 1$, $t > 0$ and $a, b > 0$ consider the function

$$f(t) = t^{1-p}a^p + (1-t)^{1-p}b^p.$$

Simple calculations show that $f'(t) = (1-p)\big(t^{-p}a^p - (1-t)^{-p}b^p\big)$, and that f has a minimum at $t = a/(a+b)$. In other words $f(t) \ge f\big(a/(a+b)\big)$, with equality if and only if $t = a/(a+b)$, that is

$$(a+b)^p \le t^{1-p}a^p + (1-t)^{1-p}b^p, \tag{15}$$

with equality if and only if $t = a/(a+b)$.

Now using (15)

$$\begin{aligned}\sum_{i=1}^n (a_i + b_i)^p &\le \sum_{i=1}^n \left(t^{1-p}a_i^p + (1-t)^{1-p}b_i^p\right)\\ &= t^{1-p}\left(\Big(\sum_{i=1}^n a_i^p\Big)^{1/p}\right)^p + (1-t)^{1-p}\left(\Big(\sum_{i=1}^n b_i^p\Big)^{1/p}\right)^p.\end{aligned}$$

Hence the left-hand side is not greater than the minimum of the right-hand side and the result follows by another application of (15). The case of equality is easily obtained from that in (15).

(*iv*) [*Matkowski & Rätz 1993*] Use the same method as in proof (*iii*) of (H), 2.1 Theorem 1, but now take $h(t) = (1 + t^{1/p})^p$, $p > 1$.

(*v*) [*Soloviov*] Let $f(\underline{a}) = \left(\sum_{i=1}^n a_i^p\right)^{1/p}$ then f is strictly convex and homogeneous, see I 4.6 Example (vii). So (M) is just I 4.6 (19). □

REMARK (i) Of course basic properties of addition extends (M) in a trivial way to the case $p = 1$.

An important use of (M) is the proof of the *triangle inequality, (T)*. If $p > 1$ then,

$$\begin{aligned}\Big(\sum_{i=1}^{n}|a_i-b_i|^p\Big)^{1/p} &= \Big(\sum_{i=1}^{n}\big|(a_i-c_i)+(c_i-b_i)\big|^p\Big)^{1/p}\\ &\le \Big(\sum_{i=1}^{n}(|a_i-c_i|+|c_i-b_i|)|)^p\Big)^{1/p},\\ &\le \Big(\sum_{i=1}^{n}(|a_i-c_i|^p\Big)^{1/p}+\Big(\sum_{i=1}^{n}(|c_i-b_i|^p\Big)^{1/p}, \quad \text{by (M).}\end{aligned}$$

This inequality holds of course when $p = 1$ being a property of the absolute value. So if $\rho_p(\underline{a}, \underline{b}) = \left(\sum_{i=1}^{n}|a_i - b_i|^p\right)^{1/p}$, $p \ge 1$, then we have the triangle inequality,

$$\rho_p(\underline{a}, \underline{b}) \le \rho_p(\underline{a}, \underline{c}) + \rho_p(\underline{c}, \underline{b}), \tag{T}$$

the essential property for showing that ρ_p is a metric on the space of n-tuples, or equivalently that $||\underline{a}||_p = \left(\sum_{i=1}^{n} a_i^p\right)^{1/p}$, $p \ge 1$, is a norm,

$$||\underline{a} + \underline{b}||_p \le ||\underline{a}||_p + ||\underline{b}||_p. \tag{T_N}$$

Remark (ii) (M) was first used in the form (T_N) by F. Riesz, and for this reason Minkowski's inequality is sometimes called the *Minkowski-Riesz inequality*. For a similar reason the Hölder inequality is sometimes called the *Hölder-Riesz inequality* when written in the form

$$||\underline{a}\,\underline{b}||_1 \le ||\underline{a}||_p ||\underline{b}||_{p'};$$

see [*MPF pp.473–513*], [*Maligranda 2001*].

The following extension of (M) is analogous to the extension of (H) given in 2.1 Corollary 2.

Corollary 10 *If $\underline{a}_i = (a_{i1}, \ldots, a_{in})$, $1 \le i \le m$, and $p > 1$ then*

$$\left(\sum_{j=1}^{n}\Big(\sum_{i=1}^{m} a_{ij}\Big)^p\right)^{1/p} \le \sum_{i=1}^{m}\Big(\sum_{j=1}^{n} a_{ij}^p\Big)^{1/p}, \tag{16}$$

with equality if and only if the n-tuples $\underline{a}_i$, $1 \le i \le m$, are pairwise dependent.

□ Inequality (16), and so (M), can be proved by induction, as was the case for (H), see 2.1 Corollary 2 proof (*ii*).

We start with the simplest case, the case $m = n = 2$ of (16):

$$\big((a_1 + b_1)^p + (a_2 + b_2)^p\big)^{1/p} \le (a_1^p + a_2^p)^{1/p} + (b_1^p + b_2^p)^{1/p}.$$

This can be obtained from 2.1(3), the case $n = 2$ of (H), in the same way that (M) was obtained from (H). Then, as with the inductive proof of 2.1 Corollary 2, we can either fix $m = 2$ and give an inductive proof of (16) for all n, and then for all m, or we can proceed the other way round; see [*HLP p.38*]. □

REMARK (iii) Much work has been done on this inequality; see for instance [*Szilárd; Zorio*].

2.5 REFINEMENTS OF THE HÖLDER, CAUCHY AND MINKOWSKI INEQUALITIES
The inequalities (H), (C) and (M) have been subjected to considerable investigation, resulting in many refinements; some of these are taken up in this section.

2.5.1 A RADO TYPE REFINEMENT The simplest proof of (H), or of 2.1 Corollary 2, depends on (GA), and it is natural to ask if it is possible to refine (H) by using (R); [*Bullen 1974*].

THEOREM 11 *With the hypotheses and notations of 2.1 Corollary 2 put*

$$\Xi_m(\underline{a}) = \frac{\sum_{j=1}^n \left(\prod_{i=1}^m a_{ij}\right)^{\rho_m}}{\prod_{i=1}^m \left(\sum_{j=1}^n a_{ij}^{r_i}\right)^{\rho_m/r_i}};$$

then if $m \ge 2$

$$\frac{1}{\rho_m}\Xi_m(\underline{a}) \le \frac{1}{\rho_{m-1}}\Xi_{m-1}(\underline{a}) + \frac{1}{r_m}, \tag{17}$$

with equality if and only if for $j = 1, \ldots, n$,

$$\frac{a_{mj}}{\sum_{j=1}^n a_{mj}} = \left(\prod_{i=1}^m \frac{a_{ij}}{\sum_{j=1}^n a_{ij^{1/r_i}}}\right)^{1/\rho_{n-1}}.$$

□ From (R)

$$\frac{1}{\rho_m}\left(\rho_m \sum_{i=1}^m \frac{b_i}{r_i} - \left(\prod_{i=1}^m b_i^{1/r_i}\right)^{\rho_m}\right) \ge \frac{1}{\rho_{m-1}}\left(\rho_{m-1} \sum_{i=1}^{m-1} \frac{b_i}{r_i} - \left(\prod_{i=1}^{m-1} b_i^{1/r_i}\right)^{\rho_{m-1}}\right).$$

Putting

$$b_i = \frac{a_{ij}^{r_i}}{\sum_{j=1}^n a_{ij}^{r_i}}, \; j = 1, \ldots, n \tag{18}$$

and summing the resulting inequalities over j leads to

$$\rho_m\big(1 - \Xi_m(\underline{a})\big) \ge \rho_{m-1}\big(1 - \Xi_{m-1}(\underline{a})\big),$$

which is equivalent to (17).

The case of equality follows from that for (R) given in II 3.1 Theorem 1. □

REMARK (i) If $m = 2$ inequality (17) reduces to $\Xi_2(\underline{a}) \le 1$, which is just (H).

REMARK (ii) Repeated application of (17) leads to $\Xi_m(\underline{a}) \le 1$, which is just 2.1(2).

REMARK (iii) By terminating the process in (ii) one step earlier and allowing for rearrangements of a_{ij}, together with similar rearrangements of r_i the result in (ii) can be improved as in II 3.1 Corollary 3.

REMARK (iv) For a Rado type extension of (M) see 3.2.6 Corollary 26.

2.5.2 INDEX SET EXTENSIONS Define the following functions on the index sets $\mathcal{I}$: if $I \in \mathcal{I}$,

$$\chi(I) = \Big(\sum_{i\in I} a_i^p\Big)^{1/p}\Big(\sum_{i\in I} b_i^{p'}\Big)^{1/p'} - \sum_{i\in I} a_i b_i;$$

$$\mu(I) = \bigg(\Big(\sum_{i\in I} a_i^p\Big)^{1/p} + \Big(\sum_{i\in I} b_i^p\Big)^{1/p}\bigg)^p - \sum_{i\in I}(a_i + b_i)^p.$$

The inequalities (H) and (M) show that if $p > 1$ then $\chi, \mu \ge 0$. In fact the functions χ and μ are also increasing and super-additive as the following more precise theorem shows; [*Everitt 1961; McLaughlin & Metcalf 1967a,1968b,1970*].

THEOREM 12 *If $p > 1$, $I, J \in \mathcal{I}$ then*

$$\chi(I) + \chi(J) \le \chi(I \cup J), \tag{19}$$

$$\mu(I) + \mu(J) \le \mu(I \cup J). \tag{20}$$

There is equality in (19) if and only if $\big(\sum_{i\in I} a_i^p, \sum_{i\in J} a_i^p\big) \sim \big(\sum_{i\in I} b_i^{p'}, \sum_{i\in J} b_i^{p'}\big)$, and in (20) if and only if $\big(\sum_{i\in I} a_i^p, \sum_{i\in I} b_i^p\big) \sim \big(\sum_{i\in J} a_i^p, \sum_{i\in J} b_i^p\big)$.

□ We first consider (19).
By 2.1 (3)

$$a_1^{1/p} b_1^{1/p'} + a_2^{1/p} b_2^{1/p'} \le (a_1 + a_2)^{1/p}(b_1 + b_2)^{1/p'},$$

with equality if and only if $(a_1, a_2) \sim (b_1, b_2)$.
Substituting $a_1 = \sum_{i\in I} a_i^p$, $a_2 = \sum_{i\in J} a_i^p$, $b_1 = \sum_{i\in I} b_i^{p'}$, $b_2 = \sum_{i\in J} b_i^{p'}$ leads immediately to (19), and the case of equality.

Now consider (20); by (~M),

$$\big((a_1 + b_1)^{1/p} + (a_2 + b_2)^{1/p}\big)^p \ge (a_1^{1/p} + a_2^{1/p})^p + (b_1^{1/p} + b_2^{1/p})^p,$$

with equality if and only if $(a_1, a_2) \sim (b_1, b_2)$. The result, and the case of equality, follow on putting $a_1 = \sum_{i\in I} a_i^p$, $a_2 = \sum_{i\in I} b_i^p$, $b_1 = \sum_{i\in J} a_i^p$, $b_2 = \sum_{i\in J} b_i^p$. □

REMARK (i) In the paper quoted above Everitt pointed out that the index set function

$$\Big(\sum_{i\in I} a_i^p\Big)^{1/p} + \Big(\sum_{i\in I} b_i^p\Big)^{1/p} - \Big(\sum_{i\in I}(a_i + b_i)^p\Big)^{1/p}$$

associated with (M) is not monotonic in either sense. In the second paper mentioned above McLaughlin & Metcalf studied the function μ, as well as a certain ratio associated with (M); see also [*Dragomir 1995a; Vasić & Pečarić 1983*].

2.5.3 AN EXTENSION OF KOBER-DIANANDA TYPE With the notations of 2.1 Corollary 2 assume $\rho_m = 1$ and define $R = \max_{1\le i\le m} r_i$, $r = \min_{1\le i\le m} r_i$ and

$$K_m = \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j,k=1}^{n}\sqrt{\frac{a_{ij}^{r_i}}{\sum_{i=1}^{m} a_{ij}^{r_i}}} - \sqrt{\frac{a_{ik}^{r_i}}{\sum_{i=1}^{m} a_{ik}^{r_i}}}\right)^2;$$

$$D_m = \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j,k=1}^{n}\frac{1}{r_i r_k}\sqrt{\frac{a_{ij}^{r_i}}{\sum_{i=1}^{m} a_{ij}^{r_i}}} - \sqrt{\frac{a_{ik}^{r_i}}{\sum_{i=1}^{m} a_{ik}^{r_i}}}\right)^2.$$

THEOREM 13 *With the above notations*

$$\Big(1 - \frac{RD_m}{1-R}\Big)\prod_{i=1}^{m}\Big(\sum_{j=1}^{n} a_{ij}^{r_i}\Big)^{1/r_i} \ge \sum_{j=1}^{n}\prod_{i=1}^{m} a_{ij} \ge \max\Big\{0,\ (1 - RD_m)\prod_{i=1}^{m}\Big(\sum_{j=1}^{n} a_{ij}^{r_i}\Big)^{1/r_i}\Big\};$$

$$\Big(1 - \frac{K_m}{R(m-1)}\Big)\prod_{i=1}^{m}\Big(\sum_{j=1}^{n} a_{ij}^{r_i}\Big)^{1/r_i} \ge \sum_{j=1}^{n}\prod_{i=1}^{m} a_{ij} \ge \max\Big\{0,\ \Big(1 - \frac{K_m}{r}\Big)\prod_{i=1}^{m}\Big(\sum_{j=1}^{n} a_{ij}^{r_i}\Big)^{1/r_i}\Big\}.$$

□ The proof is based on the method used to obtain 2.1 Corollary 2 and Theorem 1. Use inequalities II 3.5(34), (35) applied to a sequence $\underline{b}$, make the identification 2.5.1(18), and proceed as in the proof of 2.1 Corollary 2. □

REMARK (i) The cases of equality are discussed by both Kober and Diananda; Diananda gives a similar refinement of (M); [*Kober; Diananda 1963a,b*].

2.5.4 A CONTINUUM OF EXTENSIONS As has been remarked (C) is a particular case of (H). In fact (H) can be used to prove much more. Let a_{ij}, r_i, ρ_m be as in 2.1 Corollary 2; then we have the following result.

THEOREM 14 *If $r \in \mathbb{R}^*$ and $\underline{w}$ is an n-tuple define the function f as follows*

$$f(x) = \prod_{i=1}^{m}\Big(\sum_{j=1}^{n} w_j a_{ij}^{r_i x/r}\prod_{k=1}^{m} a_{kj}^{r-x}\Big)^{1/r_i}.$$

Then f is log-convex, increasing on $[0,\infty[$, and decreasing on $]-\infty,0]$, strictly unless f is constant.

□ If the m-tuples are not proportional then by 2.3 Lemma 6(b) and 2.3 Remark(iii)

$$\begin{aligned} g_i(x) =& \sum_{j=1}^{n} w_j a_{ij}^{r_i x/r} \prod_{k=1}^{m} a_{kj}^{r-x} \\ =& \sum_{j=1}^{n} \big(w_j \prod_{k=1}^{m} a_{kj}^{r}\big)\big(a_{ij}^{r_i/r} \prod_{k=1}^{m} a_{kj}^{-1}\big)^{x}, \end{aligned}$$

is log-convex, $1 \le i \le m$. Hence, I 4.1 Theorem 4(d),

$$\log f(x) = \sum_{i=1}^{n} \frac{1}{r_i} \log g_i(x),$$

is convex.

The rest of the proof is similar to that of I 4.2 Theorem 18; see [*PPT pp.90,118*], [*Mitrinović & Pečarić 1987*]. □

Remark (i) Beside the paper just mentioned this result is discussed in [*Daykin & Eliezer 1968; Eliezer & Daykin; Eliezer & Mond; Flor; Godunova & Čebaevskaya; Mon, Sheu & Wang 1992b*]; see also IV 5 Example (xiv).

Remark (ii) The most interesting case of Theorem 13 occurs on putting $r = \rho_m$ when $f(0)$ is the left-hand side of a weighted form of 2.1(2), while $f(\rho_m)$ is the right-hand side. Since $f(0) \le f(x) \le f(\rho_m)$, $0 \le x \le \rho_m$, we get a continuum of generalizations of 2.1(2), and so of (H): if $0 \le x \le \rho_m$ then

$$\Big(\sum_{j=1}^{n} w_j \Big(\prod_{i=1}^{m} a_{ij}\Big)^{\rho_m}\Big)^{1/\rho_m} \le \prod_{i=1}^{m} \Big(\sum_{j=1}^{n} w_j a_{ij}^{r_i x/\rho_m} \prod_{k=1}^{m} a_{kj}^{\rho_m - x}\Big)^{1/r_i} \le \prod_{i=1}^{m} \Big(\sum_{j=1}^{n} w_j a_{ij}^{r_i}\Big)^{1/r_i}.$$

In this case the function f is a constant if the n-tuples $\underline{a}_i^{r_i}$, $1 \le i \le m$ are proportional.

Remark (iii) The particular case, $m = 2, r_1 = r_2 = 2$, of the result in the previous remark is in [*Callebaut*] and is sometimes called *Callebaut's inequality*, [*PPT p.118*]: if $0 \le x \le y \le 1$ then

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \le \sum_{i=1}^{n} a_i^{1+x} b_i^{1-x} \sum_{i=1}^{n} a_i^{1-x} b_i^{1+x} \le \sum_{i=1}^{n} a_i^{1+y} b_i^{1-y} \sum_{i=1}^{n} a_i^{1-y} b_i^{1+y} \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

This result is an illustration of the comment that precedes 2.1 Theorem 3. Callebaut proved one half of his inequality using (H) and remarked that the other half was not apparently deducible from (H), and so gave another proof. However a simple proof using (H) was given almost immediately, [*McLaughlin & Metcalf 1967b*]; see also [*Steiger*].

THEOREM 15 *If $0 \le x \le 1$ then*

$$\Big(\sum_{i=1}^{n} a_ib_i + x\sum_{\substack{i,j=1\\ i\ne j}}^{n} a_ib_j\Big)^2 \le \Big(\sum_{i=1}^{n} a_i^2 + 2x\sum_{\substack{i,j=1\\ i<j}}^{n} a_ia_j\Big)\Big(\sum_{i=1}^{n} b_i^2 + 2x\sum_{\substack{i,j=1\\ i<j}}^{n} b_ib_j\Big). \tag{21}$$

□ Using the identities $\sum_{i,j=1}^{n} a_ib_j = \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_ib_i$, $2\sum_{\substack{i=1\\ i<j}}^{n} a_ia_j = \big(\sum_{i=1}^{n} a_i\big)^2 - \sum_{i=1}^{n} a_i^2$, and $2\sum_{\substack{i=1\\ i<j}}^{n} b_ib_j = \big(\sum_{i=1}^{n} b_i\big)^2 - \sum_{i=1}^{n} b_i^2$, and putting $y = 1 - x$, (21) can be written as

$$\Big(x\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i + y\sum_{i=1}^{n} a_ib_i\Big)^2 \le \Big(x\big(\sum_{i=1}^{n} a_i\big)^2 + y\sum_{i=1}^{n} a_i^2\Big)\Big(x\big(\sum_{i=1}^{n} b_i\big)^2 + y\sum_{i=1}^{n} b_i^2\Big).$$

On multiplying this out we obtain the equivalent inequality

$$xy\sum_{j=1}^{n}\Big(b_j\sum_{i=1}^{n} a_i - a_j\sum_{i=1}^{n} b_i\Big)^2 + y^2\Big(\big(\sum_{i=1}^{n} a_i^2\big)\big(\sum_{i=1}^{n} b_i^2\big) - \big(\sum_{i=1}^{n} a_ib_i\big)^2\Big) \ge 0,$$

which is an immediate consequence of (C). □

REMARK (iv) If $x = 0$ then (21) reduces to (C).

REMARK (v) This result is due to S.S. Wagner, but the above simple proof is by Flor; [*MPF p.85*], [*Andreescu, Andrica & Drimbe; Flor; Wagner S*].

2.5.5 BECKENBACH'S INEQUALITIES The first part of 2.1 Theorem 1 can be interpreted as follows.

Given $\underset{\sim}{a}_1$ and $\underset{\sim}{b}$ then (H) holds for all $a_2, \ldots, a_n$, with equality if and only if $a_i^p b_i^{p'} = a_i^p b_1^{p'} = a_1^p b_i^{p'}$, $2 \le i \le n$.

This has been generalized as follows in [*Beckenbach 1966*].

THEOREM 16 *Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be n-tuples, $p > 1$, and $1 \le m < n$; define the n-tuple $\underset{\sim}{\tilde{a}}$ as follows:*

$$\tilde{a}_i = \begin{cases} a_i, & \text{if } 1 \le i \le m,\\[2ex] \left(\dfrac{b_i \sum_{j=1}^{m} a_j^p}{\sum_{j=1}^{m} a_j b_j}\right)^{p'/p}, & \text{if } m < i \le n; \end{cases}$$

then given $a_1, \ldots, a_m$ *and* $\underline{b}$*,*

$$\frac{\left(\sum_{i=1}^n a_i^p\right)^{1/p}}{\sum_{i=1}^n a_i b_i} \geq \frac{\left(\sum_{i=1}^n \tilde{a}_i^p\right)^{1/p}}{\sum_{i=1}^n \tilde{a}_i b_i}, \tag{22}$$

for all $a_{m+1}, \ldots, a_n$*.*
If $p < 1, p \neq 0$*, (∼22) holds. There is equality in both cases if and only if* $a_i = \tilde{a}_i$*,* $m+1 \leq i \leq n$*.*

□ Assume that $p > 1$ and consider the left-hand side of (22) as function f of the variables $a_{m+1}, \ldots, a_n$. Then $f_j' = PQ_j$, $m+1 \leq j \leq n$, where

$$P = \frac{\left(\sum_{i=1}^n a_i^p\right)^{-1/p'}}{\left(\sum_{i=1}^m a_i b_i\right)^2}; \qquad Q_j = \Big(\sum_{i=1}^n a_i b_i\Big) a_j^{p-1} - \Big(\sum_{i=1}^n a_i^p\Big) b_j.$$

Since $P > 0$ we have that the partial derivatives f_j' are zero if and only if the Q_j are zero; that is if and only if $\dfrac{a_j^p}{a_j b_j} = \dfrac{\sum_{i=1}^n a_i^p}{\sum_{i=1}^n a_i b_i}$, $m+1 \leq j \leq n$. Equivalently, if and only if $\dfrac{a_j^p}{a_j b_j} = \dfrac{\sum_{i=1}^m a_i^p}{\sum_{i=1}^m a_i b_i}$, $m+1 \leq j \leq n$; that is $a_j = \tilde{a}_j$, $m+1 \leq j \leq n$. It is easily seen that this is a unique minimum of f. □

Remark (i) As we have seen above, the case $m = 1$ is just (H).

Remark (ii) A simple proof has been given in [*Pečarić & Beesack 1987a*].
Another result of Beckenbach is the following.

Theorem 17 *Suppose* $p > 1$*,* $\alpha > 0$*,* $\beta > 0$*,* $\gamma > 0$*, and* $\underline{a}$*,* $\underline{b}$ *are non-negative* n*-tuples; for* m*,* $0 \leq m < n$*, define the* $(n-m)$*-tuple* $\underline{c}$ *by* $c_i = (\alpha b_i/\beta)^{p'/p}$*,* $i = m+1, \ldots, n$*. Then*

$$\frac{\left(\alpha + \gamma \sum_{i=m+1}^n a_i^p\right)^{1/p}}{\beta + \gamma \sum_{i=m+1}^n a_i b_i} \geq \frac{\left(\alpha + \gamma \sum_{i=m+1}^n c_i^p\right)^{1/p}}{\beta + \gamma \sum_{i=m+1}^n b_i c_i},$$

with equality if and only if $a_i = c_i$*,* $m+1 \leq i \leq n$*.*

□

$$\begin{aligned}
\beta + \gamma \sum_{i=m+1}^n a_i b_i &= \alpha^{1/p}(\beta\alpha^{-1/p}) + \sum_{i=m+1}^n (\gamma^{1/p} a_i)(\gamma^{1/p'} b_i) \\
&\leq \Big(\alpha + \gamma \sum_{i=m+1}^n a_i^p\Big)^{1/p} \Big(\alpha^{-p'/p}\beta^{p'} + \gamma \sum_{i=m+1}^n b_i^{p'}\Big)^{1/p'}, \quad \text{by (H)}, \\
&= \Big(\alpha + \gamma \sum_{i=m+1}^n a_i^p\Big)^{1/p} \frac{\beta + \gamma \sum_{i=m+1}^n b_i c_i}{(\alpha + \gamma \sum_{i=m+1}^n c_i^p)^{1/p}}.
\end{aligned}$$

The case of equality follows from that of (H). □

REMARK (iii) A converse for this inequality has been given by Zhuang; [*Zhuang 1993*].

REMARK (iv) All of these inequalities have been given considerable attention; [*Iwamoto & Wang; Wang C L 1977,1979b,d,1981a,1982,1983,1988b*].

2.5.6 OSTROWSKI'S INEQUALITY This inequality is an extension of (C); [*Ostrowski p.289*].

THEOREM 18 *Let $\underline{a}$ and $\underline{b}$ be n tuples such that $\underline{a} \not\sim \underline{b}$ and define the n-tuple $\underline{c}$ by $\underline{a}.\,\underline{c} = 0$ and $\underline{b}.\,\underline{c} = 1$, then*

$$\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n c_i^2} \le \Big(\sum_{i=1}^n a_i^2\Big)\Big(\sum_{i=1}^n b_i^2\Big) - \Big(\sum_{i=1}^n a_i b_i\Big)^2,$$

with equality if and only if

$$c_k = \frac{b + k\sum_{i=1}^n a_i^2 - a_k \sum_{i=1}^n a_i b_i}{\big(\sum_{i=1}^n a_i^2\big)\big(\sum_{i=1}^n b_i^2\big) - \big(\sum_{i=1}^n a_i b_i\big)^2},\ 1 \le k \le n.$$

REMARK (i) A proof can be found in [*AI pp. 66–70*], where several extensions due to Ky Fan & Todd are also proved; [*Fan & Todd*]; see also [*Madevski; Mitrović 1973*]

REMARK (ii) Beesack has pointed out that this result can be regarded as a special case of a Bessel inequality for non-orthonormal vectors; [*Beesack 1975*]; see also [*Šikić & Šikić*]. It can also be regarded as an extension of 3.1.2 (7) in the case $r = q = 2, s = 1$.

2.5.7 ACZÉL-LORENTZ INEQUALITIES As was pointed out in 2.4 (M) is essential for certain properties of norms on $\mathbb{R}^n$. If a different norm is defined then we can ask if similar properties hold. A so-called *Lorentz norm*[4] is defined as

$$||\underline{a}||_p^L = (a_1^p - \sum_{i=2}^n a_i^p)^{1/p},\ p > 1;$$

this is defined on the set L_p of positive n-tuples for which $a_1 > (\sum_{i=2}^n a_i^p)^{1/p}$; [*BB pp.38–39*]. Various inequalities using these expressions are then called *Lorentz inequalities*, [*Wang C L 1984a*]. However the first such inequality was proved by

[4] This is H.A.Lorentz; see [*EM6 p.46–47*].

Aczél, [*Aczél 1956b*], so such inequalities have also been called *Aczél inequalities*; [*DI pp.16–17, PPT pp.124–126*].
It turns out that the natural analogue in this situation is not (M) but (~M). As a result of this the Lorentz triangle inequalities are analogous to (~T), (~T_N), and so the Lorentz norm is not a norm in the usual sense. The various Aczél-Lorentz inequalities are examples of reverse inequalities similar to those in I 4.4.

THEOREM 19 *(a)* [HÖLDER-LORENTZ] *If $\underline{a}, \underline{b}$ are n-tuples in L_p, $L_{p'}$ respectively, where $p > 1$ then*

$$a_1b_1 - \sum_{i=1}^n a_ib_i \geq (a_1^p - \sum_{i=1}^n a_i^p)^{1/p}(b_1^{p'} - \sum_{i=1}^n b_i^{p'})^{1/p'}. \tag{23}$$

If $0 < p < 1$ then (~23) holds; in either case there is equality if and only if $\underline{a}^p \sim \underline{b}^{p'}$.
(b) [MINKOWSKI-LORENTZ] *If $\underline{a}, \underline{b}$ are n-tuples in L_p, $p > 1$, then*

$$\Big((a_1 + b_1)^p - \sum_{i=1}^n (a_i + b_i)^p\Big)^{1/p} \geq (a_1^p - \sum_{i=1}^n a_i^p)^{1/p} + (b_1^p - \sum_{i=1}^n b_i^p)^{1/p}. \tag{24}$$

If $0 < p < 1$ then (~24) holds, and in either case there is equality if and only if $\underline{a}$ and $\underline{b}$ are dependent.
(c) [BECKENBACH-LORENTZ] *Suppose $p > 1$, $\alpha > 0, \beta > 0, \gamma > 0$, and $\underline{a}$, $\underline{b}$ are non-negative n-tuples; for m, $0 \leq m < n$, define the $(n-m)$-tuple $\underline{c}$ by $c_i = (\alpha b_i/\beta)^{p'/p}$, $i = m+1, \dots, n$. Further assume that $\alpha - \gamma\sum_{i=m+1}^n a_i^p > 0$, and that $\alpha - \gamma\sum_{i=m+1}^n c_i^p > 0$. Then*

$$\frac{\big(\alpha - \gamma\sum_{i=m+1}^n a_i^p\big)^{1/p}}{\beta - \gamma\sum_{i=m+1}^n a_ib_i} \geq \frac{\big(\alpha - \gamma\sum_{i=m+1}^n c_i^p\big)^{1/p}}{\beta - \gamma\sum_{i=m+1}^n b_ic_i},$$

with equality if and only if $a_i = c_i$, $m+1 \leq i \leq n$.

REMARK (i) The case $p = 2$ of (23), the "Cauchy" case, is the original result of Aczél.

REMARK (ii) The above results are given by Wang C L but other authors have given different proof of certain results; [*AI pp.57–59*], [*Mond & Pečarić 1995b; Popoviciu 1959a; Wang C L 1984a*].

2.5.8 VARIOUS RESULTS We first state an extension of (H) that is in [*Mudholkar, Freimer & Subbaiah*]. First note, as is shown in Lemma 1 of the quoted paper, that if $\underline{b}$ is a decreasing n-tuple, and $m \leq n$, then for some k, $0 \leq k \leq m-1$,

$$\frac{B_n - B_{m-k-1}}{k+1} \leq b_{m-k-1}, \quad b_{m-k} \leq \frac{B_n - B_{m-k}}{k}. \tag{25}$$

THEOREM 20 *If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are decreasing n-tuples, $p > 1$, $m \le n$, then*

$$\sum_{i=1}^{n} a_i b_i \le \Big(\sum_{i=1}^{m} a_i^p\Big)^{1/p} \Big(\sum_{i=1}^{m-k-1} b_i^{p'} + \big(\frac{B_n - B_{m-k-1}}{(k+1)^{1/p}}\big)^{p'}\Big)^{1/p'},$$

where k is given by (25).

REMARK (i) An extension of this result has been given in [*Iwamoto, Tomkins & Wang 1986b*], and a simpler proof of this extension, using Steffensen's inequality, VI 1.3.6 Theorem 18, can be found in [*Pearce & Pečarić 1995a*].

The following result is in [*Mikolás*]; it reduces to (M) if $pr = 1$; see [*AI p.369*], [*Alzer 1991i*].

THEOREM 21 *If $\underset{\sim}{a}_j$, $1 \le j \le n$, are m-tuples and $0 < r \le 1$, $0 < pr \le 1$, then*

$$\sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij}^r\Big)^p \le n^{1-pr} \sum_{j=1}^{n} \Big(\sum_{i=1}^{m} a_{ij}\Big)^r. \tag{26}$$

If $r > 1$ and $pr \ge 1$ then ($\sim$26) holds.

The next result is a problem set by Klamkin, [*Klamkin & Hashway*]. It is a particular case of Callebaut's inequality, 2.5.4 Remark(iii).

THEOREM 22 *If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are n-tuples and if*

$$f(k) = \Big(\sum_{i=1}^{n} a_i^{2-\frac{k}{n}} b_i^{\frac{k}{n}}\Big)\Big(\sum_{i=1}^{n} a_i^{\frac{k}{n}} b_i^{2-\frac{k}{n}}\Big), \ 0 \le k \le n;$$

then f is decreasing.

REMARK (ii) The extreme inequality given by Theorem 22, $f(0) \ge f(n)$, is just (C)

The following is due to Abramovich; [*Abramovich*].

THEOREM 23 *If $p > 1$ and $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ are n-tuples with $A_n = C_n = 1$ and for some m, $1 \le m < n$, $a_i \ge c_i$, $\quad a_j \le c_j$, $\dfrac{c_i}{b_i} \ge \dfrac{c_j}{b_j}$, $1 \le i \le m$, $m+1 \le j \le n$, then*

$$\sum_{i=1}^{n} a_i^{1/p} b_i^{1/p'} \le \sum_{i=1}^{n} b_{[i]}^{1/p'} c_{[i]}^{1/p},$$

where $(d_{[1]}, \dots d_{[n]})$ denotes a decreasing rearrangement of $(d_1, \dots d_n)$.

REMARK (iii) On putting $\underset{\sim}{b} = \underset{\sim}{c}$ this theorem reduces to 2.1 Theorem 4(b).

THEOREM 24 *If $\underline{a}, \underline{b}, \underline{u}, \underline{v}$ are n-tuples with $\underline{u} \le \underline{v}$ then*

$$\Big(\sum_{i=1}^n u_i a_i^2\Big)^{\frac12}\Big(\sum_{i=1}^n u_i b_i^2\Big)^{\frac12} - \sum_{i=1}^n u_i a_i b_i \le \Big(\sum_{i=1}^n v_i a_i^2\Big)^{\frac12}\Big(\sum_{i=1}^n v_i b_i^2\Big)^{\frac12} - \sum_{i=1}^n v_i a_i b_i.$$

REMARK (iv) This extension of (C), due to Dragomir & Arslanagić has been extended to (H) by Pečarić; [*Dragomir & Arslanagić 1991; Pečarić 1993*].

The next result is in [*Atanassov*].

THEOREM 25 *Let $\underline{a}$ be an increasing n-tuple and $\underline{k}$ an increasing m-tuple of positive integers then*

$$\prod_{i=1}^m \Big(\sum_{j=1}^n a_j^{k_i}\Big) \le n^{m-1} \sum_{j=1}^n a_j^{K_m}.$$

□ The proof is by induction, using the inequality

$$A_n B_n \le n \sum_{i=1}^n a_i b_i,$$

where $\underline{a}, \underline{b}$ are increasing real n-tuples. □

THEOREM 26 *If $\underline{a}$ is an n-tuple such that $a_1 \le a_2/2 \le \ldots \le a_n/n$ and $\underline{b}$ is a decreasing n-tuple then*

$$\Big(\sum_{i=1}^n a_i b_i\Big)^2 \le \sum_{i=1}^n b_i \sum_{i=1}^n \Big(a_i^2 - \frac{a_i a_{i-1}}{4}\Big) b_i,$$

where $a_0 = 0$. There is equality if and only if $a_i = i a_1$, $1 \le i \le n$, and $\underline{b}$ is constant.

REMARK (v) This result of Alzer has a proof that is quite long and complicated; [*Alzer 1992d, 1999b*]. It generalizes the following variant of (C):

$$\Big(\sum_{i=1}^n a_i b_i\Big)^2 \le \sum_{i=1}^n b_i \sum_{i=1}^n a_i^2 b_i$$

The following is a very simple extension of (C) is due to Hovenier, [*Hovenier 1994, 1995*].

THEOREM 27 *If* $0 \leq m \leq n-1$, *and* $\underline{a}, \underline{b}$ *are n-tuples define*

$$\alpha_m = \begin{cases} \sum_{i=1}^m a_i b_i + \left(\sum_{i=m+1}^n a_i^2\right)^{1/2}\left(\sum_{i=m+1}^n b_i^2\right)^{1/2} & \text{if } m \neq 0, \\ \left(\sum_{i=1}^n a_i^2\right)^{1/2}\left(\sum_{i=1}^n b_i^2\right)^{1/2} & \text{if } m = 0. \end{cases}$$

Then $\alpha_m \leq \alpha_{m-1}$, $0 \leq m \leq n-1$.

□ The proof uses the remark that if $1 \leq m \leq n-1$ then $\alpha_{n-1} \leq \alpha_m$, because

$$\sum_{i=1}^n a_i b_i = \sum_{i=1}^m a_i b_i + \sum_{i=m+1}^n a_i b_i \leq \sum_{i=1}^m a_i b_i + \sqrt{\sum_{i=m+1}^n a_i^2 \sum_{i=m+1}^n b_i^2}, \text{ by (C).}$$

□

REMARK (vi) The inequality $\alpha_{n-1} \leq \alpha_0$ is just (C). A similar extension of (H) has been made; [*Abramovich, Mond & Pečarić 1995*].

The next result is related to inequalities involving arithmetic means of powers rather than power means, quantities that have played an important role in statistics.

THEOREM 28 *If* $\underline{a}$ *is a real n-tuple and* $\alpha_m = \frac{1}{n}\sum_{i=1}^n a_i^m$, $m \in \mathbb{N}^*$, *and if* $\alpha_1 = 0$, $\alpha_2 = 1$ *then*

$$\alpha_4 \geq 1 + \alpha_3^2; \quad \alpha_{2m} \geq \alpha_{m-1}^2 + \alpha_m^2; \quad \alpha_3 \leq \frac{n-2}{\sqrt{n-1}}, \quad \alpha_4 \leq n-2+\frac{1}{n-1}\alpha.$$

with equality in the last two inequalities if and only if $a_1 = \sqrt{n-1}$ *and* $a_k = -1/\sqrt{n-1}$, $2 \leq k \leq n$.

The first result is in [*Pearson*], and given a different proof in [*Chakrabarti; Wilkins*]; these authors gave the bounds for the last two inequalities. The second inequality is in [*Lakshmanamurti*]; see also [*Madevski*].

REMARK (vii) There are of course many other references of interest of which we mention the following: [*MPF pp.83–189*], [*Abramovich & Pečarić; Carroll, Cordner & Evelyn; Dragomir 1987,1988,1994b; Dragomir & Arslanagić 1992; Dragomir, Milošević & Arslanagić; Dragomir & Mond; Eliezer, Mond & Pečarić; Liu; Mitrinović & Pečarić 1990b; Páles 1990c; Persson; Yang X*].

3 Inequalities Between the Power Means

3.1 THE POWER MEAN INEQUALITY

3.1.1 THE BASIC RESULT The results of the previous section imply some relations between the power means introduced in the first section. Here we give the basic result, a fundamental generalization of (GA), called the *power mean inequality*. We will refer to this inequality as (r;s), where $-\infty \le r < s \le \infty$. Some authors refer to (r;s) as *Jensen's inequality*, see [*HLP*], [*Beckenbach 1946; Cooper 1927a*], although this name is usually given to (J).

THEOREM 1 [POWER MEAN INEQUALITY] *Given two n-tuples $\underline{a}, \underline{w}$ and $r, s \in \overline{\mathbb{R}}$ with $r < s$ then*

$$\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) \le \mathfrak{M}_n^{[s]}(\underline{a};\underline{w}), \tag{r; s}$$

with equality if and only if $\underline{a}$ is constant.

The first proof for arbitrary weights was given in 1879 by Besso, although the result had been formulated in 1840 by Bienaymé; [*Besso; Bienaymé*]. In the case of equal weights a proof was given in 1858 by Schlömilch for r and s positive integers or reciprocals of positive integers, [*Schlömilch 1858b*]. Later, 1888, a proof of the equal weight case for arbitrary r and s was given by Simon; [*Simon*]. In 1902 Pringsheim gave a proof for the case $s = 1$ and $0 < r < 1$ that he attributed to Lüroth and Jensen; [*Jensen 1906; Pringsheim*].

□ If $r = -\infty$, or $s = \infty$, (r;s) is just the strict internality of the power means, inequality 1(2).

Using 1(4) we readily see that (GA) implies (0;s), $0 < s < \infty$, and (r;0), $-\infty < r < 0$, and hence (r;s) when $-\infty < r \le 0 \le s < \infty$.

Further a simple use of the second identity in 1(3) shows that (r;s), $-\infty < r < s < 0$, follows from the case (r;s), $0 < r < s < \infty$.

Another use of the identity 1(4) shows that we need only consider the two cases (1;s), $1 < s < \infty$, and (r;1), $0 < r < 1$.

Finally suppose that we have proved (1;s), $1 < s < \infty$ and that $0 < r < 1$; then by (1;1/r) and 1(4) $\mathfrak{M}_n^{[1/r]}(\underline{a}^r;\underline{w}) \ge \mathfrak{A}_n(\underline{a}^r\underline{w}) = \left(\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\right)^r$, and by 1(4) $\mathfrak{M}_n^{[1/r]}(\underline{a}^r;\underline{w}) = \left(\mathfrak{A}_n(\underline{a};\underline{w})\right)^r$. These two equations give (1;r), showing that we need only consider the case (1;s), $1 < s < \infty$.

Following these remarks we give several proofs of (r;s).

(*i*) Assume $s > 1$ and without loss in generality that $W_n = 1$.

$$\mathfrak{A}_n(\underline{a};\underline{w}) = \sum_{i=1}^{n} (a_i^s w_i)^{1/s} w_i^{1-(1/s)} \le \mathfrak{M}_n^{[s]}(\underline{a};\underline{w}), \text{ by (H)},$$

proving (1;s).

(ii) Assume without loss in generality that $1 \le a_i$, $1 \le i \le n$, $W_n = 1$, and that $\underline{a}$ is not constant and define

$$f(r) = \mathfrak{M}_n^{[r]}(\underline{a};\underline{w}), \quad \phi(r) = r^2 \frac{f'(r)}{f(r)}, \ r \in \mathbb{R}^*. \tag{1}$$

Then

$$\phi(r) = \sum_{i=1}^{n} \Big(\frac{a_i}{f(r)}\Big)^r w_i \log\Big(\frac{a_i}{f(r)}\Big)^r; \tag{2}$$

$$\phi'(r) = \frac{r}{\big(\sum_{i=1}^{n} w_i a_i^r\big)^2}\Bigg(\Big(\sum_{i=1}^{n} w_i a_i^r\Big)\Big(\sum_{i=1}^{n} w_i a_i^r (\log a_i)^2\Big) - \Big(\sum_{i=1}^{n} w_i a_i^r \log a_i\Big)^2\Bigg) \tag{3}$$

Hence, by (C), $\operatorname{sign}\phi' = \operatorname{sign} r$, which implies that $\phi(r) > 0, r \neq 0$. This in turn implies that $f'(r) > 0$, $r \neq 0$, from which (r;s) is immediate.
This proof can be given without appealing to (C), [*Burrows & Talbot; Wahlund*].
(iii) Make the same assumptions as in proof (i).
Define the n-tuple $\underline{b}$ by $\underline{b} = \dfrac{\underline{a}}{\mathfrak{A}_n(\underline{a};\underline{w})}$. Then it is sufficient to prove that

$$\mathfrak{M}_n^{[s]}(\underline{b};\underline{w}) \ge 1. \tag{4}$$

From the definition of $\underline{b}$, $\sum_{i=1}^{n} w_i b_i = 1$, so if we put $b_i = 1 + \beta_i$, $1 \le i \le n$, then $\beta_i > -1$ and $\sum_{i=1}^{n} w_i \beta_i = 0$.
Hence, using (B),

$$\sum{}_{i=1}^{n} w_i b_i^s = \sum_{i=1}^{n} w_i (1+\beta_i)^s \ge \sum_{i=1}^{n} w_i(1 + s\beta_i) = 1.$$

This immediately gives (4). See [*Herman, Kučera & Šimša pp.167–168*].
(iv)[*Orts*] Assume that $\underline{a}$ is not constant.
(i) First assume that $r, s \in \mathbb{N}^*$.
$\mathfrak{M}_n^{[1]}(\underline{a};\underline{w}) < \mathfrak{M}_n^{[2]}(\underline{a};\underline{w})$ follows by (C).
Now assume that $\mathfrak{M}_n^{[k]}(\underline{a};\underline{w}) \le \mathfrak{M}_n^{[k+1]}(\underline{a};\underline{w})$, $1 \le k \le m-1$. Then

$$\begin{aligned}\Big(\mathfrak{M}_n^{[m]}(\underline{a};\underline{w})\Big)^{2m} &= \Big(\sum_{i=1}^{n} w_i a_i^m\Big)^2 < \Big(\sum_{i=1}^{n} w_i a_i^{m+1}\Big)\Big(\sum_{i=1}^{n} w_i a_i^{m-1}\Big), \text{ by (C)},\\ &= \Big(\mathfrak{M}_n^{[m+1]}(\underline{a};\underline{w})\Big)^{m+1}\Big(\mathfrak{M}_n^{[m-1]}(\underline{a};\underline{w})\Big)^{m-1}\\ &< \Big(\mathfrak{M}_n^{[m+1]}(\underline{a};\underline{w})\Big)^{m+1}\Big(\mathfrak{M}_n^{[m]}(\underline{a};\underline{w})\Big)^{m-1}, \text{by the induction hypothesis.}\end{aligned}$$

This gives $\mathfrak{M}_n^{[m]}(\underline{a};\underline{w}) \le \mathfrak{M}_n^{[m+1]}(\underline{a};\underline{w})$, and completes the proof for this case.

(ii) Now suppose that $r, s \in \mathbb{Q}$, and that $r = y/x, s = z/x$ where $x, y, z \in \mathbb{N}^*$ and $y < z$; then

$$\begin{aligned}\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) &= \Big(\sum_{i=1}^n w_i \big(a_i^{1/x}\big)^y\Big)^{1/r}\\ &< \Big(\sum_{i=1}^n w_i \big(a_i^{1/x}\big)^z\Big)^{y/zr} = \mathfrak{M}_n^{[s]}(\underline{a};\underline{w}).\end{aligned}$$

(iii) The general case of positive real r, s follows by taking limits, and as can easily be verified strict inequality is obtained.
(*v*) Assume that $W_n = 1$ and consider the extreme values of the function $\sum_{i=1}^n w_i a_i^s$ as a function of $\underline{a}$ subject to the condition $\sum_{i=1}^n w_i a_i = A$ we easily see that if $s > 1$ then $\mathfrak{A}_n(\underline{a};\underline{w}) \le \mathfrak{M}_n^{[s]}(\underline{a};\underline{w})$, while if $s < 1$ the opposite inequality holds.
(*vi*) We now give an inductive proof of the equal weight case of (1;s) that has been attributed to Steinitz; see [*Paasche*]. In the case $n = 2$ assume that $0 < a < b$ when we have to show that $2\big((a+b)/2\big)^s < a^s + b^s$, or

$$\big((a+b)/2\big)^s - a^s < b^s - \big((a+b)/2\big)^s \tag{5}$$

This is immediate from the strict convexity of $f(y) = y^s$, $y > 0$ and the property of I 4.1 Lemma 2. Alternatively, putting $\beta = (b-a)/2$, (5) can be written as

$$(b-\beta)^s - (b-2\beta)^s < b - (b-\beta)^s$$

which follows by noting that the function $g(y) = y^s - (y-\beta)^s, y > \beta$, is strictly increasing, the last inequality being just $g(b-\beta) < g(b)$.
Now let $\underline{a}$ be a non-constant n-tuple, $n \ge 3$, and without loss in generality assume that $\max \underline{a} = a_n$, $\min \underline{a} = a_{n-1}$ when by the strict internality of the arithmetic mean $a_{n-1} < \mathfrak{A}_n(\underline{a}) < a_n$. Now write $A = \mathfrak{A}_n(\underline{a}), a_{n-1} = A - \delta,\ a_n = A + \Delta$ when it is easily seen that $(n-1)A = \sum_{i-1}^{n-2} a_i + (A - \delta + \Delta)$. Assuming (1;s) for all $(n-1)$-tuples this last identity gives

$$nA^s = (n-1)A^s + A^s \le \sum_{i=1}^{n-2} a_i^s + (A-\delta+\Delta)^s + A^s.$$

However $(A-\delta+\Delta)^s + A^s < (A-\delta)^s + (A+\Delta)^s$ since this inequality can be written $(A-\delta+\Delta)^s - (A+\Delta^s) < (A-\delta)^s - A^s$ which is a consequence either of the strict convexity of the function f above, or of the strict monotonicity of the function g above. This shows that $nA^s < \sum_{i=1}^n a_i^s$ and completes the induction.

(vii) Cauchy's proof of (GA), II 2.4.1 (ii), can be adapted to give a proof for the equal weight case of (r;s).
Suppose that $n = 2^{m+1}$, and we assume the result for $n = 2$ and $n = 2^m$, then with $\underline{a}, \underline{b}, \underline{c}$ as in II 2.4.1 (ii),

$$\begin{aligned}\mathfrak{A}_{2^{m+1}}(\underline{a}) =& \frac{\mathfrak{A}_{2^m}(\underline{b}) + \mathfrak{A}_{2^m}(\underline{c})}{2} \le \Big(\frac{\mathfrak{A}^s_{2^m}(\underline{b}) + \mathfrak{A}^s_{2^m}(\underline{c})}{2}\Big)^{1/s}, \text{ by the case } n = 2,\\ \le& \Big(\frac{\big(\mathfrak{M}^{[s]}_{2^m}(\underline{b})\big)^s + \big(\mathfrak{M}^{[s]}_{2^m}(\underline{c})\big)^s}{2}\Big)^{1/s}, \text{ by the case } n = 2^m\\ =& \mathfrak{M}^{[s]}_{2^{m+1}}(\underline{a}).\end{aligned}$$

The proof for all n is completed by the method of II 2.2.4 Lemma 6.
($viii$) The inequality to be proved is equivalent to

$$\Big(\mathfrak{A}_n(\underline{a};\underline{w})\Big)^s \le \mathfrak{A}_n(\underline{a}^s;\underline{w}),\ s > 1,$$

and so is an immediate consequence of (J), see II 1.1 Remark (xi), and the strict convexity of the function x^s, $s > 1$.
(ix) A simple proof can be given along the lines of Soloviov's proof of (GA), see II 2.4.5 proof (lix). By I 4.6 Example (vii) the function $\gamma(\underline{a}) = \big(\sum_{i=1}^n w_i a_i^s\big)^{1/s}$ is strictly convex and homogeneous on $(\mathbb{R}^*_+)^n$. Now assume that $W_n = 1$ and then $\nabla\chi(\underline{e}) = \underline{w}$, so (1;s) follows from the support inequality, I 4.6 (24); [*Soloviov*].
(x) If, in the thermodynamic proof of (GA), II 2.4.5 proof (l), the heat capacity is allowed to be a function of the temperature then the argument there is readily generalized to give a proof of (r;s) in the case $0 < r < s = r+1$; see [*Landsberg 1980b; Sidhu*]. A similar idea occurs earlier in [*Cashwell & Everett 1967,1968,1969*].
(xi) Gao's proof of (GA), II 2.4.6 proof ($lxxii$), can be adapted to give a proof of (r;s) in the case (1;s) and (r;1), $r \ne 0$; [*Gao*]. Assume without loss in generality that $\underline{a}$ is an increasing n-tuple with $a_1 \ne a_n$ and for $t \in \mathbb{R}^*$ let $\underline{x}_i = (\overbrace{x,\ldots,x}^{i \text{ terms}}, a_{i+1},\ldots a_n)$, and $D_i(x) = \mathfrak{A}_n(\underline{x}_i;\underline{w}) - \mathfrak{M}^{[t]}_n(\underline{x}_i;\underline{w})$, $,0 < x < a_{i+1}$, $1 \le i \le n$. Then

$$D_i'(x) = \frac{W_i}{W_n}\bigg(1 - \Big(\frac{\mathfrak{M}^{[t]}_n(\underline{x}_i;\underline{w})}{x}\Big)^{1-t}\bigg) < 0, \text{by internality, 1(2).}$$

Hence D_i is strictly increasing if $t < 1$, and strictly decreasing if $t > 1$, and so, as in the quoted proof of (GA), $D_1(a_1) > 0$ if $t < 1$, and $D_1(a_1) < 0$ if $t > 1$.
(xii) A very simple proof of the equal weight case with $r = 1$ and s an integer can be found in [*Brenner 1980*].

(*xiii*) A proof of the finite non-zero index case has been given by Qi; see VI 1.2.2 Remark(ix), [*Qi, Mei, Xia & Xu*].
(*xiv*) A completely different proof is given by Segre, see VI 4.5 Example (iv); [*Segre*].
(*xv*) Another proof can be found in VI 4.6 Remark(vii); [*Ben-Tal, Charnes & Teboulle*].
(*xvi*) See also 4.1 Remark (iv).
In all of these proofs the case of equality is easily discussed. □

REMARK (i) Proof (*ii*) is in [*Norris 1935*]; an alternative calculus proof can be found in [*Wang C L 1977*]. A geometric version of the equal weight case of proof (*viii*) has been given in [*Gagan*]. An alternative proof of the case when r and s are integers can be found in [*Ben-Chaim & Rimer; Rimer & Ben-Chaim*].

REMARK (ii) If $0 < r < s < \infty$ and $W_n \le 1$ then an easy deduction from (r;s) is that $\mathcal{R}(r, \underline{a}; \underline{w}) \le \mathcal{R}(s, \underline{a}; \underline{w})$; further if $W_n < 1$ this inequality is strict; see 2.3 Remark (iii) and 2.3 Lemma 6(b). It is easy to see that if this inequality holds for all $\underline{a}$ then $W_n \le 1$; see [*HLP p.29*].

REMARK (iii) If $s = 2r$ then (r;s) is equivalent to

$$\Big(\sum_{i-1}^{n} w_i a_i^r\Big)^2 \le W_n\Big(\sum_{i-1}^{n} w_i a_i^{2r}\Big),$$

which follows from (C); and the case of equality also is given by that of (C). Of course this is the basis of the first step in the induction in proof (*iv*) above.

REMARK (iv) Using $s = 2r = 1/2^m, m \in \mathbb{N}$, (r;s) gives yet another proof of (GA); [*Schlömilch 1858a*]. Assume $\underline{a}$ is not constant: then by (r;s) and 1 Theorem 2(b),

$$\mathfrak{A}_n(\underline{a}; \underline{w}) = \mathfrak{M}_n^{[1]}(\underline{a}; \underline{w}) > \mathfrak{M}_n^{[1/2]}(\underline{a}; \underline{w}) > \cdots \ge \lim_{m\to\infty} \mathfrak{M}_n^{[2^{-m}]}(\underline{a}) = \mathfrak{G}_n(\underline{a}; \underline{w}).$$

REMARK (v) If (GA) has been proved, without the case of equality Paley used an idea similar to that in the previous remark to complete the proof; see II 2.2.3 Lemma 5. Suppose then $\underline{a}$ is not constant, proceed as in Remark (iv):

$$\mathfrak{A}_n(\underline{a}; \underline{w}) > \mathfrak{M}_n^{[1/2]}(\underline{a}; \underline{w}) = \Big(\mathfrak{A}_n(\underline{a}^{1/2}; \underline{w})\Big)^2 \ge \Big(\mathfrak{G}_n^{[1]}(\underline{a}^{1/2}; \underline{w})\Big)^2 = \mathfrak{G}_n(\underline{a}; \underline{w}),$$

using the weaker form of (GA), obtained say by a limit argument from the rational weight case.

REMARK (vi) If $r = 1, s = 2$ the equal weight case of (r;s) is particularly easy to prove.

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{1\le i<j\le n} a_i a_j \le \sum_{i=1}^{n} a_i^2 + \sum_{1\le i<j\le n} (a_i^2 + a_j^2), \text{ by (GA)},$$
$$= n\sum_{i=1}^{n} a_i^2.$$

See also [*Iles & Wilson*].

REMARK (vii) In II 2.2.1 Figure 3 the segment FD is $\mathfrak{Q}(a,b)$, this is also the value of segment CD in II 2.2.1 Figure 4. Hence these diagrams give proofs of the $n = 2$ case of (r;s) for the values $r, s = -1, 0, 1, 2$.

REMARK (viii) If $s = 1$, $r = 1/p < 1$, $\underline{w} = \underline{c}^{p'}, \underline{a}\,\underline{w} = \underline{b}^p$ then (r;s) becomes

$$\sum_{i=1}^{n} b_i c_i \le \Big(\sum_{i=1}^{n} b_i^p\Big)^{1/p}\Big(\sum_{i=1}^{n} c_i^{p'}\Big)^{1/p'},$$

that is (H). Since several proofs of (r;s) are independent of (H) they give further proofs of (H).

REMARK (ix) If $1 < r < s < \infty$ then (r;s) is pictured geometrically in Figure 1, where $0 < a_1 < \cdots < a_n$, and the curve HL has equation $x = t^r, y = t^s$, $a_1 \le t \le a_n$, and R and S have coordinates $1/W_n \sum_{i=1}^{n} w_i a_i^r$, $1/W_n \sum_{i=1}^{n} w_i a_i^s$ respectively[5], so G is the centroid of the points $P_1 \dots, P_n$ on the curve HL.

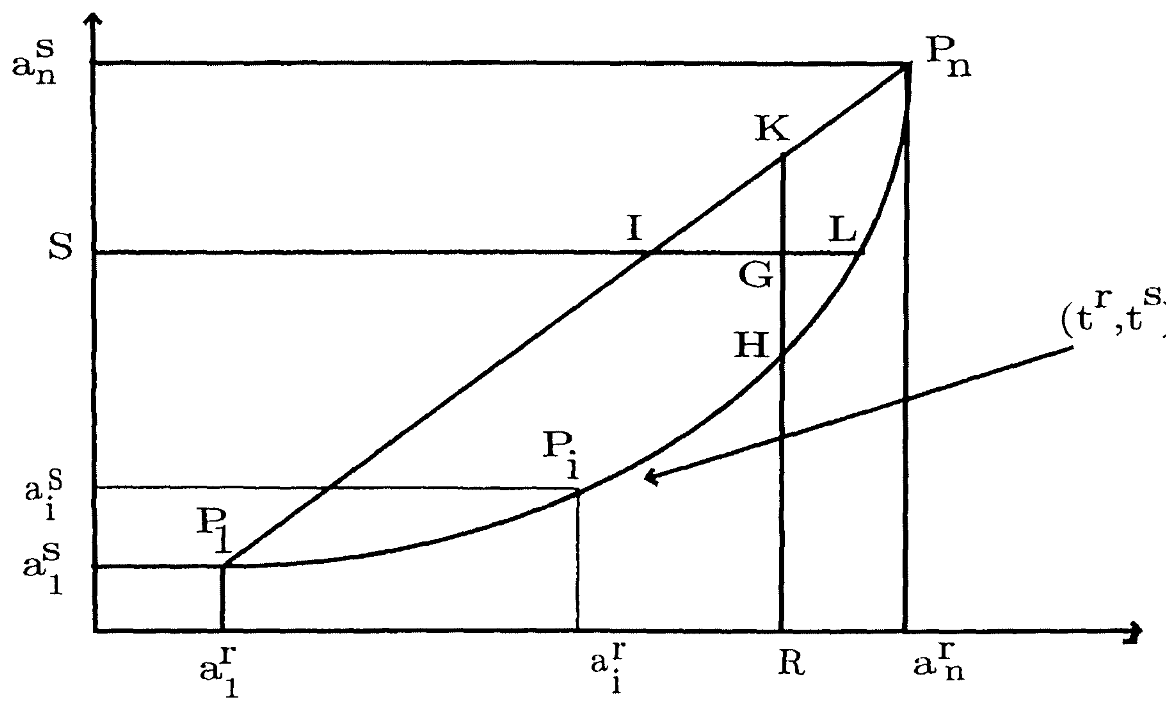

Figure 1

[5] See also VI 4.1 Example (iv).

REMARK (x) A proof of (r;s) in the case $s = mr$, $m \in \mathbb{N}^*$, that uses Čebišev's inequality is given in 3.1.4 Remark (i).

The inequality (r;s) with $r = 1, s = 2$ has been used to prove the following converse of (T). *If* $\pi/4 < \theta < \pi/2$, $\alpha - \theta < \arg z_k < \alpha + \theta$, $1 \le k \le n$, *then*

$$\left|\sum_{k=1}^{n} z_k\right| \ge \frac{1}{\sqrt{2}} \sum_{k=1}^{n} |z_k|.$$

See [*Janić, Kečkić & Vasić; Kocić & Maksimović; Vasić & Kečkić, 1971*].

Burrows and Talbot have used proof (*ii*) of (r;s) to obtain another interesting inequality. If f is as defined in that proof then $f'(r) \ge 0$, and substituting in (2) $b_i = \big(a_i/f(r)\big)^r$, $1 \le i \le n$, leads to

$$\frac{1}{W_n} \sum_{i=1}^{n} w_i b_i \log b_i \ge 0.$$

That is: $\mathfrak{A}_n(\underline{b};\underline{w}) = 1 \implies \mathfrak{A}_n(\underline{b}\log b;\underline{w}) \ge 0$; or equivalently

$$\sum_{i=1}^{n} w_i b_i = W_n \implies \Big(\prod_{i=1}^{n} b_i^{w_i b_i}\Big)^{1/W_n} \ge 1.$$

There is equality if and only if $\underline{b}$ is constant; [*Burrows & Talbot*].

The following result is due to Bronowski; [*Bronowski*]. *If* $a > b > 1$ *and if* $\underline{w}$ *is a real n-tuple with* $W_n = 0$ *then* $\sum_{i=1}^{n} a^{w_i} > \sum_{i=1}^{n} b^{w_i}$.

Several proofs are given in the reference of which the following is the simplest. Let $r > 1$ be defined by $a = b^r$ when:

$$\begin{aligned}\left(\frac{\sum_{i=1}^{n} a^{w_i}}{\sum_{i=1}^{n} b^{w_i}}\right)^{1/(r-1)} &= \left(\frac{\sum_{i=1}^{n} b^{rw_i}}{\sum_{i=1}^{n} b^{w_i}}\right)^{1/(r-1)} = \left(\frac{\sum_{i=1}^{n} b^{(r-1)w_i} b^{w_i}}{\sum_{i=1}^{n} b^{w_i}}\right)^{1/(r-1)}\\ &\ge \Big(\prod_{i=1}^{n} (b^{w_i})^{b^{w_i}}\Big)^{1/\sum_{i=1}^{n} b^{w_i}}, \quad \text{by (r-1;0)},\\ &> \Big(\prod_{i=1}^{n} b^{w_i}\Big)^{1/\sum_{i=1}^{n} b^{w_i}} = 1, \quad \text{since } b > 1 \text{ and } W_n = 0.\end{aligned}$$

REMARK (xi) Inequality (r;s) can be used to give a very simple proof of 1 Theorem 2(b) in the case of equal weights; [*Schaumberger 1992*]. Note first that if $r \ne 0$ then $\mathfrak{M}_n^{[-r]}(\underline{a};\underline{a}^r) = \mathfrak{M}_n^{[r]}(\underline{a})$. So if $r > 0$, we have by (r;s) that

$$\mathfrak{G}_n(\underline{a};\underline{a}^r) \ge \mathfrak{M}_n^{[-r]}(\underline{a};\underline{a}^r) = \mathfrak{M}_n^{[r]}(\underline{a}) \ge \mathfrak{G}_n(\underline{a}),$$

and the inequalities are reversed if $r < 0$. Letting $r \to 0$ in these inequalities gives $\lim_{r\to 0} \mathfrak{M}_n^{[r]}(\underline{a}) = \mathfrak{G}_n(\underline{a})$.

The function $m(r) = \left(\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\right)^r$ is strictly log-convex unless $\underline{a}$ is constant, when m is constant; see 2.3 Lemma 6(b), and 2.3 Remark(iii) The same result follows by considering ϕ of (1) since, using (3), $\phi'(r)/r = \left(\log \circ m(r)\right)''$; see [*Beckenbach 1966; Norris 1935*]. The log-convexity property is due to Liapunov. Popoviciu has proved that $\log \mathfrak{M}_n^{[r]}(\underline{a};\underline{w})$ is a convex function of $1/r$, $r > 0$, and so $\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})$ is a convex function of $1/r$, $r > 0$. This last result was pointed out by Beesack, who used it to give a simple proof of the following result due Hsu; see [*Julia*], [*Beesack 1972; Hsu; Liapunov; Rahmail 1976*].

THEOREM 2 *If* $0 < r < s < t$ *and* $\underline{a}$ *is not constant then*

$$1 < \frac{\mathfrak{M}_n^{[t]}(\underline{a};\underline{w}) - \mathfrak{M}_n^{[r]}(\underline{a};\underline{w})}{\mathfrak{M}_n^{[t]}(\underline{a};\underline{w}) - \mathfrak{M}_n^{[s]}(\underline{a};\underline{w})} < \frac{s(t-r)}{r(t-s)};$$

and if $0 < r < \infty$,

$$1 < \frac{\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) - \mathfrak{M}_n^{[-r]}(\underline{a};\underline{w})}{\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})} < \frac{W_n}{\min \underline{w}}.$$

REMARK (xii) The equal weight case of the second inequality was stated by Hsu and given a proof by Rahmail; the general case can be found in [*Pečarić & Wang*]. This result should be compared with 6.4 Theorem 8.

REMARK (xiii) Further remarks and results on these convexity properties have been made in [*Beckenbach 1942; Shniad*].

Inequality (r;s), together with 1 Theorem 2(c), has been used to approximate roots of algebraic equations; [*Netto pp.290–297*], [*Dunkel 1909/10*]. Suppose for simplicity that all the roots are positive and denote them, in increasing order, by $a_1, \ldots, a_n$. The numbers $\mathfrak{M}_n^{[r]}(\underline{a})$, $r \in \mathbb{Z}$, form an increasing sequence with $\lim_{r\to\infty} \mathfrak{M}_n^{[r]} = a_n$, $\lim_{r\to-\infty} \mathfrak{M}_n^{[r]} = a_1$.

Now consider $\left(\binom{n}{2}^{-1} \sum_2 !(a_{i_1}a_{i_2})^r\right)^{1/r}$, $r \in \mathbb{N}^*$, we get an increasing sequence with limit $a_{n-1}a_n$; equivalently

$$\lim_{r\to\infty} \left(\frac{n \sum_2 !(a_{i_1}a_{i_2})^r}{\binom{n}{2} \sum_{i=1}^n a_i^r}\right)^{1/r} = a_{n-1}.$$

In general

$$\lim_{r\to\infty}\left(\frac{p\sum\limits_{p}!\big(\prod_{j=1}^{p} a_{i_j}\big)^r}{(n-p+1)\sum\limits_{p-1}!\big(\sum_{j=1}^{p-1} a_{i_j}\big)^r}\right)^{1/r} = a_{n-p+1}. \tag{6}$$

The values of the numerator in (6) are easy to obtain if we take $r = 2, 4, 8, \ldots$. This is done by forming equations whose roots are the sequences of those of the original equation, and iterating this process. If the k-th equation by this procedure is $x^n + c_1^{(k)}x^{n-1} + \cdots + c_{n-1}^{(k)}x + c_n^{(k)} = 0$, and then the ratio in (6) is just

$$\left(\frac{pc_p^{(k)}}{(n-p+1)c_{p-1}^{(k)}}\right)^{1/2^k}.$$

In this way, by forming all such expressions, every root of the original equation can be approximated simultaneously.

A different kind of generalization of Theorem 1 is the following result which contains the equal weight (r;s) as a special case; see [*Marshall, Olkin & Proschan*].

THEOREM 3 *If $\underline{a}$ and $\underline{b}$ are n-tuples such that $\underline{b}$ is decreasing, but $\underline{b}/\underline{a}$ is increasing then $\left(\sum_{i=1}^n a_i^r \,/\, \sum_{i=1}^n b_i^r\right)^{1/r}$ increases with r*

Various rewritings of (r;s) can be found in [*Kim Y 2000b*].

3.1.2 HÖLDER'S INEQUALITY AGAIN In this section we will consider the inequality

$$\mathfrak{M}_n^{[s]}(\underline{a}\,\underline{b};\underline{w}) \le \mathfrak{M}_n^{[q]}(\underline{a};\underline{w})\mathfrak{M}_n^{[r]}(\underline{b};\underline{w}). \tag{7}$$

Similar inequalities for power sums have been considered earlier; see 2.2 Remark (i), 2.3 Corollary 8.

It is convenient to deal with various simple cases of this inequality and then state a theorem that will cover the remaining more interesting situations.

As there is no loss in generality we will assume that $q \le r$, $q, r \in \overline{\mathbb{R}}$. Further since (7) is an identity if both $\underline{a}$ and $\underline{b}$ are constant we assume that this is not the case.

(a) If either $\underline{a}$ or $\underline{b}$ is constant then (7) reduces to a case of (r;s). For instance, if $\underline{a}$ is constant, when by the above assumption $\underline{b}$ is not constant, then (7) holds as (s;q) if $s \le q$, strictly if $s < q$, while ($\sim$7) holds if $s \ge q$, strictly if $s > q$.

(b) If $q = r = \infty$ then (7) holds by I 2.2 (16) and (r;s), strictly unless $s = \infty$ and for some i, $1 \le i \le n$, $a_i = \max \underline{a}$, $b_i = \max \underline{b}$. If $q = r = -\infty$ then ($\sim$7) holds by (r;s), strictly unless $s = \infty$ and for some i, $1 \le i \le n$, $a_i = \min \underline{a}$, $b_i = \min \underline{b}$.

(c) If $q < r = \infty$ then (7) holds strictly if $s \le q$. If $q = -\infty < r$ then ($\sim$7) holds strictly if $r \le s$.

(d) If $q = 0$ then (7) holds for $s = 0$ and hence for all $s \le 0$, strictly if $s < 0$ or if $s = 0$ and $r > 0$.
(e) If $r = 0$ then ($\sim$7) holds for all $s \ge 0$, strictly if $s > 0$ or if $s = 0$ and $q < 0$.
(f) If $q = r = s = 0$ then (7) is an identity.

THEOREM 4 *If $\underline{a}, \underline{b}$ and $\underline{w}$ are n-tuples, $\underline{a}\,\underline{b}$ not constant, then (7) holds if $q, r, s \in \mathbb{R}^*$ and satisfy either (a) $0 < q \le r$ and $1/q + 1/r \le 1/s$, or (b) $q < 0 < r$ and $1/q + 1/r \le 1/s < 0$. If $q < 0 < r$ and $1/q + 1/r \ge 1/s > 0$, or $q \le r < 0$ and $1/q + 1/r \ge 1/s$ then ($\sim$7) holds. The inequality is strict in both cases unless $1/q + 1/r = 1/s$ and $\underline{a}^q \sim \underline{b}^r$.*

□ Let $\dfrac{1}{q} + \dfrac{1}{r} = \dfrac{1}{t}$ then $\dfrac{t}{q} + \dfrac{t}{r} = 1$ so t/q and t/r are conjugate indices.
In (a) $t > 0$ and $s \le t$ so

$$\begin{aligned}\mathfrak{M}_n^{[s]}(\underline{a}\,\underline{b};\underline{w}) \le \mathfrak{M}_n^{[t]}(\underline{a}\,\underline{b};\underline{w}) &= \big(\mathfrak{A}_n((\underline{a}\,\underline{b})^t;\underline{w})\big)^{1/t}, \text{ by(r;s) and 1 (4)}\\ &\le \mathfrak{M}_n^{[q]}(\underline{a};\underline{w})\mathfrak{M}_n^{[r]}(\underline{b};\underline{w}), \text{ by (H)}\end{aligned}$$

Similar arguments, using (H) or ($\sim$H), give the remaining cases.
The cases of equality follow from that for (r;s) and (H). □

REMARK (i) The only case not obtained above is the case $q = -r$. Taking limits suggests that the correct result is

$$\min \underline{a}\,\underline{b} \le \mathfrak{M}_n^{[-r]}(\underline{a};\underline{w})\mathfrak{M}_n^{[r]}(\underline{b};\underline{w}) \le \max \underline{a}\,\underline{b}.$$

By the use of 2.1 Corollary 2 inequality (7) can be generalized.

COROLLARY 5 *If $r_i > 0$, $1 \le i \le m$, $1/s \ge \sum_{i=1}^m 1/r_i$ and if $\underline{a}^{(i)}$, $1 \le i \le m, \underline{w}$, are n-tuples with $\prod_{i=1}^m \underline{a}^{(i)}$ not constant then*

$$\mathfrak{M}_n^{[s]}\Big(\prod_{i=1}^m \underline{a}^{(i)};\underline{w}\Big) \le \prod_{i=1}^m \Big(\mathfrak{M}_n^{[r_i]}(\underline{a}^{(i)};\underline{w})\Big),$$

with equality if and only if $1/s = \sum_{i=1}^m 1/r_i$ and $\big(\underline{a}^{(i)}\big)^{r_i}$, $1 \le i \le m$ are proportional.

REMARK (ii) See also IV 2 Theorem 12 for another way of stating this result.

It has been remarked earlier that certain of the basic inequalities are equivalent. It is convenient to collect these equivalencies here.

THEOREM 6 (a) (B) $\iff$ (GA): (b) (GA)$\Longrightarrow$ (H): (c) (H) $\iff$ (C):
(d) (H) $\iff$ (r;s): (e) (H) $\Longrightarrow$ (M): (f) (M) $\Longrightarrow$ (T);
(g) (T)$\Longrightarrow$ (C); (h) (r;s), $r, s > 0, \Longrightarrow$ (GA).

□ (a) For the one implication see II 2.4.5 proof (*lvi*); for the other implication let $0 < \alpha < 1$, then using (GA) we have:

$$(1+x)^\alpha = (1+x)^\alpha 1^{1-\alpha} \le \alpha(1+x) + (1-\alpha)1 = 1 + \alpha x.$$

(b) 2.1 Theorem 1, proof (*i*). (c) 2.2 Theorem 5.
(d) 3.1.1 Theorem 1, proof (*i*) and 3.1.1 Remark (viii).
(e) 2.4 Theorem 9 proof (*i*). (f) See 2.4
(g)(T_N) with $p = 2$ is

$$\sqrt{\sum_{i=1}^n (a_i + b_i)^2} \le \sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2},$$

which on squaring gives (C).
(h) Take $s = 1$ in (r;s) and let $r \to 0$. □

REMARK (iii) Better results are possible since for instance (GA) is equivalent to (GA) with equal weights; (B) with $0 < \alpha < 1$ is equivalent to (B) with $\alpha > 1$; etc. This topic is discussed in [*MPF pp.191–209*], [*Infantozzi; Maligranda 2001*].

3.1.3 MINKOWSKI'S INEQUALITY AGAIN

THEOREM 7 *(a) Let $\underline{a}, \underline{b}$ and $\underline{w}$ be n-tuples and suppose that $1 \le r \le \infty$, then*

$$\mathfrak{M}_n^{[r]}(\underline{a} + \underline{b}; \underline{w}) \le \mathfrak{M}_n^{[r]}(\underline{a}; \underline{w}) + \mathfrak{M}_n^{[r]}(\underline{b}; \underline{w}); \tag{8}$$

with equality if and only if either (i) $r = 1$, or (ii) $1 < r < \infty$ and $\underline{a} \sim \underline{b}$, or (iii) $r = \infty$ and for some i, $1 \le i \le n$, $\max \underline{a} = a_i$ and $\max \underline{b} = b_i$.
If $r < 1$ then (∼8) holds.
(b) If $\underline{a}^{(j)} = (a_{1j}, \ldots, a_{mj}) > 0$, $1 \le j \le n$, and $\underline{a}_{(i)} = (a_{i1}, \ldots, a_{in}) > 0$, $1 \le i \le m$, $\underline{u}$ an m-tuple and $\underline{v}$ an n-tuple, and suppose $r, s \in \overline{\mathbb{R}}$, $r < s$, then,

$$\mathfrak{M}_n^{[s]}\left(\mathfrak{M}_m^{[r]}(\underline{a}^{(j)}; \underline{u}); \underline{v}\right) \le \mathfrak{M}_m^{[r]}\left(\mathfrak{M}_n^{[s]}(\underline{a}_{(i)}; \underline{v}); \underline{u}\right). \tag{9}$$

□ (a) If $r \ne 0$ then this is just (M), 2.4 Theorem 9 and Remark (i). Alternatively we can use I 4.6 Example (vii) which shows that $M(\underline{a}) = \mathfrak{M}_n^{[r]}(\underline{a}; \underline{w}) : (\mathbb{R}_n^*)^n \mapsto \mathbb{R}$ is strictly convex. The case $r = 0$ follows by taking limits.

However these methods fail to give the cases of equality when $r = 0$ so we give two independent proofs of this case.
(i) By (GA) in a quasi-linearization form, see 2.1 Remark (vii), and assuming without loss in generality that $W_n = 1$,

$$\mathfrak{G}_n(\underline{a}+\underline{b};\underline{w}) = \inf_{\underline{c}\in C}\sum_{i=1}^{n} w_i c_i(a_i+b_i), \text{ where } C = \{\underline{c};\ \mathfrak{G}_n(\underline{c};\underline{w}) = 1\}.$$

But $\inf_{\underline{c}\in C}\sum_{i=1}^{n} w_i c_i(a_i+b_i) \ge \inf_{\underline{c}\in C}\sum_{i=1}^{n} w_i c_i a_i + \inf_{\underline{c}\in C}\sum_{i=1}^{n} w_i c_i b_i$, by I 2.2 (e)($\sim$15) so we get

$$\mathfrak{G}_n(\underline{a}+\underline{b};\underline{w}) \ge \mathfrak{G}_n(\underline{a};\underline{w}) + \mathfrak{G}_n(\underline{b};\underline{w}). \tag{10}$$

the case $r = 0$ of (a).
More simply: with $\underline{c} = \underline{a}$ or $\underline{b}$ and using (GA),

$$\frac{\mathfrak{G}_n(\underline{c};\underline{w})}{\mathfrak{G}_n(\underline{a}+\underline{b};\underline{w})} = \mathfrak{G}_n(\underline{c}/(\underline{a}+\underline{b});\underline{w}) \le \mathfrak{A}_n(\underline{c}/(\underline{a}+\underline{b});\underline{w});$$

now add the two inequalities obtained by taking the two values of $\underline{c}$.
The case of equality follows from that of (GA).
(ii) Since by I 4.6 Example (viii) the function $\chi(\underline{a}) = \mathfrak{G}_n(\underline{a};\underline{w})$ is strictly concave (10) follows from II 4.6 ($\sim$19); see [*Soloviov*].
(b) Assume $r, s \in \mathbb{R}$, $rs \ne 0$, $r < s$, and without loss in generality that $U_m = V_n = 1$ when (9) is

$$\Big(\sum_{j=1}^{n} v_j\big(\sum_{i=1}^{m} u_i a_{ij}^r\big)^{s/r}\Big)^{1/s} \le \Big(\sum_{i=1}^{m} u_i\big(\sum_{i=1}^{m} v_j a_{ij}^s\big)^{r/s}\Big)^{1/r},$$

or

$$\Big(\sum_{j=1}^{n}\big(\sum_{i=1}^{m} u_i v_j^{r/s} a_{ij}^r\big)^{s/r}\Big)^{r/s} \le \sum_{i=1}^{m}\big(\sum_{j=1}^{n} u_i^{s/r} v_j a_{ij}^s\big)^{r/s};$$

but this is just 2.4(16).
The other cases are proved by taking limits. □

Remark (i) The cases of equality in (9) are discussed in [*HLP p.31*]. This inequality is sometimes called *Jessen's inequality*, [*MPF p.108*]; it has been much generalized, see [*Toader 1987b; Toader & Dragomir*].

Remark (ii) These results have been proved by many authors; see [*BB p.26*], [*Beckenbach 1942; Besso; Bienaymé; Giaccardi 1955; Jessen 1931a; Liapunov; Norris 1935; Schlömilch 1858b; Simon*].

Remark (iii) Inequality (10) is sometimes called *Hölder's inequality*.
A simple proof of (10) can be given using the inverse geometric means of Nanjundiah, II 3.4. The proof in fact gives a Rado-Popoviciu type extension of (10).

THEOREM 8 *If $n > 1$ then*

$$\left(\frac{\mathfrak{G}_n(\underline{a};\underline{w}) + \mathfrak{G}_n(\underline{b};\underline{w})}{\mathfrak{G}_n(\underline{a}+\underline{b};\underline{w})}\right)^{W_n} \le \left(\frac{\mathfrak{G}_{n-1}(\underline{a};\underline{w}) + \mathfrak{G}_{n-1}(\underline{b};\underline{w})}{\mathfrak{G}_{n-1}(\underline{a}+\underline{b};\underline{w})}\right)^{W_{n-1}},$$

with equality only if $a_n\mathfrak{G}_{n-1}(\underline{b};\underline{w}) = b_n\mathfrak{G}_{n-1}(\underline{a};\underline{w})$

□ By II 3.4 (22) and II 3.4 Lemma 10 (a)

$$\begin{aligned}\mathfrak{G}_n^{-1}(\underline{\mathfrak{G}}(\underline{a};\underline{w})+&\underline{\mathfrak{G}}(\underline{b};\underline{w});\underline{w})\\ &\le \mathfrak{G}_n^{-1}(\underline{\mathfrak{G}}(\underline{a};\underline{w});\underline{w}) + \mathfrak{G}_n^{-1}(\underline{\mathfrak{G}}(\underline{b};\underline{w});\underline{w})\\ &= a_n + b_n = \mathfrak{G}_n^{-1}(\underline{\mathfrak{G}}(\underline{a}+\underline{b};\underline{w});\underline{w}),\end{aligned}$$

which gives the above inequality.
The case of equality follows from that of II 3.4 Lemma 10 (c). □

REMARK (iv) A simple proof of the right-hand inequality of II 2.5.3(36) can be given using (10).

$$\begin{aligned}1 + \mathfrak{G}_n(\underline{a};\underline{w}) =&\mathfrak{G}_n(\underline{e};\underline{w}) + \mathfrak{G}_n(\underline{a};\underline{w}) \le \mathfrak{G}_n(\underline{e}+\underline{a};\underline{w}),\ \text{by (10)},\\ \le&\mathfrak{A}_n(\underline{e}+\underline{a};\underline{w}),\ \text{by (GA)}, = 1 + \mathfrak{A}_n(\underline{a};\underline{w}).\end{aligned}$$

More can be proved by using (8) rather than (10), and appealing to (r;s) instead of (GA). *If* $0 \le r < 1$ *then*

$$1 + \mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) \le \mathfrak{M}_n^{[r]}(\underline{e}+\underline{a};\underline{w}) \le 1 + \mathfrak{A}_n(\underline{a};\underline{w}).$$

Conversely, using the equal weight case of left-hand inequality of II 2.5.3(36) we can prove the equal weight case of (10); [*Dragomir, Comănescu & Pearce*].

$$\frac{\mathfrak{G}_n(\underline{a}) + \mathfrak{G}_n(\underline{b})}{\mathfrak{G}_n(\underline{b})} = 1 + \mathfrak{G}_n(\underline{a}/\underline{b}) \le \mathfrak{G}_n(\underline{e}+\underline{a}/\underline{b}) = \frac{\mathfrak{G}_n(\underline{a}+\underline{b})}{\mathfrak{G}_n(\underline{b})}.$$

3.1.4 ČEBIŠEV'S INEQUALITY

We can extend Cebišev's inequality, II 5.3 (1), to power means.

THEOREM 9 *(a) If $0 \le r < \infty$, and if $\underline{a}$, $\underline{b}$ and $\underline{w}$ are positive n-tuples with $\underline{a}$ and $\underline{b}$ similarly ordered then,*

$$\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\mathfrak{M}_n^{[r]}(\underline{b};\underline{w}) \le \mathfrak{M}_n^{[r]}(\underline{a}\,\underline{b};\underline{w}), \tag{11}$$

If $\underline{a}$ and $\underline{b}$ are oppositely ordered then (~11) holds.
(b) If $-\infty < r \le 0$, and if $\underline{a}$ and $\underline{b}$ oppositely, [similarly], ordered, then (11) ,

[(∼11)], holds.
There is equality in (11) when $r = 0$, and in all other cases there is equality if and only if either $\underline{a}$, or $\underline{b}$ is constant.

□ The case $r = 0$ is trivial and is in any case pointed out in II 1.2 (9). If $r \neq 0$ then by 1(4) we need only consider the case $r = 1$. This however is just II 5.3 Theorem 4. □

REMARK (i) If the n-tuples $\underline{a}^{(k)}$, $1 \le k \le m$ are similarly ordered and $r > 0$ then from (11)

$$\prod_{k=1}^{m} \mathfrak{M}_n^{[r]}(\underline{a}^{(k)}; \underline{w}) \le \mathfrak{M}_n^{[r]}\Bigl(\prod_{k=1}^{m} \underline{a}^{(k)}; \underline{w}\Bigr).$$

Now putting $s = mr$ and $\underline{a}^{(1)} = \cdots = \underline{a}^{(m)}$ this inequality is just (r;s).

3.2 REFINEMENTS OF THE POWER MEAN INEQUALITY

The inequality (r;s) has a similar form to (GA) and so it is natural to consider extensions similar to those considered in II 3.

3.2.1 THE POWER MEAN INEQUALITY WITH GENERAL WEIGHTS As we have seen, when $r, s \in \mathbb{R}$ the inequality (r;s) is a particular case of (J), so we can use the Jensen-Steffensen inequality, I 4.3, to extend II 3.7 Theorem 22 to allow general weights in (r;s) that satisfy II 3.7 (49). In this more general situation however we must take full note of the remarks in I 4.2 theorem 12 and I 4.3 Theorem 19 and ensure that we are in the domains of the functions being used; that is we must require that the sums $\sum_{i=1}^{n} w_i a_i^r$, $\sum_{i=1}^{n} w_i a_i^s$ be positive. We will not pursue this further but consider the case $n = 2$, that is the extension of II 3.7 Theorem 23.

THEOREM 10 *If $n = 2$, $w_1 w_2 < 0$, $W_2 = 1$ $\underline{a}$ a 2-tuple, $r, s \in \mathbb{R}^*$, $r < s$, and if $w_1 a_1^r + w_2 a_2^r > 0$, $w_1 a_1^s + w_2 a_2^s > 0$ then (∼r;s) holds.*

REMARK (i) If we assume that $a_1 < a_2$ this theorem can be put more simply since then all we need is that $w_1 a_1^s + w_2 a_2^s > 0$, or $w_2 > -a_1^s/(a_2^s - a_1^s)$. Further we can extend II 3.7 (51) and (52) as follows: if $w_2 < 0$ then $\mathfrak{M}^{[s]}(\underline{a}; \underline{w}) < \mathfrak{M}^{[r]}(\underline{a}; \underline{w}) < a_1$, while if $w_1 < 0$ then $\mathfrak{M}^{[r]}(\underline{a}; \underline{w}) > \mathfrak{M}^{[s]}(\underline{a}; \underline{w}) > a_2$.

REMARK (ii) This simple theorem is behind the interesting inequalities of Nanjundiah, see below 3.2.6.

3.2.2 DIFFERENT WEIGHT EXTENSION

THEOREM 11 *If $\underline{a}, \underline{u}$ and $\underline{v}$ are n-tuples, and if $r, s \in \mathbb{R}^*$, $r < s$, define the n-tuple $\underline{w}$ by $\underline{w}^{s-r} = \underline{u}\,\underline{v}^{-1}$; then*

$$\mathfrak{M}_n^{[r]}(\underline{a};\underline{u}) \le \frac{\mathfrak{M}_n^{[r]}(\underline{w};\underline{u})}{\mathfrak{M}_n^{[s]}(\underline{w};\underline{v})}\mathfrak{M}_n^{[s]}(\underline{a};\underline{v}),$$

with equality if and only if $\underline{a}\,\underline{w}$ is constant.

□ Since

$$\frac{\mathfrak{M}_n^{[r]}(\underline{a};\underline{u})}{\mathfrak{M}_n^{[s]}(\underline{a};\underline{v})} = \left(\frac{\mathfrak{M}_n^{[r]}(\underline{a}\,\underline{w};\underline{u}^{s/(s-r)}\underline{v}^{r/(r-s)})}{\mathfrak{M}_n^{[s]}(\underline{a}\,\underline{w};\underline{u}^{s/(s-r)}\underline{v}^{r/(r-s)})}\right)\left(\frac{\mathfrak{M}_n^{[r]}(\underline{w};\underline{u})}{\mathfrak{M}_n^{[s]}(\underline{w};\underline{v})}\right),$$

the result is an immediate consequence of (r;s). □

REMARK (i) See [*Bullen 1967; Mitrinović & Vasić 1966a*].

The following theorem is an application of this result;[*Vasić & Kečkić 1971*].

THEOREM 12 *If $\underline{z}$ is a complex n-tuple, $\underline{w}$ a positive n-tuple, and if $p > 1$ then*

$$\left|\sum_{k=1}^{n} z_k\right| \le \left(\sum_{k=1}^{n} w_k^{p'/p}\right)^{1/p'} \mathfrak{M}_n^{[p]}(|\underline{z}|;\underline{w}),$$

with equality if and only if $\underline{w}\,|\underline{z}|^{p-1}$ is constant, and $z_k\overline{z_j} \ge 0$, $1 \le j,k \le n$.

A direct proof of this inequality using (H) can be found in [*Bullen 1972*].

3.2.3 EXTENSIONS OF RADO-POPOVICIU TYPE

An extension of (R) and (P) to power means is given by the following result.

THEOREM 13 *Assume that $\underline{a}$ and $\underline{w}$ are n-tuples, $n \ge 2$, and $r, s \in \overline{\mathbb{R}}$, $r \ne s$.*
(a) If $r < s$ and $\lambda = r$ or s but $\lambda \ne 0$, then

$$\begin{aligned} W_n\left(\left(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})\right)^{\lambda} - \left(\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\right)^{\lambda}\right)& \\ \ge W_{n-1}\left(\left(\mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})\right)^{\lambda} - \left(\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})\right)^{\lambda}\right)&, \end{aligned} \tag{12}$$

with equality when $\lambda = s$ if and only if $a_n = \mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})$, and when $\lambda = r$ if and only if $a_n = \mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})$.
(b) If $r \le 0 \le s$ then

$$\left(\frac{\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})}{\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})}\right)^{W_n} \ge \left(\frac{\mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})}{\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})}\right)^{W_{n-1}}, \tag{13}$$

with equality, when $s = 0$, if and only if $a_n = \mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})$, when $r = 0$, if and only if $a_n = \mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})$, and when $r < 0 < s$, if and only if both conditions hold.

□ (a) The case $r \neq 0, s = \infty$, when $\lambda = r$, or $s \neq 0, r = -\infty$, when $\lambda = s$ are almost immediate.

As in 3.1 Theorem 1 we can if $s \neq 0$ use 1(4) to assume $s = 1$, or if $r \neq 0$ that $r = 1$. Further if either $s = 0$, or $r = 0$, then (12) reduces to (R). So it suffices to consider the cases (i) $r = 1 < s < \infty$, and (ii) $-\infty < r < 1 = s$, $r \neq 0$.

`Case (i)` In this case if $\lambda = s$ then (12) is equivalent to

$$W_{n-1}^{1-s}\Big(\sum_{i=1}^{n-1} w_i a_i\Big)^s + w_n^{1-s}\big(w_n a_n\big)^s \geq W_n^{1-s}\Big(\sum_{i=1}^{n} w_i a_i\Big)^s,$$

and when $\lambda = 1$ to

$$W_n^{1-1/s}\Big(\sum_{i=1}^{n} w_i a_i^s\Big)^{1/s} \geq W_{n-1}^{1-1/s}\Big(\sum_{i=1}^{n-1} w_i a_i^s\Big)^{1/s} + w_n^{1-1/s}\big(w_n a_n^s\big)^{1/s}.$$

An application of (B), or (H), confirms these inequalities and gives the cases of equality.

`Case (ii)` To consider this case put $x = a_n$ assume that $\lambda = s = 1$, and put $f(x) = \mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{M}_n^{[r]}(\underline{a};\underline{w})$. Then

$$f(x) = \Big(\frac{W_{n-1}}{W_n}\mathfrak{A}_{n-1}(\underline{a};\underline{w}) + \frac{w_n}{W_n}x\Big) - \Big(\frac{W_{n-1}}{W_n}\big(\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})\big)^r + \frac{w_n}{W_n}x^r\Big)^{1/r},$$

and

$$f'(x) = \frac{w_n}{W_n}\bigg(1 - \Big(\frac{w_n}{W_n} + \frac{W_{n-1}}{W_n}\big(\frac{\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})}{x}\big)^r\Big)^{1/(r-1)}\bigg).$$

So f has a unique minimum at $x = x_0 = \mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})$.

As a result $f(x) > f(x_0)$ if $x \neq x_0$, and simple calculations show that this is the desired result.

(b) The cases of non-finite r and, or, s are straightforward. Further, if $r, s \in \mathbb{R}$ and either r or s is zero the result reduces to (P) by using 1(4). So we may assume

that $r, s \in \mathbb{R}^*$. Then

$$
\begin{aligned}
\log\Big(\frac{\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})}{\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})}\Big)^{W_n} = W_n\Big(\frac{1}{s}\log\Big(\frac{W_{n-1}}{W_n}\big(\frac{1}{W_{n-1}}\sum_{i=1}^{n} w_i a_i^s\big) + \frac{w_n}{W_n}a_n^s\Big)\\
- \frac{1}{r}\log\Big(\frac{W_{n-1}}{W_n}\big(\frac{1}{W_{n-1}}\sum_{i=1}^{n} w_i a_i^r\big) + \frac{w_n}{W_n}a_n^r\Big)\Big)\\
\ge W_{n-1}\Big(\frac{1}{s}\log\Big(\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i a_i^s\Big) - \frac{1}{r}\log\Big(\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i a_i^r\Big)\Big),\\
\text{by the concavity of the logarithmic function,}\\
= \log\Big(\frac{\mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})}{\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})}\Big)^{W_{n-1}}.
\end{aligned}
$$

The case of equality follows since the logarithmic function is strictly concave. □

REMARK (i) Another proof of one case of part (a) of this theorem is given below, see 3.2.6 Theorem 24.

REMARK (ii) The technique used in II 4.1 Theorem 1 proof (i) can be applied to the proof of (a) case (ii) to obtain inequalities converse to (r;s). This will not be done here as the whole topic will be taken up in 4 below.

REMARK (iii) Many similar results follow from more general theorems to be proved later, see IV 3.2, and so we will not discuss them at this point; see [*Bullen 1968; Mitrinović & Vasić 1966a,b,c*],

A simple deduction from Theorem 13 is the following.

COROLLARY 14 *If $\underline{a}$ and $\underline{w}$ are n-tuples, $n \ge 2$, and $-\infty \le r \le 1 \le s \le \infty$, $r \ne s$ then*

$$W_n\big(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) - \mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\big) \ge W_{n-1}\big(\mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w}) - \mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})\big), \tag{14}$$

with equality when $s = 1$ if and only if $a_n = \mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})$, when $r = 1$ if and only if $a_n = \mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})$, and when $r < 1 < s$ if and only if both of these conditions hold.

□ Take $\lambda = r = 1$ and $\lambda = s = 1$ in (12) and add the resulting inequalities. The cases of equality are immediate. □

REMARK (iv) Inequality (14) and its analogue (13) are the simplest extensions of (R) and (P).

If we do not assume that $r \le 1 \le s$ then (14) need not hold as the following example of Diananda shows.

EXAMPLE (i) Assume that $1 < r < s < \infty$, $0 < \delta < 1$, $w_1 = \cdots = w_n = 1$, $n \ge 3$, $a_1 = \cdots = a_{n-3} = a_n = 1$, $a_{n-2} = (1+\delta)^{1/s}$, $a_{n-1} = (1-\delta)^{1/s}$. Then (14) reduces to $1 + (n-1)\mathfrak{M}^{[r]}_{n-1}(\underline{a};\underline{w}) \ge n\mathfrak{M}^{[r]}_n(\underline{a};\underline{w})$ or equivalently $\mathfrak{A}_n(\underline{b}) \ge \mathfrak{M}^{[r]}_n(\underline{b})$, where the n-tuple $\underline{b} = (k,\ldots,k,1)$, with $k = (n-1)^{1/r}\big(n-3+(1+\delta)^{r/s}+(1-\delta)^{r/s}\big)$. Since $k \ne 1$ this is false by (r;s). If $r < s < 1$ repeat the above replacing s by r in a_{n-2}, and in a_{n-1}.

3.2.4 INDEX SET EXTENSIONS We now prove a general theorem that implies many extensions of the results in II 3.2.2 ; see [*Mitrinović & Vasić 1968d*].

THEOREM 15 *Let $\lambda,\mu > 0$, $\lambda+\mu \ge 1$, $\underline{a},\underline{b},\underline{u},\underline{v}$ sequences, I_1, I_2, J_1, J_2 index sets with I_1, J_1 disjoint, I_2, J_2 disjoint, then*

$$\begin{aligned} &U^{\lambda}_{I_1\cup J_1}V^{\mu}_{I_2\cup J_2}\big(\mathfrak{M}^{[r]}_{I_1\cup J_1}(\underline{a};\underline{u})\big)^{\lambda r}\big(\mathfrak{M}^{[s]}_{I_2\cup J_2}(\underline{b};\underline{v})\big)^{\mu s} \\ &\quad\ge U^{\lambda}_{I_1}V^{\mu}_{I_2}\big(\mathfrak{M}^{[r]}_{I_1}(\underline{a};\underline{u})\big)^{\lambda r}\big(\mathfrak{M}^{[s]}_{I_2}(\underline{b};\underline{v})\big)^{\mu s} + U^{\lambda}_{J_1}V^{\mu}_{J_2}\big(\mathfrak{M}^{[r]}_{J_1}(\underline{a};\underline{u})\big)^{\lambda r}\big(\mathfrak{M}^{[s]}_{J_2}(\underline{b};\underline{v})\big)^{\mu s}. \end{aligned} \tag{15}$$

If $\lambda+\mu > 1$ (15) is strict; if $\lambda+\mu = 1$ (15) is strict unless

$$U^{\lambda}_{I_1}V^{\mu}_{J_2}\big(\mathfrak{M}^{[r]}_{I_1}(\underline{a};\underline{u})\big)^{\lambda r}\big(\mathfrak{M}^{[s]}_{J_2}(\underline{b};\underline{v})\big)^{\mu s} = U^{\lambda}_{J_1}V^{\mu}_{I_2}\big(\mathfrak{M}^{[r]}_{J_1}(\underline{a};\underline{u})\big)^{\lambda r}\big(\mathfrak{M}^{[s]}_{I_2}(\underline{b};\underline{v})\big)^{\mu s}. \tag{16}$$

If $\lambda\mu < 0$, $\lambda+\mu = 1$ then (~15) holds, with the same conditions for equality.

□ This follows immediately from 2.3 Corollary 8(b) with $n = 2$ on putting $\lambda = 1/p$, $\mu = 1/p'$ and $a_1^p = U_{I_1}\big(\mathfrak{M}^{[r]}_{I_1}(\underline{a};\underline{u})\big)^r$, $a_2^p = U_{J_1}\big(\mathfrak{M}^{[r]}_{J_1}(\underline{a};\underline{u})\big)^r$; and $b_1^{p'} = V_{I_2}\big(\mathfrak{M}^{[s]}_{I_2}(\underline{b};\underline{v})\big)^s$, $b_2^{p'} = V_{J_2}\big(\mathfrak{M}^{[s]}_{J_2}(\underline{b};\underline{v})\big)^s$. The cases of equality are immediate. □

REMARK (i) Since Theorem 15 is such an immediate consequence of a very simple case of (H) any deduction from this theorem can also be obtained directly and easily from (H).

REMARK (ii) Inequality (15) is easily extended to allow for m pairs of disjoint index sets.

REMARK (iii) Put $\mathrm{I}_1 = I_2 = I$, $J_1 = J_2 = J$, $r = p, s = p'$, $\lambda = 1/p$, $\mu = 1/p'$, $u_i = 1 = v_i$, $i \in \mathbb{N}^*$ then (15) becomes

$$\Big(\sum_{i\in I\cup J} a_i^p\Big)^{1/p}\Big(\sum_{i\in I\cup J} b_i^{p'}\Big)^{1/p'} \ge \Big(\sum_{i\in I} a_i^p\Big)^{1/p}\Big(\sum_{i\in I} b_i^{p'}\Big)^{1/p'} + \Big(\sum_{i\in J} a_i^p\Big)^{1/p}\Big(\sum_{i\in J} b_i^{p'}\Big)^{1/p'}.$$

Since clearly

$$\sum_{I\cup J} a_ib_i = \sum_I a_ib_i + \sum_J a_ib_i$$

the last inequality implies the first part of 2.5.2 Theorem 12, and the case of equality in that theorem is an easy consequence of that in 2.4 Corollary 10.

COROLLARY 16 *If I, J are disjoint index sets, $\underline{a}, \underline{u}, \underline{v}$ are sequences, $r, s \in \mathbb{R}$, $r < 0 < s$, then*

$$\begin{aligned}\frac{U_{I\cup J}^{s/(s-r)}}{V_{I\cup J}^{r/(s-r)}} &\left(\frac{\mathfrak{M}_{I\cup J}^{[r]}(\underline{a};\underline{u})}{\mathfrak{M}_{I\cup J}^{[s]}(\underline{a};\underline{v})}\right)^{rs/(s-r)} \\ &\geq \frac{U_{I}^{s/(s-r)}}{V_{I}^{r/(s-r)}}\left(\frac{\mathfrak{M}_{I}^{[r]}(\underline{a};\underline{u})}{\mathfrak{M}_{I}^{[s]}(\underline{a};\underline{v})}\right)^{rs/(s-r)} + \frac{U_{J}^{s/(s-r)}}{V_{J}^{r/(s-r)}}\left(\frac{\mathfrak{M}_{J}^{[r]}(\underline{a};\underline{u})}{\mathfrak{M}_{J}^{[s]}(\underline{a};\underline{v})}\right)^{rs/(s-r)}.\end{aligned} \tag{17}$$

The inequality is strict unless

$$\frac{U_I}{V_I}\frac{\left(\mathfrak{M}_{I}^{[r]}(\underline{a};\underline{u})\right)^r}{\left(\mathfrak{M}_{I}^{[s]}(\underline{a};\underline{v})\right)^s} = \frac{U_J}{V_J}\frac{\left(\mathfrak{M}_{J}^{[r]}(\underline{a};\underline{u})\right)^r}{\left(\mathfrak{M}_{J}^{[s]}(\underline{a};\underline{v})\right)^s}.$$

If $rs > 0$, $r \neq s$ ($\sim$17) holds and is strict under the same conditions.

□ This is an immediate consequence of Theorem 15, by taking $\underline{a} = \underline{b}$, $I_1 = I_2 = I$, $J_1 = J_2 = J$, $\lambda = s/(s-r)$, $\mu = -r/(s-r)$. □

REMARK (iv) Defining the following function on the index sets

$$\nu(I) = \frac{U_{I}^{s/(s-r)}}{V_{I}^{r/(s-r)}}\left(\frac{\left(\mathfrak{M}_{I}^{[r]}(\underline{a};\underline{u})\right)^r}{\left(\mathfrak{M}_{I}^{[s]}(\underline{a};\underline{v})\right)^s}\right)^{rs/(s-r)},$$

then Corollary 16 says that ν is super-additive if $rs < 0$ and sub-additive if $rs > 0$.

REMARK (v) Taking $I = \{1, 2, \ldots, n-1\}$, $J = \{n\}$ (17) becomes

$$\frac{U_{n}^{s/(s-r)}}{V_{n}^{r/(s-r)}}\left(\frac{\mathfrak{M}_{n}^{[r]}(\underline{a};\underline{u})}{\mathfrak{M}_{n}^{[s]}(\underline{a};\underline{v})}\right)^{rs/(s-r)} \geq \frac{U_{n-1}^{s/(s-r)}}{V_{n-1}^{r/(s-r)}}\left(\frac{\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{u})}{\mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{v})}\right)^{rs/(s-r)} + \frac{u_{n}^{s/(s-r)}}{v_{n}^{r/(s-r)}}, \tag{18}$$

with equality if and only if

$$\left(\frac{U_{n-1}v_n}{u_nV_{n-1}}\right)a_n^{s-r} = \frac{\left(\mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{v})\right)^s}{\left(\mathfrak{M}_{n-1}^{[u]}(\underline{a};\underline{u})\right)^u}.$$

REMARK (vi) Inequalities of this type for power means seem to have been first discussed by McLaughlin & Metcalf, and independently by Bullen; see [*Bullen 1968; McLaughlin & Metcalf 1967a,b,c*].

COROLLARY 17 *If $I, J, \underline{a}, \underline{u}, \underline{v}$ are as in Corollary 16 and $r < 0 < s$ then*

$$\left(\frac{U_{I\cup J}}{V_{I\cup J}}\Big(\frac{\mathfrak{M}^{[r]}_{I\cup J}(\underline{a};\underline{u})}{\mathfrak{M}^{[s]}_{I\cup J}(\underline{a};\underline{v})}\Big)^{s}\right)^{U_{I\cup J}} \geq \left(\frac{U_I}{V_I}\Big(\frac{\mathfrak{M}^{[r]}_{I}(\underline{a};\underline{u})}{\mathfrak{M}^{[s]}_{I}(\underline{a};\underline{v})}\Big)^{s}\right)^{U_I}\left(\frac{U_J}{V_J}\Big(\frac{\mathfrak{M}^{[r]}_{J}(\underline{a};\underline{u})}{\mathfrak{M}^{[s]}_{I}(\underline{a};\underline{v})}\Big)^{s}\right)^{U_J}. \tag{19}$$

Inequality (19) is strict unless $\mathfrak{M}^{[r]}_I(\underline{a};\underline{u}) = \mathfrak{M}^{[r]}_J(\underline{a};\underline{u})$.

□ The right-hand side of (17), divided by $U_{I\cup J}$ can be written in the form

$$\frac{U_I}{U_{I\cup J}}\left(\frac{U_I}{V_I}\Big(\frac{\mathfrak{M}^{[r]}_{I}(\underline{a};\underline{u})}{\mathfrak{M}^{[s]}_{I}(\underline{a};\underline{v})}\Big)^{s}\right)^{r/(s-r)} + \frac{U_J}{U_{I\cup J}}\left(\frac{U_J}{V_J}\Big(\frac{\mathfrak{M}^{[r]}_{J}(\underline{a};\underline{u})}{\mathfrak{M}^{[s]}_{J}(\underline{a};\underline{v})}\Big)^{s}\right)^{r/(s-r)},$$

which by (GA) is greater than or equal to

$$\left(\frac{U_I}{V_I}\Big(\frac{\mathfrak{M}^{[r]}_{I}(\underline{a};\underline{u})}{\mathfrak{M}^{[s]}_{I}(\underline{a};\underline{v})}\Big)^{s}\right)^{rU_I/(s-r)U_{I\cup J}}\left(\frac{U_J}{V_J}\Big(\frac{\mathfrak{M}^{[r]}_{J}(\underline{a};\underline{u})}{\mathfrak{M}^{[s]}_{J}(\underline{a};\underline{v})}\Big)^{s}\right)^{rU_J/(s-r)U_{I\cup J}}.$$

On taking suitable powers of this inequality (19) follows. □

REMARK (vii) Taking $I = \{1,\ldots,n\}$, $J = \{n+1,\ldots,n+m\}$, $\underline{u} = \underline{v}$, $s = 1$ and then letting $r \to 0-$, (19) reduces to II 3.2.2 Theorem 8(b), together with the case of equality, making Corollary 17 a generalization of that result.

Define the following function on the index sets, $\sigma(I) = W_I\mathfrak{M}^{[s]}_I(\underline{a};\underline{w})$, then the following is an analogue of 2.5.2 Theorem 12; [*Everitt 1963*].

COROLLARY 18 *If $s > 1$ and I, J are disjoint index sets then*

$$\sigma(I\cup J) \geq \sigma(I) + \sigma(J), \tag{20}$$

with equality if and only if $\mathfrak{M}^{[s]}_I(\underline{a};\underline{w}) = \mathfrak{M}^{[s]}_J(\underline{a};\underline{w})$. If $s < 1$ then ($\sim$ 20) holds with the same case of equality. If $s = 1$ then (20) is an identity.

□ The case $s = 1$ is trivial so suppose first that $s > 1$.

In Theorem 15 take $I_1 = I_2 = I$, $J_1 = J_2 = J$, $r = 0$, $\lambda = 1-\mu$, $\mu = 1/s$, and $\underline{u} = \underline{v} = \underline{w}$; the result is then immediate.

If $s < 1$, $s \neq 0$ there is a similar proof.

If $s = 0$ the right-hand side of (20) is

$$\begin{aligned} W_{I\cup J}\Big(\frac{W_I}{W_{I\cup J}}\mathfrak{G}_I(\underline{a};\underline{w}) + \frac{W_J}{W_{I\cup J}}\mathfrak{G}_J(\underline{a};\underline{w})\Big) & \\ \leq W_{I\cup J}\mathfrak{G}_I(\underline{a};\underline{w})^{W_I/W_{I\cup J}}\mathfrak{G}_J(\underline{a};\underline{w})\Big)^{W_J/W_{I\cup J}}, & \text{ by (GA)},\\ = W_{I\cup J}\mathfrak{G}_{I\cup J}(\underline{a};\underline{w}). & \end{aligned}$$

The case of equality follows from that of (GA). □

The following generalizes II 3.2.2 Theorem 8 and the notation used is introduced there; see [*Bullen 1968; McLaughlin & Metcalf 1967b, 1968b; Mitrinović & Vasić, 1968d*].

THEOREM 19 *If $\underline{a}, \underline{u}$ and $\underline{v}$ are $(n+m)$-tuples, $\lambda \in \mathbb{R}$, $0 < \lambda^{r/(s-r)}\overline{U}_m < U_{n+1}$, $s/r > 1$ then*

$$\frac{V_{n+m}}{\overline{V}_m}\Big(\mathfrak{M}^{[s]}_{n+m}(\underline{a};\underline{v})\Big)^s - \frac{U_{n+m}}{\overline{U}_m}\Big(\mathfrak{M}^{[r]}_{n+m}(\underline{a};\underline{u})\Big)^s \tag{21}$$
$$\geq \frac{V_n}{\overline{V}_m}\Big(\mathfrak{M}^{[s]}_n(\underline{a};\underline{v})\Big)^s - \frac{U_n^{s/r}}{\overline{U}_m}\Big(\mathfrak{M}^{[r]}_n(\underline{a};\underline{u})\Big)^s\big(U_{n+m} - \overline{U}_m\lambda^{r/(s-r)}\big)^{(r-s)/r}$$
$$+\Big(\overline{\mathfrak{M}}^{[s]}_m(\underline{a};\underline{v})\Big)^s - \frac{1}{\lambda}\Big(\overline{\mathfrak{M}}^{[r]}_m(\underline{a};\underline{u})\Big)^s;$$

and

$$\frac{V_{n+m}}{\overline{V}_m}\Big(\mathfrak{M}^{[s]}_{n+m}(\underline{a};\underline{v})\Big)^s - \lambda\frac{U_{n+m}}{\overline{U}_m}\Big(\mathfrak{M}^{[r]}_{n+m}(\underline{a};\underline{u})\Big)^s\Big(\frac{\overline{\mathfrak{M}}^{[s]}_m(\underline{a};\underline{v})}{\overline{\mathfrak{M}}^{[r]}_m(\underline{a};\underline{u})}\Big)^s \tag{22}$$
$$\geq \frac{V_n}{\overline{V}_m}\Big(\mathfrak{M}^{[s]}_n(\underline{a};\underline{v})\Big)^s - \frac{U_n^{s/r}}{\overline{U}_m}\Big(\mathfrak{M}^{[r]}_n(\underline{a};\underline{u})\Big)^s\big(U_{n+m} - \overline{U}_m\lambda^{r/(s-r)}\big)^{(r-s)/r}\Big(\frac{\overline{\mathfrak{M}}^{[s]}_m(\underline{a};\underline{v})}{\overline{\mathfrak{M}}^{[r]}_m(\underline{a};\underline{u})}\Big)^s$$

If $0 \le s/r \le 1$ then ($\sim$ 21) and ($\sim$22) hold.
Equality occurs if $s = 0$, when no restriction need be placed on λ, or if $s \neq 0$ if $U_n\mathfrak{M}^{[r]}_n(\underline{a};\underline{u}) = \big(\lambda^{r/(r-s)}U_{n+m} - \overline{U}_m\big)^{1/r}\overline{\mathfrak{M}}^{[r]}_m(\underline{a};\underline{u})$.
Letting $r \to 0$ in (21), (21), gives inequalities that hold for all s, that are strict except under the conditions obtained from the above conditions on letting $r \to 0$.

□ `Proof of (21) case (i):` $r \neq 0, s \neq 0$. In this case (21) is equivalent to

$$\frac{1}{\overline{V}_m}\sum_{i=1}^{n+m} v_i a_i^s - \frac{U_{n+m}}{\overline{U}_m}\Big(\sum_{i=1}^{n+m} u_i a_i^r\Big)^{s/r}$$
$$\geq \frac{1}{\overline{V}_m}\sum_{i=1}^{n} v_i a_i^s - \frac{1}{\overline{U}_m}\Big(\sum_{i=1}^{n} u_i a_i^r\Big)^{s/r}\big(U_{n+m} - \overline{U}_m\lambda^{r/(s-r)}\big)^{(r-s)/r}$$
$$+\frac{1}{\overline{V}_m}\sum_{i=n+1}^{n+m} v_i a_i^s - \frac{1}{\lambda\overline{U}_m^{s/r}}\Big(\sum_{i=n+1}^{n+m} u_i a_i^r\Big)^{s/r}.$$

This, in turn, is equivalent to

$$\Big(\sum_{i=1}^{n} u_i a_i^r\Big)^{s/r}\big(U_{n+m} - \overline{U}_m\lambda^{r/(s-r)}\big)^{(r-s)/r} \tag{23}$$
$$+\big(\overline{U}_m\lambda^{r/(s-r)}\big)^{(r-s)/r}\Big(\sum_{i=n+1}^{n+m} u_i a_i^r\Big)^{s/r} \geq U_{n+m}^{(r-s)/r}\Big(\sum_{i=1}^{n+m} u_i a_i^r\Big)^{s/r};$$

and this last inequality follows from the $n = 2$ case of (H), as does the case of equality.

`Proof of (21) case (ii):` $r \neq 0, s = 0$. Now (21) is equivalent to

$$\frac{V_{n+m}}{\overline{V}_m} - \frac{U_{n+m}}{\overline{U}_m} \geq \frac{V_n}{\overline{V}_m} - \frac{1}{\overline{U}_m}(U_{n+m} - \overline{U}_m\lambda^{-1}) + 1 + \lambda^{-1},$$

which reduces to an identity for all λ. Equivalently we could use (23).

`Proof of (21) case (iii):` $r = 0, s \neq 0$. Letting $r \to 0$ (21) becomes

$$\begin{aligned}\frac{V_{n+m}}{\overline{V}_m}\Big(\mathfrak{M}^{[s]}_{n+m}(\underline{a};\underline{v})\Big)^s - \frac{U_{n+m}}{\overline{U}_m}\Big(\mathfrak{G}_{n+m}(\underline{a};\underline{u})\Big)^s \qquad\qquad (24)\\ \geq \frac{V_n}{\overline{V}_m}\Big(\mathfrak{M}^{[s]}_n(\underline{a};\underline{v})\Big)^s - \frac{U_n}{\overline{U}_m}\Big(\mathfrak{G}_n(\underline{a};\underline{u})\Big)^s\lambda^{\overline{U}_m/U_n}\\ + \Big(\overline{\mathfrak{M}}^{[s]}_m(\underline{a};\underline{v})\Big)^s - \lambda^{-1}\Big(\overline{\mathfrak{G}}_m(\underline{a};\underline{u})\Big)^s.\end{aligned}$$

But (24) is equivalent to

$$\begin{aligned}\frac{1}{\overline{V}_m}\sum_{i=1}^{n+m} v_i a_i^s - \frac{U_{n+m}}{\overline{U}_m}\Big(\mathfrak{G}_{n+m}(\underline{a};\underline{u})\Big)^s \geq \frac{1}{\overline{V}_m}\sum_{i=1}^{n} v_i a_i^s - \frac{U_n}{\overline{U}_m}\Big(\mathfrak{G}_n(\underline{a};\underline{u})\Big)^s\lambda^{\overline{U}_m/U_n}\\ + \frac{1}{\overline{V}_m}\sum_{i=n+1}^{n+m} v_i a_i^s - \lambda^{-1}\Big(\mathfrak{G}_n(\underline{a};\underline{u})\Big)^s\lambda^{\overline{U}_m/U_n}, \qquad (25)\end{aligned}$$

or to

$$\frac{\overline{U}_m}{U_{n+m}}\Big(\overline{\mathfrak{G}}_m(\underline{a};\underline{u})\Big)^s + \frac{U_n}{U_{n+m}}\Big(\mathfrak{G}_n(\underline{a};\underline{u})\Big)^s\lambda^{U_{n+m}/U_n} \geq \lambda\Big(\mathfrak{G}_{n+m}(\underline{a};\underline{u})\Big)^s,$$

which is a special case of (GA). The case of equality follows from that of (GA).

`Proof of (20) case (iv)` $r = 0, s = 0$. This case is covered by the previous case where the assumption $s \neq 0$ is not used.

`Proof of (22)`

In all four cases (22) holds under the same circumstances as (21); we see this as follows in two important cases.

`Case` $r \neq 0, s \neq 0$. After some simplification (22) is seen to be equivalent to (21).

`Case` $r = 0, s \neq 0$. Letting $r \to 0$ (22) becomes:

$$\begin{aligned}\frac{V_{n+m}}{\overline{V}_m}\Big(\mathfrak{M}^{[s]}_{n+m}(\underline{a};\underline{v})\Big)^s - \lambda\frac{U_{n+m}}{\overline{U}_m}\Big(\mathfrak{G}_{n+m}(\underline{a};\underline{u})\Big)^s\left(\frac{\overline{\mathfrak{M}}^{[s]}_m(\underline{a};\underline{v})}{\overline{\mathfrak{G}}_m(\underline{a};\underline{v}}\right)^s\\ \geq \frac{V_n}{\overline{V}_m}\Big(\mathfrak{M}^{[s]}_n(\underline{a};\underline{v})\Big)^s - \lambda^{U_{n+m}/\overline{U}_m}\frac{U_n}{\overline{U}_m}\Big(\mathfrak{G}_n(\underline{a};\underline{u})\Big)^s\left(\frac{\overline{\mathfrak{M}}^{[s]}_m(\underline{a};\underline{v})}{\overline{\mathfrak{G}}_m(\underline{a};\underline{v}}\right)^s.\end{aligned}$$

which after some simplification is seen to be equivalent to (25). □

REMARK (viii) It should be remarked that neither (21) nor (22) depend on $\underline{v}$.

REMARK (ix) If we take $\lambda = 1$, $\underline{u} = \underline{v}$ then (22) shows that if $s/r > 1$ then

$$\sigma(I) = U_I\Big(\big(\mathfrak{M}_I^{[s]}(\underline{a};\underline{u})\big)^s - \big(\mathfrak{M}_I^{[r]}(\underline{a};\underline{u})\big)^s\Big)$$

is super-additive, which generalizes II 3.2.2 Theorem 7.
Included as special cases of these results are the inequalities in II 3.2.1, in particular those of Example (iii); and for further results see below 3.2.5 and IV 3.2.

3.2.5 THE LIMIT THEOREM OF EVERITT The result of II 3.3 is easily extended to power means as was pointed out by [*Everitt 1967,1969*]. His results are included in the following theorem, [*Diananda 1973.*]

THEOREM 20 *If $\underline{a}$ is sequence and r, s are real numbers with $-\infty < r \le 1 \le s < \infty$, $r \ne s$, then $\lim_{n\to\infty} n\big(\mathfrak{M}_n^{[s]}(\underline{a}) - \mathfrak{M}_n^{[r]}(\underline{a})\big)$ is finite if and only if: either (a) $r = 1$ and $\sum_{n=1}^{\infty} a_n$ converges , or (b) for some positive α, $\sum_{n=1}^{\infty}(a_n - \alpha)^2$ converges.*

The above limit exists because of 3.2.3 (14), but as we saw in 3.2.3 Example (i) the limit need not exist if r and s do not straddle 1. However it could still happen that Theorem 20 holds under these circumstances. A partial answer to this question was given by Diananda.

THEOREM 21 *If $-\infty < r < s < \infty$ and the sequence $\underline{a}$ converges to the positive number α then $\lim_{n\to\infty} n\big(\mathfrak{M}_n^{[s]}(\underline{a}) - \mathfrak{M}_n^{[r]}(\underline{a})\big)$ exists and is finite if and only if the series $\sum_{n=1}^{\infty}(a_n - \alpha)^2$ converges.*

In the same paper Diananda discusses the analogous $\lim_{n\to\infty}\big(\mathfrak{M}_n^{[s]}(\underline{a})/\mathfrak{M}_n^{[r]}(\underline{a})\big)^n$. By 3.2.3 (13) this limit exists if $-\infty < r \le 0 \le s < \infty$, $r \ne s$, but not in general if r and s do not straddle 0.

THEOREM 22 *If $\underline{a}$ is sequence and r, s satisfy $-\infty < r \le 0 \le s < \infty$, $r \ne s$, then $\lim_{n\to\infty}\big(\mathfrak{M}_n^{[s]}(\underline{a})/\mathfrak{M}_n^{[r]}(\underline{a})\big)^n$ is finite if and only if for some positive α, $\sum_{n=1}^{\infty}(a_n - \alpha)^2$ converges. If $\lim_{n\to\infty} a_n = \alpha, 0 < \alpha < \infty$, the limit exists, finite, under the same condition if $-\infty < r < s < \infty$.*

3.2.6 NANJUNDIAH INEQUALITIES

As might be expected the results of II 3.4 can be extended to power means. The arguments being similar to the earlier ones will not always be given in detail.

If $r \in \mathbb{R}$, then the *Nanjundiah inverse rth power mean*, or just *rth Nanjundiah mean, of the sequence* $\underline{a}$ *with weight* $\underline{w}$ is:

$$\mathfrak{N}_n^{[r]}(\underline{a};\underline{w}) = \begin{cases} \left(\dfrac{W_n}{w_n}a_n^r - \dfrac{W_{n-1}}{w_n}a_{n-1}^r\right)^{1/r}, n \in \mathbb{N}^*, & \text{if } r \neq 0, \\ \mathfrak{G}_n^{-1}(\underline{a};\underline{w}), & \text{if } r = 0. \end{cases}$$

Of course $\mathfrak{N}_n^{[1]}(\underline{a};\underline{w}) = \mathfrak{A}_n^{-1}(\underline{a};\underline{w})$, and we define the sequences of rth power means $\underline{\mathfrak{M}}^{[r]}(\underline{a};\underline{w})$, and inverse power means $\underline{\mathfrak{N}}^{[r]}(\underline{a};\underline{w})$, in a manner analogous to that used in II 3.4 for the cases $r = 0, 1$.

We easily extend II 3.4 Lemma 10 (a) and (b).

LEMMA 23 *(a) With the above notations*

$$\mathfrak{M}^{[r]}(\underline{\mathfrak{N}}^{[r]};\underline{w}) = \mathfrak{N}^{[r]}(\underline{\mathfrak{M}}^{[r]};\underline{w}) = a_n,\ n \in \mathbb{N}^*.$$

(b) If $n > 1$ and $r, s \in \mathbb{R}$ with $r < s$, and if $W_n a_n^s$, $n \in \mathbb{N}$ is increasing, then

$$\mathfrak{N}_n^{[r]}(\underline{a};\underline{w}) \geq \mathfrak{N}_n^{[s]}(\underline{a};\underline{w}), \tag{26}$$

with equality if and only if $a_{n-1}^s = a_n^s$.

□ (a) Elementary.

(b) First note that $\mathfrak{N}_n^{[r]}(\underline{a};\underline{w}) = \mathfrak{M}^{[r]}(a_{n-1}, a_n; -W_{n-1}/w_n, W_n/w_n)$, and apply 3.2.1 Theorem 10, noting 3.2.1 Remark (i). □

We can now give a very simple extension of (R), a particular case of 3.2.3 Theorem 13 (a).

THEOREM 24 *If $r, s \in \mathbb{R}$ with $r < s$ then*

$$\begin{aligned} W_n\Big(\big(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})\big)^s - \big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\big)^s\Big) \\ \geq W_{n-1}\Big(\big(\mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})\big)^s - \big(\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})\big)^s\Big), \end{aligned} \tag{27}$$

with equality if and only if $a_n = \mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})$.

□ This follows as in proof (v) of II 3.1 Theorem 1, but using (26) and noting that $W_n(\mathfrak{M}_n^{[s]})^s$, $n \in \mathbb{N}$, is increasing;

$$\mathfrak{N}_n^{[s]}(\underline{\mathfrak{M}_n^{[s]}};\underline{w}) = a_n = \mathfrak{N}_n^{[r]}(\underline{\mathfrak{M}_n^{[r]}};\underline{w}) \geq \mathfrak{N}_n^{[s]}(\underline{\mathfrak{M}_n^{[r]}};\underline{w}),$$

which is what we have to prove. The case of equality follows from that for Lemma 23(b). □

We can also give an analogue for (M), and use it to prove a Rado type extension of (M).

THEOREM 25 *If $r > 1$ and $W_n a_n^r$, $W_n b_n^r$, $n \in \mathbb{N}$ are increasing then*

$$\mathfrak{N}_n^{[r]}(\underline{a}+\underline{b};\underline{w}) \geq \mathfrak{N}_n^{[r]}(\underline{a};\underline{w}) + \mathfrak{N}_n^{[r]}(\underline{b};\underline{w}); \tag{28}$$

if $r < 1$ then ($\sim$28) holds.

□ Assume that $r > 1$ and let

$$c_n = \frac{a_n}{a_n+b_n},\ d_n = \frac{b_n}{a_n+b_n},\ U_n = W_n(a_n+b_n)^r, u_n = \tilde{\Delta}U_n,\ n \in \mathbb{N}^*.$$

Simple calculations then give:

$$\begin{aligned}\frac{\mathfrak{N}_n^{[r]}(\underline{a};\underline{w}) + \mathfrak{N}_n^{[r]}(\underline{b};\underline{w})}{\mathfrak{N}_n^{[r]}(\underline{a}+\underline{b};\underline{w})} &= \mathfrak{N}_n^{[r]}(\underline{c};\underline{u}) + \mathfrak{N}_n^{[r]}(\underline{d};\underline{u}) \leq \mathfrak{A}_n^{-1}(\underline{c};\underline{u}) + \mathfrak{A}_n^{-1}(\underline{d};\underline{u}), \text{ by (26)}\\ &= \mathfrak{A}_n^{-1}(\underline{c}+\underline{d};\underline{u}) = 1.\end{aligned}$$

□

COROLLARY 26 *If $r > 1$ then for $n \geq 1$,*

$$\begin{aligned}W_n\Big(\big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) + \mathfrak{M}_n^{[r]}(\underline{b};\underline{w})\big)^r - \big(\mathfrak{M}_n^{[r]}(\underline{a}+\underline{b};\underline{w})\big)^r\Big)\\ \geq W_{n-1}\Big(\big(\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w}) + \mathfrak{M}_{n-1}^{[r]}(\underline{b};\underline{w})\big)^r - \big(\mathfrak{M}_{n-1}^{[r]}(\underline{a}+\underline{b};\underline{w})\big)^r\Big).\end{aligned} \tag{29}$$

□ Replace $\underline{a}, \underline{b}$ by $\underline{\mathfrak{M}}^{[r]}(\underline{a};\underline{w}), \underline{\mathfrak{M}}^{[r]}(\underline{a};\underline{w})$ respectively in (28). □

We now will generalize II 3.4 Lemma 11 and Corollary 13.

LEMMA 27 *If $r, s \in \mathbb{R}$, $r < s$ then*

$$\mathfrak{N}_n^{[r]}(\underline{\mathfrak{N}}^{[s]};\underline{w}) \geq \mathfrak{N}_n^{[s]}(\underline{\mathfrak{N}}^{[r]};\underline{w}). \tag{30}$$

□ We can assume that $r, s \in \mathbb{R}^*$ since the other cases follow from II 3.4 Lemma 11. Further we will assume that $r = 1$, $s > 1$, when (30) is

$$\begin{aligned}\frac{W_n}{w_n}\Big(\frac{W_n}{w_n}a_n^s - \frac{W_{n-1}}{w_n}a_{n-1}^s\Big)^{1/s} - \frac{W_{n-1}}{w_n}\Big(\frac{W_{n-1}}{w_{n-1}}a_{n-1}^s - \frac{W_{n-2}}{w_{n-1}}a_{n-2}^s\Big)\\ \geq \left(\frac{W_n}{w_n}\Big(\frac{W_n}{w_n}a_n - \frac{W_{n-1}}{w_n}a_{n-1}\Big)^s - \frac{W_{n-1}}{w_n}\Big(\frac{W_{n-1}}{w_{n-1}}a_{n-1} - \frac{W_{n-1}}{w_{n-1}}a_{n-2}\Big)^s\right)^{1/s},\end{aligned}$$

or putting, $a_n = a$, $a_{n-1} = b$, $a_{n-2} = c$ and $p = W_n/w_n$, $q = W_{n-1}/w_{n-1}$,

$$\begin{aligned}p\big(pa^s - (p-1)b^s\big)^{1/s} - (p-1)\big(qb^s - (q-1)c^s\big)^{1/s}\\ \geq \Big(p\big(pa - (p-1)b\big)^s - (p-1)\big(qb - (q-1)c\big)^s\Big)^{1/s}.\end{aligned} \tag{31}$$

Now if $(p-1)\gamma = (p-q)b + (q-1)c$ the right-hand side of (31) can be written

$$\Big(\big(p(pa-(p-1)b\big)^s - (p-1)\big(pb-(p-1)\gamma\big)^s \Big)^{1/s},$$

which by (28) is less than or equal to

$$p\big(pa^s-(p-1)b^s\big)^{1/s} - (p-1)\big(pb^s-(p-1)\gamma^s\big)^{1/s}.$$

So we will have proved (31) if we show that

$$pb^s - (p-1)\gamma^s \geq qb^s - (q-1)c^s;$$

equivalently

$$\gamma^s = \Big(\frac{p-q}{p-1}b + \frac{q-1}{p-1}c\Big)^s \leq \frac{p-q}{p-1}b^s + \frac{p-q}{p-1}c^s.$$

This however follows from (J) and the convexity of $f(x) = x^s$. □

THEOREM 28 *If $r, s \in \mathbb{R}$, $r < s$ and $n \geq 1$ then*

$$W_n\big(\mathfrak{M}_n^{[r]}(\underline{\mathfrak{M}}^{[s]};\underline{w}) - \mathfrak{M}_n^{[s]}(\underline{\mathfrak{M}}^{[r]};\underline{w})\big) \geq W_{n-1}\big(\mathfrak{M}_{n-1}^{[r]}(\underline{\mathfrak{M}}^{[s]};\underline{w}) - \mathfrak{M}_{n-1}^{[s]}(\underline{\mathfrak{M}}^{[r]};\underline{w})\big).$$

□ The proof follows that of II 3.4 Theorem 15, but using Lemma 27, rather than II 3.4 Lemma 11. □

COROLLARY 29 *If $r, s \in \mathbb{R}$, $r < s$ and $n \geq 1$ then*

$$\mathfrak{M}_n^{s}(\underline{\mathfrak{M}}^{[r]};\underline{w}) \leq \mathfrak{M}_n^{[r]}(\underline{\mathfrak{M}}^{[s]};\underline{w}), \quad r < s. \tag{32}$$

□ Immediate consequence of Theorem 28. □

REMARK (iii) The above results are stated in [*Nanjundiah 1952*] with some extra conditions on the weights, and was proved in the author's unpublished thesis that was communicated privately to the author who then published Nanjundiah's proof; [*Bullen 1996b*]. A different proof given in [*Rassias pp.27–37*], [*Mond & Pečarić 1996a,c; Tarnavas & Tarnavas*] is based on the lemma of Kedlaya, VI 5 Lemma 5(a); [*Kedlaya 1994, 1999*].
The following deduction from (32) was made by Nanjundiah.

COROLLARY 30 [HARDY'S INEQUALITY] *If $p > 1$ and $\underline{b}$ is an n-tuple then*

$$\mathfrak{M}_n^{[p]}\big(\underline{\mathfrak{A}}(\underline{b})\big) < \frac{p}{p-1}\mathfrak{M}_n^{[p]}(\underline{b}), \quad \text{or} \quad \sum_{i=1}^{n}\mathfrak{A}_i^p(\underline{b}) < \Big(\frac{p}{p-1}\Big)^p \sum_{i=1}^{n} b_i^p.$$

□ In the equal weight case of (32) take $s = 1$, $r = 1/p$ and $\underline{a} = \underline{b}^p$ when we get :

$$\sum_{i=1}^{n} \mathfrak{A}^p(\underline{b}) \le n^{1-p}\Big(\sum_{i=1}^{n} i^{-1/p}\mathcal{S}_i^{1/p}(p,\underline{b})\Big)^p,$$

where the notation $\mathcal{S}_i(p,\underline{b})$ is defined in 2.3. Now if we put $\underline{w}=(1, 2^{-1/p}, \dots n^{-1/p})$, and $\underline{\mathcal{S}}(p,\underline{b}) = \big(\mathcal{S}_i(p,\underline{b}), 1 \le i \le n\big)$, this becomes

$$\begin{aligned}\sum_{i=1}^{n} \mathfrak{A}^p(\underline{b}) &\le n^{1-p} W_n^p \mathfrak{M}_n^{[1/p]}(\underline{\mathcal{S}}(p,\underline{b});\underline{w}), \ , \\ &\le n^{1-p} W_n^p \mathcal{S}_n(p,\underline{b}), \text{ by internality, 1(2),} \\ &= \frac{1}{n^{p-1}}\big(\sum_{i=1}^{n} i^{-1/p}\big)^p \sum_{i=1}^{n} b_i^p \\ &< \big(\frac{p}{p-1}\big)^p \sum_{i=1}^{n} b_i^p.\end{aligned}$$

□

REMARK (iv) This inequality has been the object of considerable research; for further information see [*AI p.131; DI pp.111-112; EM4 p.369; HLP pp.229–239*].

4 Converse Inequalities

We extend the discussion of II 4 by considering the quantities

$$\mathbb{Q}_n^{r,s}(\underline{a};\underline{w}) = \frac{\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})}{\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})}, \quad \mathbb{D}_n^{r,s}(\underline{a};\underline{w}) = \mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) - \mathfrak{M}_n^{[r]}(\underline{a};\underline{w}),$$

defined for all $r, s \in \mathbb{R}$. Then the power mean inequality, (r;s), says that if $r \le s$ then

$$\mathbb{Q}_n^{r,s}(\underline{a};\underline{w}) \ge 1; \quad \mathbb{D}_n^{r,s}(\underline{a};\underline{w}) \ge 0.$$

with equality if and only if either $r = s$ or $\underline{a}$ is constant. We are looking for upper bounds for these quantities assuming that

$$0 < m \le \underline{a} \le M < \infty; \tag{1}$$

upper bounds that depend only on M and m and that improve the trivial ones

$$\frac{M}{m} \ge \mathbb{Q}_n^{r,s}(\underline{a};\underline{w}); \quad M - m \ge \mathbb{D}_n^{r,s}(\underline{a};\underline{w}).$$

Various other forms of converse inequalities are possible; consider for instance 3.1.1 Figure 1. We have the simple inequalities $RH < RG < RK$ and $SI < SG < SL$. These imply that if $1 < r < s < \infty$, $\underline{a}$ is not a constant and (1) holds, then

$$\left(\left(\frac{M^r - m^r}{M^s - m^s}\right)(\mathfrak{M}_n^{[s]}(\underline{a}))^s + \frac{M^s m^r - M^r m^s}{M_s - m^s}\right)^{1/r} < \mathfrak{M}_n^{[r]}(\underline{a})$$
$$< \mathfrak{M}_n^{[s]}(\underline{a}) < \left(\left(\frac{M^s - m^s}{M^r - m^r}\right)\left(\mathfrak{M}_n^{[r]}(\underline{a})\right)^s + \frac{M^r m^s - M^s m^r}{M_r - m^r}\right)^{1/s}.$$

Such inequalities have been used, for instance in actuarial mathematics; see [*Blackwell & Girschick p.31*], [*Giaccardi 1955*].

4.1 Ratios of Power Means

THEOREM 1 *(a) If $r, s \in \mathbb{R}^*$, $r \le s$ then*

$$\mathbb{Q}_n^{r,s}(\underline{a};\underline{w}) \ge \min_{1\le i\le n} \mathbb{Q}_{n-1}^{r,s}(\underline{a}_i';\underline{w}).$$

(b) If $r, s \in \mathbb{R}^*$, $r \le s$ and for some $k, 1 \le k \le n$, $\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) \le a_k \le \mathfrak{M}_n^{[s]}(\underline{a};\underline{w})$, then

$$\mathbb{Q}_n^{r,s}(\underline{a};\underline{w}) \le \max_{1\le i\le n} \mathbb{Q}_{n-1}^{r,s}(\underline{a}_i';\underline{w}).$$

□ (a) it is easily seen that

$$\mathbb{Q}_n^{r,s}(\underline{a};\underline{w}) = \frac{\left((n-1)^{-1}\sum_{i=1}^n (W_n - w_i)\left(\mathfrak{M}_{n-1}^{[r]}(\underline{a}_i';\underline{w})\right)^r\left(\mathbb{Q}_{n-1}^{r,s}(\underline{a}_i';\underline{w})\right)^s\right)^{1/s}}{\left((n-1)^{-1}\sum_{i=1}^n (W_n - w_i)\left(\mathfrak{M}_{n-1}^{[r]}(\underline{a}_i';\underline{w})\right)^r\right)^{1/r}}$$
$$\ge \min_{1\le i\le n} \mathbb{Q}_{n-1}^{r,s}(\underline{a}_i';\underline{w}) \frac{\left((n-1)^{-1}\sum_{i=1}^n (W_n - w_i)\left(\mathfrak{M}_{n-1}^{[r]}(\underline{a}_i';\underline{w})\right)^s\right)^{1/s}}{\left((n-1)^{-1}\sum_{i=1}^n (W_n - w_i)\left(\mathfrak{M}_{n-1}^{[r]}(\underline{a}_i';\underline{w})\right)^r\right)^{1/r}}$$
$$\ge \min_{1\le i\le n} \mathbb{Q}_{n-1}^{r,s}(\underline{a}_i';\underline{w}),$$

by (r;s), and by noting that $(n-1)^{-1}\sum_{i=1}^n (W_n - w_i) = 1$.

(b) Assume, without loss in generality, that $\underline{a}$ is increasing, when the hypothesis implies that

$$W_n\left(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})\right)^s \le \sum_{\substack{i=1\\ i\ne k}}^n w_i a_i^s + w_k\left(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})\right)^s,$$

and so

$$\mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) \le \left(\frac{1}{W_n - w_k}\sum_{\substack{i=1\\ i\ne k}}^n w_i a_i^s\right)^{1/s}.$$

Similarly

$$\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) \ge \left(\frac{1}{W_n - w_k}\sum_{\substack{i=1\\ i\ne k}}^n w_i a_i^r\right)^{1/r}.$$

Hence

$$\mathbb{Q}_n^{r,s}(\underline{a};\underline{w}) \le \mathbb{Q}_{n-1}^{r,s}(\underline{a}'_k;\underline{w}),$$

and this completes the proof. □

REMARK (i) This result is due to Gleser, who gives similar results obtained by removing arbitrary sub-collections of $\underline{a}$; [*Gleser*].

REMARK (ii) Beckenbach, [*Beckenbach 1964*], has generalized Theorem 1 by assuming that $m \le a_i \le M$ only for i such that $0 \le n_0 < i \le n$; this result is proved later, see IV 6 Corollary 3.

THEOREM 2 *Assume that $r, s \in \mathbb{R}^*$ and that $\underline{a}$ is an n-tuple that is not constant. If r is fixed, $r > 0$, there is a unique $s = s_0$, $s_0 = r\theta, 0 < \theta < 1$, at which $\left(\mathbb{Q}_n^{r,s}(\underline{a};\underline{w})\right)^{rs}$ attains its unique minimum value; if $r < 0$ there is a unique such s_0 at which the unique maximum is attained.*

□ Assume, without loss in generality, that the non-constant $\underline{a}$ is increasing, and put $g_r(s) = g(s) = \log\left(\mathbb{Q}_n^{r,s}(\underline{a};\underline{w})\right)^{rs}$. Simple calculations give:

$$g'(s) = \frac{r\sum_{i=1}^n w_i a_i^s \log a_i}{\sum_{i=1}^n w_i a_i^s} - \log\frac{1}{W_n}\sum_{i=1}^n w_i a_i^r,$$

$$g''(s) = \frac{r}{(\sum_{i=1}^n w_i a_i^s)^2}\left(\Big(\sum_{i=1}^n w_i a_i^s \log^2 a_i\Big)\Big(\sum_{i=1}^n w_i a_i^s\Big) - \Big(\sum_{i=1}^n w_i a_i^s \log a_i\Big)^2\right).$$

If then $r > 0$, (C) gives that $g'' > 0$ and so g' is strictly increasing. Further

$$\lim_{s\to-\infty} g'(s) = \log\left(\frac{W_n a_1^r}{\sum_{i=1}^n w_i a_i^r}\right) < 0; \quad \lim_{s\to\infty} g'(s) = \log\left(\frac{W_n a_n^r}{\sum_{i=1}^n w_i a_i^r}\right) > 0;$$

$$g'(0) = \log\frac{\mathfrak{G}_n(\underline{a}^r;\underline{w})}{\mathfrak{A}_n(\underline{a}^r;\underline{w})} < 0, \quad \text{by (GA)};$$

$$g'(r) = \frac{\sum_{i=1}^n w_i a_i^r \log a_i^r}{\sum_{i=1}^n w_i a_i^r} - \log\left(\frac{1}{W_n}\sum_{i=1}^n w_i a_i^r\right) > 0,$$

by (J), and the concavity of the logarithmic function.

These facts are sufficient to establish the result when $r > 0$; the case $r < 0$ is similar. □

REMARK (iii) Some properties of $g_r, r > 0$ are: there is a unique minimum at $s = s_0$, $0 < s_0 < r$; $g_r(r) = 0$; if $r_0 < s < s'$ then $g_r(s) < g_r(s')$.

REMARK (iv) Since $\mathbb{Q}_n^{r,s}(\underline{a};\underline{w})\mathbb{Q}_n^{s,r}(\underline{a};\underline{w}) = 1$ we can easily state a similar theorem with s fixed.

REMARK (v) From the Remark (ii) if $0<r<s$, $\big(\mathbb{Q}_n^{r,s}(\underline{a};\underline{w})\big)^{rs} \ge \big(\mathbb{Q}_n^{r,r}(\underline{a};\underline{w})\big)^{r^2} = 1$, which is equivalent to (r;s).

REMARK (vi) Suppose that $1 \le r < s$ then from the properties in Remark (ii), applied to g_1: $\big(\mathbb{Q}_n^{s,1}(\underline{a};\underline{w})\big)^{s} \ge \big(\mathbb{Q}_n^{r,1}(\underline{a};\underline{w})\big)^{r}$, or

$$\big(\mathfrak{A}_n(\underline{a};\underline{w})\big)^{s-r} \le \frac{\sum_{i=1}^n w_i a_i^s}{\sum_{i=1}^n w_i a_i^r},$$

a result that, in the equal weight case, can be found in [*Berkolaĭko*]. An alternative proof has been given in [*Mitrović 1970*]. This is a particular case of the fundamental inequality between the Gini means, 5.2.1(7).

THEOREM 3 *Assume that $n \ge 2$, $\underline{a}$ an n-tuple with $0 < m \le \underline{a} \le M$, $\underline{w}$ an n-tuple with $W_n = 1$, $r, s \in \mathbb{R}$, $r < s$, and let $\beta = M/m$, then*

$$\mathbb{Q}_n^{r,s}(\underline{a};\underline{w}) \le \Gamma^{r,s}(\beta), \tag{2}$$

where

$$\Gamma^{r,s}(\beta) = \begin{cases} \left(\dfrac{s-r}{\beta^s-\beta^r}\dfrac{\beta^s-1}{s}\right)^{1/r}\left(\dfrac{\beta^s-\beta^r}{s-r}\dfrac{r}{\beta^r-1}\right)^{1/s}, & \textit{if } rs \ne 0,\\ \Gamma^{0,s}(\beta) = \lim\limits_{r\to 0-}\Gamma^{r,s}(\beta) = \left(\dfrac{\beta^s-1}{s\log\beta}\right)^{1/s}\exp\left(\dfrac{\log\beta}{\beta^s-1}-\dfrac{1}{s}\right), & \textit{if } r=0,\\ \Gamma^{r,0}(\beta) = \lim\limits_{s\to 0+}\Gamma^{r,s}(\beta) = \left(\dfrac{r\log\beta}{\beta^r-1}\right)^{1/r}\exp\left(\dfrac{1}{r}-\dfrac{\log\beta}{\beta^r-1}\right), & \textit{if } s=0. \end{cases} \tag{3}$$

Further if

$$\theta(r,s) = \begin{cases} \dfrac{1}{s-r}\left(\dfrac{r}{\beta^r-1}-\dfrac{s}{\beta^s-1}\right), & \textit{if } rs \ne 0,\\ \theta(0,s) = \lim\limits_{r\to 0-}\theta(r,s) = \dfrac{1}{s\log\beta}-\dfrac{1}{\beta^s-1}, & \textit{if } r=0,\\ \theta(r,0) = \lim\limits_{s\to 0+}\theta(r,s) = \dfrac{1}{r\log\beta}-\dfrac{1}{\beta^r-1}, & \textit{if } s=0, \end{cases} \tag{4}$$

then $0 < \theta(r,s) < 1$, *and equality occurs in (2) if and only if for some set of indices* I, $W_I = \theta$, $a_i = M, i \in I$ *and* $a_i = m, i \notin I$.

$\square$ Let us define

$$\Gamma^{r,s}(\beta) = \sup\{x;\ x = \mathbb{Q}_n^{r,s}(\underline{a};\underline{w}), W_n = 1, 0 < m \le \underline{a} \le M\},$$

assuming, as will be shown, that this quantity does not depend on $\underline{w}$.
Writing $\mathcal{A} = \{\underline{a};\ 1 \le \underline{a} \le \beta\}, \mathcal{W} = \{\underline{w};\ W_n = 1\}$, also define

$$\Gamma^{r,s}(\beta;\underline{w}) = \sup_{\underline{a}\in\mathcal{A}} \mathbb{Q}_n^{r,s}(\underline{a};\underline{w}).$$

Then using the basic properties of $\mathbb{Q}_n^{r,s}$,

$$\Gamma^{r,s}(\beta) = \sup_{\underline{w}\in\mathcal{W}} \Gamma^{r,s}(\beta;\underline{w}) = \sup_{\underline{a}\in\mathcal{A},\underline{w}\in\mathcal{W}} \mathbb{Q}_n^{r,s}(\underline{a};\underline{w}). \tag{5}$$

Clearly $\Gamma^{r,s}(\beta) \le \beta$, and simple calculations establish the following identities:

$$\Gamma^{r,s}(\beta) = \Gamma^{-s,-r}(\beta), \tag{6}$$

$$\big(\Gamma^{r,s}(\beta)\big)^r = \Gamma^{1,s/r}(\beta^r),\ 0<r<s<\infty, \quad \big(\Gamma^{0,s}(\beta)\big)^s = \Gamma^{0,1}(\beta^s),\ 0<s<\infty, \tag{7}$$

$$\big(\Gamma^{r,s}(\beta)\big)^{-r} = \Gamma^{-1,-s/r}(\beta^{-r}),\ -\infty < r < 0 < s < \infty. \tag{8}$$

Identity (6) shows that it is sufficient to evaluate $\Gamma^{r,s}$ in the following three cases:
(α) $0 < r < \infty$; (β) $r = 0$, $0 < s < \infty$; (γ) $-\infty < r < 0 < s < \infty$.
Further using (7) and (8) these three cases can be reduced respectively to a consideration of: (i) $\Gamma^{1,t}$, $t > 1$; (ii) $\Gamma^{0,1}$; (iii) $\Gamma^{-1,t}$, $0 < t$.
(i) To evaluate $\Gamma^{1,t}(\beta)$ let us first consider $\Gamma^{1,t}(\beta,\underline{w})$.
Since the set $\mathcal{A}$ is compact there is a $\underline{b} \in \mathcal{A}$ such that

$$\big(\Gamma^{1,t}(\beta,\underline{w})\big)^t = \big(\mathbb{Q}_n^{1,t}(\underline{b};\underline{w})\big)^t = \frac{\sum_{i=1}^n w_i b_i^t}{\big(\sum_{i=1}^n w_i b_i\big)^t}. \tag{9}$$

For some k, $1 \le k \le n$, put for $s = 1$ or t, $w_k = w$, $\sum_{\substack{i=1\\i\ne k}}^n w_i b_i^s = (1-w)\alpha_s$. Then if $1 \le x \le \beta$, the function ϕ,

$$\phi(x) = \big(\mathbb{Q}_n^{t,1}(b_1,\ldots,b_{k-1},x,b_{k+1},\ldots,b_n;\underline{w})\big)^t = \frac{wx^t + (1-w)\alpha_t}{\big(wx + (1-w)\alpha_1\big)^t},$$

has a maximum either at $x = 1$ or at $x = \beta$; this is immediate from a consideration of ϕ'. Since k, $1 \le k \le n$, was arbitrary the point $\underline{b}$ of $\mathcal{A}$ given by (9) is such that

$b_i = 1$ or β, $1 \le i \le n$. Write I for the set of indices for which the second alternative occurs, of course I can be empty, and let $y = W_I$, when $0 \le y \le 1$ and

$$\left(\Gamma^{1,t}(\beta; \underline{w})\right)^t = \frac{1 - y + y\beta^t}{\left(1 - y + y\beta\right)^t} = \psi(y), \text{ say.}$$

By (5) $\left(\Gamma^{1,t}(\beta)\right)^t = \sup_{0 \le y \le 1} \psi(y)$, and simple calculations with ψ' show that ψ has a maximum at y_0, $0 < y_0 = \theta(1,t) < 1$, where θ is defined by (4).

Hence I is not empty and $\Gamma^{1,t}(\beta) = \left(\psi\big(\theta(1,y)\big)\right)^{1/t}$, which is the value given by (3).

The cases of equality are immediate from this argument.

This proves the theorem when either $-\infty < r < s < 0$ or $0 < r < s < \infty$.

(ii) As in (i) there is a $\underline{b} \in \mathcal{A}$ such that

$$\Gamma^{0,1}(\beta, \underline{w}) = \mathfrak{Q}_n^{0,1}(\underline{b}; \underline{w}) = \frac{\sum_{i=1}^n w_i b_i}{\left(\exp \sum_{i=1}^n w_i \log b_i\right)}.$$

For some k, $1 \le k \le n$, put

$$w_k = w, \ \sum_{\substack{i=1\\ i\ne k}}^n w_i b_i = (1-w)\alpha_1, \quad \sum_{\substack{i=1\\ i\ne k}}^n w_i \log b_i = (1-w)\alpha_0.$$

Then if $1 \le x \le \beta$ the function ϕ,

$$\phi(x) = \frac{wx + (1-w)\alpha_1}{w \log x + (1-w)\log \alpha_0},$$

has a maximum either at $x = 1$ or at $x = \beta$. The argument then proceeds as in (i). and gives the theorem when either $-\infty < r < s = 0$ or $0 = r < s < \infty$.

(iii) As in (i), or (ii), there is a $\underline{b} \in \mathcal{A}$ such that

$$\left(\Gamma^{-1,t}(\beta, \underline{w})\right)^t = \Big(\sum_{i=1}^n w_i b_i^t\Big)\Big(\sum_{i=1}^n w_i b_i^{-1}\Big)^t.$$

For some k, $1 \le k \le n$, put

$$w_k = w, \quad \sum_{\substack{i=1\\ i\ne k}}^n w_i b_i^t = (1-w)\alpha_t \quad \text{and} \quad \sum_{\substack{i=1\\ i\ne k}}^n w_i b_i^{-1} = (1-w)\alpha_{-1}.$$

Then if $1 \le x \le \beta$ the function ϕ,

$$\phi(x) = \left(wx^t + (1-w)\alpha_t\right)\left(wx^{-1} + (1-w)\alpha_{-1}\right)^t$$

has a maximum either at $x = 1$ or at $x = \beta$. The argument then proceeds as in (i) and gives the remaining case of the theorem, $-\infty < r < 0 < s < \infty$. $\square$

REMARK (vii) The above result is due to Specht, and was later rediscovered by Cargo & Shisha who gave the cases of equality; [*Cargo & Shisha 1962; Specht*]. An alternative proof of Specht's result is in the paper by Beckenbach, [*Beckenbach 1964*]. See also [*Laohakosol & Ubolsri*].

Another proof of Theorem 3 has been given by Goldmann, based on the following inequality, that in the case $r = s = 1$ is due to Rennie, see [*Goldman; Rennie 1963*].

LEMMA 4 *If* $-\infty < r < s < \infty$, $rs < 0$ *and* $0 < m \le \underline{a} \le M$ *then*

$$(M^s - m^s)\big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\big)^r - (M^r - m^r)\big(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})\big)^s \le M^s m^r - M^r m^s. \tag{10}$$

If $rs > 0$ *then* ($\sim$10) *holds.*

□ This lemma can be deduced from I 4.4 Theorem 23; see [*Beesack & Pečarić; Pečarić & Beesack 1987b*].

□

To deduce (2) from this lemma rewrite (10) as

$$\frac{-r}{s-r}\frac{M^r - m^r}{rm^r}\left(\frac{\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})}{m}\right)^s + \frac{s}{s-r}\frac{M^s - m^s}{sm^s}\left(\frac{\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})}{m}\right)^r \le \frac{M^s m^r - M^r m^s}{(s-r)m^r m^s}$$

and use (GA) to get

$$\left(\frac{M^r - m^r}{rm^r}\left(\frac{\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})}{m}\right)^s\right)^{-r/(s-r)}\left(\frac{M^s - m^s}{sm^s}\left(\frac{\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})}{m}\right)^r\right)^{s/(s-r)} \le \frac{M^s m^r - M^r m^s}{(s-r)m^r m^s}.$$

This, on rewriting, is (2).

REMARK (viii) Theorem 3 in the special case $r = 1$, $s \ge 1$ was considered earlier; see [*Knopp 1935*].

Other references are: [*Mitrinović & Vasić p.78*], [*Cargo 1969; Diaz, Goldman & Metcalf; Diaz & Metcalf 1963; Lupaş 1972; Marshall & Olkin 1964; Mond & Shisha 1965; Newman M A; Rosenbloom; Shisha & Mond; Wang & Yang; Zhang Z H*].

The following more classical inequalities are special cases of Theorem 3; [*Kantorovič; Schweitzer P*].

THEOREM 5 *(a)* [KANTOROVIČ'S INEQUALITY] *If* $\underline{a}, \underline{w}$ *are* n*-tuples,* $n \ge 2$*, with* $W_n = 1$ *and* $0 < m \le \underline{a} \le M$*, then*

$$\left(\sum_{i=1}^n w_i a_i\right)\left(\sum_{i=s1}^n \frac{w_i}{a_i}\right) \le \frac{(M+m)^2}{4Mm}, \tag{11}$$

with equality if for some index set I, $W_I = 1/2$, $a_i = M$, $i \in I$, and $a_i = m, i \notin I$.
(b) [SCHWEITZER'S INEQUALITY] *If $\underline{a}$ is an n-tuple, $n \geq 2$, $0 < m \leq \underline{a} \leq M$, then*

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right) \leq \frac{n^2(M+m)^2}{4Mm}, \tag{12}$$

with equality if and only if there is an index set I containing $n/2$ elements, and $a_i = M$, $i \in I$, and $a_i = m$, $i \notin I$; in particular (12) is strict if n is odd.

□ (a) We give four proofs of this result.

(*i*) Put $r = -1$, $s = 1$ in Theorem 3.

(*ii*) [*Henrici*] Determine two n-tuples $\underline{\alpha}, \underline{\beta}$ as follows:

$$a_i = \alpha_i M + \beta_i m, \quad a_i^{-1} = \alpha_i M^{-1} + \beta_i m^{-1}, \qquad 1 \leq i \leq n.$$

It is easily seen that both of these n tuples are non-negative, and further

$$1 = a_i a_i^{-1} = (\alpha_i + \beta_i)^2 + \alpha_i\beta_i \frac{(M-m)^2}{Mm}; \tag{13}$$

so in particular $\underline{\alpha} + \underline{\beta} \leq \underline{e}$.
Now let $A = \sum_{i=1}^{n} w_i\alpha_i$ and $B = \sum_{i=1}^{n} w_i\beta_i$ when

$$A + B = \sum_{i=1}^{n} w_i(\alpha_i + \beta_i) \leq W_n = 1. \tag{14}$$

Hence the left-hand side of (11) is equal to

$$\begin{aligned}(AM + Bm)(AM^{-1} + Bm^{-1}) =&(A+B)^2 + AB\frac{(M-m)^2}{Mm}\\ \leq&(A+B)^2\Big(1 + \frac{(M-m)^2}{4Mm}\Big), \qquad \text{by (GA)},\\ =&(A+B)^2\frac{(M+m)^2}{4Mm} \leq \frac{(M+m)^2}{4Mm}.\end{aligned}$$

This completes the proof of (11). There is a equality if and only if $A + B = 1$ and, using the case of equality in (GA), $A = B$. From (14) the first condition implies that $\underline{\alpha} + \underline{\beta} = \underline{e}$, which from (13) implies that for $1 \leq i \leq n$ either $\alpha_i = 0$ or $\beta_i = 0$; equivalently $a_i = M$ or m. So if I is the set of indices for which $a_i = M$ the condition $A = B$ implies that $W_I = 1/2$.

(*iii*) [*Pták*]

$$\begin{aligned}\sum_{i=1}^{n} w_i a_i \sum_{i=1}^{n} \frac{w_i}{a_i} =&\sum_{i=1}^{n} w_i\Big(\frac{a_i}{\sqrt{mM}}\Big)\sum_{i=1}^{n} w_i\Big(\frac{\sqrt{mM}}{a_i}\Big)\\ \leq&\frac{1}{4}\Big(\sum_{i=1}^{n} w_i\Big(\frac{a_i}{\sqrt{mM}}\Big) + \Big(\frac{\sqrt{mM}}{a_i}\Big)\Big)^2, \qquad \text{by (GA)},\\ \leq&\frac{1}{4}\Big(\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}}\Big)^2, \qquad \text{by I 2.2 (23)}.\end{aligned}$$

(*iv*) It is possible to deduce (11) from (12). We can without loss in generality assume that that $w_i = v_i/V$, $V, v_i \in \mathbb{N}^*$, $1 \le i \le n$, $V_n = V$. Then simple calculations show that (11) reduces to a case of (12).

(b) This is just a special case of (a). □

REMARK (ix) Proof (*ii*) is based on a method used in [*Pólya & Szegö 1972 p.71*] to prove inequality (20) below.

REMARK (x) The observation that (12) implies the more general (11) is due to Henrici; see [*Henrici*].

REMARK (xi) Theorem 3 has been used to obtain a matrix inequality that generalizes Kantorovič's inequality; see [*Mond 1965a*] and VI 5.

REMARK (xii) Kantorovič's inequality has been obtained in the form

$$\frac{\mathfrak{A}_n(\underline{a}) - \mathfrak{H}_n(\underline{a})}{\mathfrak{H}_n(\underline{a})} \le \frac{(M-m)^2}{4Mm},$$

by Weiler, who then points out that because of (GA) the same inequality holds with the harmonic mean replaced by the geometric mean; [*Weiler*].

A completely different type of converse inequality has been proved by Rahmail; see [*Pečarić 1983b; Rahmail 1978; Vasić & Milovanović*].

THEOREM 6 *If $0 < r < s$, $\underline{w}$ an n-tuple, $\underline{a}$ a monotonic, concave n-tuple, and $\underline{n} = (0, 1, 2, \ldots, n-1)$, $\underline{\tilde{n}} = (n-1, n-2, \ldots, 1)$, then*

$$\mathfrak{M}_n^{[s]}(\underline{a}; \underline{w}) \le \alpha \mathfrak{M}_n^{[r]}(\underline{a}; \underline{w}); \tag{14}$$

where if $\underline{a}$ is increasing

$$\alpha = \mathfrak{M}_n^{[s]}(\underline{n}; \underline{w}) / \mathfrak{M}_n^{[r]}(\underline{n}; \underline{w}); \tag{15}$$

while if $\underline{a}$ is decreasing,

$$\alpha = \mathfrak{M}_n^{[s]}(\underline{\tilde{n}}; \underline{w}) / \mathfrak{M}_n^{[r]}(\underline{\tilde{n}}; \underline{w}).$$

If $\underline{a}$ is convex with $a_1 = 0$ and $0 < s < r$ then (~14) holds with α given by (15).

REMARK (xiii) The proof uses the monotonicity of the n-tuples $\dfrac{a_i}{i-1}, 1 \le i \le n$, and $\dfrac{a_i}{n-i}, 1 \le i \le n-1$.

REMARK (xiv) The second part of the theorem has been generalized to k-convex n-tuples; see [*Milovanović & Milovanović 1979*].

The following result is related to those in 2.5.4 and to II 3.8 Theorem 29.

THEOREM 7 *If the n-tuples $\underline{a}, \underline{b}, \underline{b}/\underline{a}$ are either all increasing, or all decreasing, and if $r < s$ then* $f(x) = \mathbb{Q}_n^{r,s}\big((1-x)\underline{b} + x\underline{a}; w\big)$, $0 \le x \le 1$, *is decreasing. In particular* $\mathbb{Q}_n^{r,s}(\underline{b}; w) \ge \mathbb{Q}_n^{r,s}(\underline{a}; w)$.

REMARK (xv) In the special case $r = 0, s = 1$ and $\underline{a}$ constant this is due to Pečarić; the general result was proved by Alzer; [*Alzer 1990ℓ; Pečarić 1984d*].

4.2 Differences of Power Means

An upper bound for $\mathbb{D}_n^{r,s}(\underline{a}; \underline{w})$ was given by Mond & Shisha, and is Theorem 10 below; [*Mond & Shisha 1967a,b; Rosenbloom; Shisha & Mond*]
We first prove two lemmas.

LEMMA 8 *If* $0 < m < x < M$ *and* $r < s$ *define* $f(x) = r(x^r - \alpha x^s - \beta)$ *where* $\alpha = \dfrac{M^r - m^r}{M^s - m^s}$, $\beta = \dfrac{M^s m^r - M^r m^s}{M^s - m^s}$, *then* $f(x) > 0$, $m < x < M$.

□ Simple computations show that if we consider $f(x)$, $x > 0$, then f' has a unique zero. Since $f(m) = f(M) = 0$ all we need to show is that $f'(m) > 0$.
However putting $y = M/m$, $f'(m) = rm^{r-1}(r - \alpha s m^{s-r}) = rg(y)/(y^s - 1)$ and so $\operatorname{sgn} f'(m) = \operatorname{sgn}(rs)\operatorname{sgn} g(y)$.
Noting that $g(1) = 0$ and $g'(y) = rsy^{r-1}(y^{s-r} - 1)$ and that $g'(y) > 0$ if $rs > 0$, $g'(y) < 0$ if $rs < 0$, completes the proof. □

LEMMA 9 *With the above notations define the function h by*

$$h(x) = \begin{cases} x^{1/s} - (\alpha x + \beta)^{1/r}, & \text{if } rs \ne 0,\\ x^{1/s} - m\big(M/m\big)^{(x-m^s)/(M^s-m^s)}, & \text{if } r = 0,\\ m\big(M/m\big)^{(x-m^r)/(M^r-m^r)} - x^{1/r}, & \text{if } s = 0, \end{cases}$$

on the interval J,

$$J = \begin{cases} \big[\min\{M^s, m^s\}, \max\{M^s, m^s\}\big], & \text{if } s \ne 0,\\ \big[M^r, m^r\big], & \text{if } s = 0. \end{cases}$$

Then there is a unique $y \in \overset{\circ}{J}$ *where h takes its maximum value. Further*

$$h(y) = \begin{cases} \big((1-\theta)m^s + \theta M^s\big)^{1/s} - \big((1-\theta)m^r + \theta M^r\big)^{1/r}, & \text{if } rs \ne 0,\\ \big((1-\theta)m^s + \theta M^s\big)^{1/s} - m^{1-\theta}M^{\theta}, & \text{if } r = 0,\\ m^{1-\theta}M^{\theta} - \big((1-\theta)m^r + \theta M^r\big)^{1/r}, & \text{if } s = 0, \end{cases}$$

where

$$\theta = \begin{cases} \dfrac{y - m^s}{M^s - m^s}, & \text{if } s \ne 0,\\ \dfrac{y - m^r}{M^r - m^s}, & \text{if } s = 0. \end{cases}$$

□ Let us consider the case $rs \neq 0$.
Since $h(m^s) = h(M^s) = 0$ and, by Lemma 8, $h(x) > 0, x \in \overset{\circ}{J}$, it is clear that for at least one $y \in \overset{\circ}{J}$ and $h'(y) = 0$.
Suppose that y is not unique; that is there are two points $y_1, y_2, y_1 < y_2$, such that h takes its maximum value at both.
Now if $h'(x) = 0$ then

$$h''(x) = \frac{1}{s(\alpha x + \beta)} x^{1/s-2}\big(\alpha x(\frac{1}{s} - \frac{1}{r}) + \beta(\frac{1}{s} - 1)\big).$$

Since $h''(y_i) \leq 0, i = 1, 2$, it follows from the above that for $i = 1, 2$ we must have that $\frac{1}{s}\big(\alpha y_i(\frac{1}{s} - \frac{1}{r}) + \beta(\frac{1}{s} - 1)\big) \leq 0$.
Now for some x, $y_1 < x < y_2$, $h'(x) = 0$ and this is a local minimum of h. However from the expression for h'' we see that $h''(x) < 0$. This is a contradiction, so y is unique.
The cases $r = 0$, or $s = 0$, can be discussed in a similar manner, and the rest of the lemma is a result of simple calculations. □

THEOREM 10 *Assume that $n \geq 2$, $\underline{a}$ an n-tuple with $0 < m \leq \underline{a} \leq M$, $\underline{w}$ an n-tuple, $r, s \in \mathbb{R}$, $r < s$, and $\alpha, \beta, h, y, \theta$ are as in Lemmas 8, 9 then*

$$\mathbb{D}_n^{r,s}(\underline{a}; \underline{w}) \leq h(y) \tag{16}$$

with equality if and only if for some index set I, $W_I = \theta, a_i = M, i \in I, a_i = m, i \notin I$.

□ (i) $rs \neq 0$
Applying Lemma 8,

$$r\sum_{i=1}^{n} w_i a_i^r > r\big(\alpha \sum_{i=1}^{n} w_i a_i^s + W_n\beta\big);$$

and so

$$\mathbb{D}_n^{r,s}(\underline{a}; \underline{w}) \leq \mathfrak{M}_n^{[s]}(\underline{a}; \underline{w}) - \Big(\alpha\big(\mathfrak{M}_n^{[s]}(\underline{a}; \underline{w})\big)^s + \beta\Big)^{1/r} = h\Big(\big(\mathfrak{M}_n^{[s]}(\underline{a}; \underline{w})\big)^s\Big), \tag{17}$$

with equality if and only if for each i, $1 \leq i \leq n$, $a_i = m$ or M. This by Lemma 9 gives (16), with equality if and only if for each i, $1 \leq i \leq n$, $a_i = m$ or M, and $y = \sum_{i=1}^{n} w_i a_i^s$.
After simple computations this completes the proof in this case.

(ii) If instead of using Lemma 8 we note that if $m \le x \le M$, $q > 0$ then, by the strict concavity of the logarithmic function,

$$x^q \ge M^{q(x^s-m^s)/(M^s-m^s)} m^{q(M^s-x^s)/(M^s-m^s)},$$

with equality if and only if $x = M$ or m, then we can obtain (16) in the case $r = 0$, or $s = 0$.

This completes the proof. □

REMARK (i) The case $r = 1$, $s = 0$ was considered earlier; see [*Knopp 1935*].

REMARK (ii) In [*Cargo & Shisha 1962*] it is shown that the maximum of $\mathbb{D}_n^{r;s}$, and also of $\mathbb{Q}_n^{r;s}$, occurs at a vertex of the n-cube; see also [*Pečarić & Mesihović 1993*].

REMARK (iii) The methods used to obtain converses of (J), I 4.4 Theorem 28, and (GA), II 4.1 Theorem 1 proof (*ii*) can also be used here; see [*Bullen 1994a*].

4.3 CONVERSES OF THE CAUCHY, HÖLDER AND MINKOWSKI INEQUALITIES

Converse inequalities for (C), (H) and (M) can be obtained as corollaries of the results obtained in sections 4.1 and 4.2.

THEOREM 11 *Let $\underline{b}$ and $\underline{c}$ be n-tuples, $p \in \mathbb{R}$, $p > 1$, $0 < m < M$, $\beta = M/m$, and suppose that, $m \le \underline{b}^{1/p'}/\underline{c}^{1/p} \le M$, then*

$$\frac{\left(\sum_{i=1}^n b_i^p\right)^{1/p}\left(\sum_{i=1}^n c_i^{p'}\right)^{1/p'}}{\left(\sum_{i=1}^n b_i c_i\right)} \le \left(\frac{\beta^p - \beta^{p'}}{p + p'}\right)\left(\frac{p'}{1 - \beta^{-p'}}\right)^{1/p}\left(\frac{p}{\beta^p - 1}\right)^{1/p'}. \tag{18}$$

Further if $\theta = \dfrac{1}{p + p'}\left(\dfrac{p'}{1 - \beta^{-p'}} - \dfrac{p}{\beta^p - 1}\right)$ then equality occurs if and only if there is a set of indices I such that, $(1 - \theta)\sum_{i\in I} b_i c_i = \theta \sum_{i\notin I} b_i c_i$ and $b_i^{1/p'} c_i^{-1/p} = M$, $i \in I$, $= m$, $i \notin I$.

If $0 < p < 1$ then ($\sim$18) holds.

□ If $p > 1$ this is an immediate consequence of 4.1 Theorem 3 by putting $s = p$, $r = -p'$, $\underline{a} = \underline{b}^{1/p'}\underline{c}^{-1/p}$ and $w_i = b_i c_i / \sum_{k=1}^n b_k c_k$, $1 \le i \le n$.

The other cases can be found in [*Beesack & Pečarić; Vasić & Pečarić 1983*]. □

REMARK (i) If $b \le \underline{b} \le B$, $c \le \underline{c} \le C$ then the right-hand side of (18) can be replaced by the simpler $(B/b)^{1/p'}(C/c)^{1/p}$; see [*Crstici, Dragomir & Neagu*]. Other results have been given in [*Barnes 1970; Laohakosol & Leerawat*].

A particular case of (18) is the following result; [*Watson; Greub & Rheinboldt; Masuyama 1982,1985*].

THEOREM 12 [CASSEL'S INEQUALITY] *If $\underline{b}, \underline{c}, \underline{w}$ are n-tuples with $0 < m_1 \le \underline{b} \le M_1$, $0 < m_2 \le \underline{c} \le M_2$, where $m_1 m_2 < M_1 M_2$ then*

$$\Big(\sum_{i=1}^n w_i b_i^2\Big)\Big(\sum_{i=1}^n w_i c_i^2\Big) \le \frac{(m_1 m_2 + M_1 M_2)^2}{4 m_1 m_2 M_1 M_2}\Big(\sum_{i=1}^n w_i b_i c_i\Big)^2. \tag{19}$$

REMARK (ii) The case of constant $\underline{w}$ can be written

$$\Big(\sum_{i=1}^n b_i^2\Big)^{1/2}\Big(\sum_{i=1}^n c_i^2\Big)^{1/2} \le \frac{1}{2}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)\Big(\sum_{i=1}^n b_i c_i\Big), \tag{20}$$

and is called the *Pólya-Szegö inequality*; see [*Pólya & Szegö 1972 p.71*]. Clearly (19) and (20) are equivalent.

REMARK (iii) Inequality (20) is a special case of Kantorovič's inequality, 4.1 (11); conversely, as Kantorovič pointed out, (11) follows from (20). For a generalization of (20) see [*Chen Y L*]

REMARK (iv) All of the inequalities (11), (19) and (20) can be deduced from an integral form of Schweitzer's inequality, VI 1.3.1(11); [*AI p.60, Remark 2*], [*Makai*].

An alternative proof of these results has been given by Diaz & Metcalf that is based on the case $p = 2$ of the following theorem; [*PPT p.115*], [*Diaz & Metcalf 1963 1964a,1965; Mond & Shisha 1967b; Shisha & Mond*].

THEOREM 13 *If $\underline{w}, \underline{b}$ and $\underline{c}$ are n-tuples, $p > 1$, and if $0 < m < \underline{c}^{1/p'}\underline{b}^{-1/p} < M$ then*

$$(M^{p'} - m^{p'})\sum_{i=1}^n w_i c_i^p + (m^{p'}M^{pp'} - M^{p'}m^{pp'})\sum_{i=1}^n w_i b_i^{p'} \le (M^{pp'} - m^{pp'})\sum_{i=1}^n w_i b_i c_i, \tag{21}$$

with equality if and only if for all i, $1 \le i \le n$, $c_i^{1/p'} b_i^{-1/p} = m$ or M. The same inequality holds if $p < 0$ but if $0 < p < 1$ then (~21) holds.

□ This is a consequence of I 4.4 Theorem 23; see [*PPT pp.114–115*]. □

Let us now see how inequality (20) follows by taking equal weights and $p = 2$ in (21). Put $m^2 = m_2/M_1$, $M^2 = M_2/m_1$ in this special case of (21) to get

$$\sum_{i=1}^n c_i^2 + \frac{m_2 M_2}{m_1 M_1}\sum_{i=1}^n b_i^2 \le \Big(\frac{M_2}{m_1} + \frac{m_2}{M_1}\Big)\sum_{i=1}^n b_i c_i,$$

or

$$\left(\frac{M_2}{m_1}+\frac{m_2}{M_1}\right)\sum_{i=1}^n b_ic_i-2\sqrt{\frac{m_2M_2}{m_1M_1}\sum_{i=1}^n b_i^2\sum_{i=1}^n c_i^2}$$
$$\geq\left(\sqrt{\sum_{i=1}^n c_i^2}-\sqrt{\frac{m_2M_2}{m_1M_1}\sum_{i=1}^n b_i^2}\right)^2\geq 0.$$

This on rewriting gives (20).

A converse of (M) can be deduced from the converse of (H) given in Theorem 11; [*Mond & Shisha 1967b*].

THEOREM 14 *Let $\underline{b},\underline{c},m,M,\beta,\theta,p$ be defined as in Theorem 11 but now with $m\leq \underline{b}^{1/p'}(\underline{b}+\underline{c})^{1/p'}$, $\underline{c}^{1/p'}(\underline{b}+\underline{c})^{1/p'}\leq M$; then*

$$\frac{\left(\sum_{i=1}^n b_i^p\right)^{1/p}+\left(\sum_{i=1}^n c_i^p\right)^{1/p}}{\left(\sum_{i=1}^n (b_i+c_i)^p\right)^{1/p}}\leq E; \tag{23}$$

where E is the right-hand side of (18).
There is equality if and only if: (i) there is a set of indices, I, such that

$$(1-\theta)\sum_{i\in I} b_i(b_i+c_i)^{p-1}=\theta\sum_{i\notin I} b_i(b_i+c_i)^{p-1},$$

with $\left(b_i/(b_i+c_i)\right)^{1/p'}=M$, or m according as $i\in I$, or $i\notin I$;
and, (ii) there is a set of indices, J, such that

$$(1-\theta)\sum_{i\in J} c_i(b_i+c_i)^{p-1}=\theta\sum_{i\notin J} c_i(b_i+c_i)^{p-1},$$

with $\left(c_i/(b_i+c_i)\right)^{1/p'}=M$, or m according as $i\in J$, or $i\notin J$.
If either $0<p<1$, or $p<0$,then ($\sim$24) holds.

□ If $p>1$ then by Theorem 11

$$\begin{aligned}\sum_{i=1}^n (b_i+c_i)^p=&\sum_{i=1}^n b_i(b_i+c_i)^{p-1}+\sum_{i=1}^n c_i(b_i+c_i)^{p-1}\\ \geq& E^{-1}\Big(\sum_{i=1}^n b_i^p\Big)^{1/p}\Big(\sum_{i=1}^n (b_i+c_i)^p\Big)^{1/p'}\\ &+E^{-1}\Big(\sum_{i=1}^n c_i^p\Big)^{1/p}\Big(\sum_{i=1}^n (b_i+c_i)^p\Big)^{1/p'},\end{aligned}$$

which gives the result in this case.
The other cases have similar proofs. □

THEOREM 15 *Let $\underline{b}$ and $\underline{c}$ be n-tuples, $p > 1$, $0 < m < \underline{b}^{1/p'}\underline{c}^{1/p} < M$; then with the notation of 4.2 Lemma 9,*

$$\left(\frac{\sum_{i=1}^{n} b_i^p}{\sum_{i=1}^{n} b_i c_i}\right)^{1/p} - \left(\frac{\sum_{i=1}^{n} b_i c_1}{\sum_{i=1}^{i} c_i^{p'}}\right)^{1/p'} \leq h(y),$$

with equality if and only if there is an index set, I, such that $(1-\theta)\sum_{i\in I} b_i c_i = \theta\sum_{i\notin I} b_i c_i$, and $b_i^{1/p'}c_i^{-1/p} = M$, or m, according as $i \in I$, or $i \notin I$,

□ This follows from 4.2 Theorem 10 in a manner similar to the way in which Theorem 11 follows from 4.1 Theorem 3. □

REMARK (v) In the cases $p = 1, 2$ converse inequalities of a different type have been obtained by Benedetti who considers the maximum possible value when the n-tuples have terms that are restricted to certain finite sets of values. However it is beyond the scope of this work to consider the converses of (C), (H) and (M) in more detail. The reader is referred to the standard works [*AI; BB; PPT*] for further references in this area; see also [*Dragomir & Goh 1997b; Izumino; Izumino & Tominaga; Tóth*].

REMARK (vi) Converse inequalities have been based on the order relation of I 3.3; see [*Pečarić 1984a*].

REMARK (vii) A converse of the equal weight case of 3.1.3 (9) has been given by Tôyama; see [*AI p.285*], [*Tôyama*].

$$\mathfrak{M}_m^{[r]}\big(\mathfrak{M}_n^{[s]}(\underline{a}_{(i)})\big) \leq \big(\min\{m,n\}\big)^{1/rs}\mathfrak{M}_n^{[s]}\big(\mathfrak{M}_m^{[r]}(\underline{a}^{(j)})\big).$$

This was given a simpler proof and extended to the weighted case in [*Alzer & Ruscheweyh 2000*].

REMARK (viii) Leindler has proved that if $1 \leq p, q, r < \infty$, and if $1/p + 1/q = 1 + 1/r$ then

$$\Big(\sum_{n=-\infty}^{\infty} a_n^p\Big)^{1/p}\Big(\sum_{n=-\infty}^{\infty} b_n^q\Big)^{1/q} \leq \sum_{n=-\infty}^{\infty}\Big(\sum_{m=-\infty}^{\infty} (a_m^r b_{n-m}^r\Big)^{1/r};$$

see [*Leindler, 1972a,b,c, 1973a,b, 1976; Uhrin 1975*].

The next result is a discrete form of a result of Zagier; [*Alzer 1992e; Pečarić 1995b; Zagier*].

THEOREM 16 *Let $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ be sequences, $\underline{a}$ and $\underline{b}$ decreasing to zero, $\underline{c}, \underline{d} \leq 1$ and let $A = \sum_{i=1}^{\infty} a_i, B = \sum_{i=1}^{\infty} b_i, C = \sum_{i=1}^{\infty} c_i, D = \sum_{i=1}^{\infty} d_i$ then*

$$\sum_{i=1}^{\infty} a_i b_i \geq \frac{\sum_{i=1}^{\infty} a_i c_i \sum_{i=1}^{\infty} d_i b_i}{\max\{C, D\}}.$$

In particular if $\underline{a}, \underline{b} \leq 1$

$$\sum_{i=1}^{\infty} a_i b_i \geq \frac{\sum_{i=1}^{\infty} a_i^2 \sum_{i=1}^{\infty} b_i^2}{\max\{A, B\}}.$$

□ For any $j \geq 1$,

$$\sum_{i=1}^{\infty} a_i c_i = C a_j + \sum_{i=1}^{\infty} (a_i - a_j) c_i \leq C a_j + \sum_{i=1}^{j} (a_i - a_j) c_i,$$

so

$$D_j \sum_{i=1}^{\infty} a_i c_i \leq j C a_j + D \sum_{i=1}^{j} (a_i - a_j) \leq \max\{C, D\} A_j.$$

Hence

$$\Delta b_j D_j \sum_{i=1}^{\infty} a_i c_i \geq \max\{C, D\} A_j \Delta b_j.$$

which on summing over j gives the required result. □

REMARK (ix) Obviously from (C),

$$\sum_{i=1}^{\infty} a_i b_i \leq \min\Big\{A, B, (\sum_{i=1}^{\infty} a_i^2)^{1/2} (\sum_{i=1}^{\infty} b_i^2)^{1/2}\Big\}.$$

The following result is in [*McLaughlin*].

THEOREM 17 *If $\underline{a}, \underline{b}$ are real $2n$-tuples then*

$$\Big(\sum_{i=1}^{n} a_{2i} b_{2i-1} - a_{2i-1} b_{2i}\Big)^2 \leq \sum_{i=1}^{2n} a_i^2 \sum_{i=1}^{2n} b_i^2 - \Big(\sum_{i=1}^{n} a_i b_i\Big)^2.$$

An inequality that is a mixture of (H) and (M) has been proved in [*Iusem, Isnard & Butnariu*].

THEOREM 18 *If $\underline{a}$ and $\underline{b}$ are two n-tuples, $p \geq 2$, define $c_i = b_i^{p-1}$, $1 \leq i \leq n$, then*

$$(\sum_{i=1}^{n} a_i^p)^{1/p} (\sum_{i=1}^{n} b_i^{p'})^{1/p'} - \sum_{i=1}^{n} a_i b_i \leq \frac{1}{p} ((\sum_{i=1}^{n} a_i^p)^{1/p} + (\sum_{i=1}^{n} c_i^p)^{1/p})^p - \sum_{i=1}^{n} (a_i^p + c_i^p).$$

If $1 < p \leq 2$ *the opposite inequality holds.*

5 Other Means Defined Using Powers

There are many other generalizations of the classical arithmetic, geometric and harmonic means besides the power means. Some generalizations are based on the very close connection of the power means with the convexity of certain functions; such generalizations are taken up in the next chapter. Other generalizations are really only defined in the case $n = 2$ and these are considered in VI 2. Here we study some generalizations that like the power means are based on the use powers, logarithms and exponentials.

5.1 Counter-harmonic Means

DEFINITION 1 *If* $\underline{a}$ *and* $\underline{w}$ *are* n*-tuples and* $r \in \overline{\mathbb{R}}$ *then the* r-th counter-harmonic mean of $\underline{a}$ with weight $\underline{w}$, *is*

$$\mathfrak{H}_n^{[r]}(\underline{a};\underline{w}) = \begin{cases} \dfrac{\sum_{i=1}^n w_i a_i^r}{\sum_{i=1}^n w_i a_i^{r-1}}, & \text{if } r \in \mathbb{R}, \\ \max \underline{a}, & \text{if } r = \infty, \\ \min \underline{a}, & \text{if } r = -\infty. \end{cases} \tag{1}$$

As with previous means we will just write $\mathfrak{H}^{[r]}(\underline{a};\underline{w})$ when $n = 2$, $\mathfrak{H}_n^{[r]}$ if the reference is unambiguous, $\mathfrak{H}_n^{[r]}(\underline{a})$ will denote the equal weight case, and if I is an index set the notation $\mathfrak{H}_I^{[r]}(\underline{a};\underline{w})$ is used in the manner of I 4.2, II 3.2.2.
The following identities are easily obtained: $\mathfrak{H}^{[1/2]}(a,b) = \mathfrak{G}(a,b)$, and

$$\mathfrak{H}_n^{[1]}(\underline{a};\underline{w}) = \mathfrak{A}_n(\underline{a};\underline{w});\ \mathfrak{H}_n^{[0]}(\underline{a};\underline{w}) = \mathfrak{H}_n(\underline{a};\underline{w});$$
$$\mathfrak{H}_n^{[r]}(\underline{a};\underline{w}) = \mathfrak{A}_n(\underline{a};\underline{a}^{r-1}\underline{w}),\ \mathfrak{H}_n^{[r]}(\underline{a};\underline{w}) = \left(\mathfrak{H}_n^{[-r+1]}(\underline{a}^{-1};\underline{w})\right)^{-1},\ r \in \mathbb{R}.$$

REMARK (i) The reader can easily check that $\mathfrak{H}_n^{[2]}(\underline{a})$ is the point at which the function $\sum_{i=1}^n \left((a_i - x)/x\right)^2$ takes its minimum value.

THEOREM 2 *(a) If* $1 \leq r \leq \infty$ *then*

$$\mathfrak{H}_n^{[r]}(\underline{a};\underline{w}) \geq \mathfrak{M}_n^{[r]}(\underline{a};\underline{w}), \tag{2}$$

and if $-\infty \leq r \leq 1$ *then* ($\sim$ 2) *holds. Inequality(2) is strict unless* $r = 1, \infty$ *or* $\underline{a}$ *is constant.*
(b) If $-\infty \leq r \leq 0$ *then the following stronger result holds:*

$$\mathfrak{H}_n^{[r]}(\underline{a};\underline{w}) \leq \mathfrak{M}_n^{[r-1]}(\underline{a};\underline{w}). \tag{3}$$

Inequality (3) is strict unless $r = -\infty, 0$ or $\underline{a}$ is constant.

□ The cases $r = \pm\infty$ are trivial so assume that $r \in \mathbb{R}$; then

$$\mathfrak{H}_n^{[r]}(\underline{a};\underline{w}) = \mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\left(\frac{\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})}{\mathfrak{M}_n^{[r-1]}(\underline{a};\underline{w})}\right)^{r-1}. \tag{4}$$

If $1 \le r < \infty$ then (r;s) and (4) imply (2), while if $-\infty < r \le 1$ ($\sim$2) is implied. This completes the proof of (a).
Now assume that $r \le 0$ then:

$$\begin{aligned}\mathfrak{H}_n^{[r]}(\underline{a};\underline{w}) = \left(\mathfrak{H}_n^{[-r+1]}(\underline{a}^{-1};\underline{w})\right)^{-1} &\le \left(\mathfrak{M}_n^{[-r+1]}(\underline{a}^{-1};\underline{w})\right)^{-1}, \quad \text{by(a)},\\ &= \mathfrak{M}_n^{[r-1]}(\underline{a};\underline{w}).\end{aligned}$$

This gives (b), and the cases of equality are immediate. □

REMARK (ii) Inequalities (2) and (3), together with 1 Theorem 2(c) justify the cases $r = \pm\infty$ of Definition 1; see [*HLP p. 62*].

REMARK (iii) Inequality (2) in the equal weight case and with $r = 2$ is due to Jacob; [*Jacob*].

THEOREM 3 *If $\underline{a}$ and $\underline{w}$ are n-tuples and if $-\infty \le r < s \le \infty$ then*

$$\mathfrak{H}_n^{[r]}(\underline{a};\underline{w}) \le \mathfrak{H}_n^{[s]}(\underline{a};\underline{w}), \tag{5}$$

with equality if and only if $\underline{a}$ is constant .

□ (*i*) By 2.3 Lemma 6(b) and 2.3 Remark (iii), $m(r) = \sum_{i=1}^n w_i a_i^r$ is strictly log-convex if $\underline{a}$ is not constant; see also 3.1.1. So by I 4.1(3), if $-\infty < r < s < \infty$,

$$\log \circ m(r) - \log \circ m(r-1) < \log \circ m(s) - \log \circ m(s-1);$$

which is just (5) in this case. If either $r = -\infty$ or $r = \infty$ the result is immediate.
(*ii*) In the case that $(s-r) \in \mathbb{N}$ the following proof has been given; see [*Angelescu*]. First note that $m(r-1)x^2 - 2m(r)x + m(r+1) = \sum_{i=1}^n w_i(x-a_i)^2 a_i^{r-1}$. Hence this quadratic does not have any real zeros; so $m^2(r) \le m(r+1)m(r-1)$, which implies (5) with s replaced by $r+1$. □

It is easy to check that these means have the properties (Ho) and (Co), and (5) shows that they are strictly internal. However counter-harmonic means do not have the properties of (Mo) and substitution. Consider the following examples; [*Beckenbach 1950; Farnsworth & Orr 1986*].

EXAMPLE (i) If $a = 1, b = 2, c = 3$ then $\mathfrak{H}_3^{[2]}(1,2,3) = 7/3$, $\mathfrak{H}_2^{[2]}(1,2) = 5/3$ but $\mathfrak{H}_3^{[2]}(5/3, 5/3, 3) = 131/57 \neq 7/3$

EXAMPLE (ii) Consider h(x) $= \mathfrak{H}_2^{[2]}(x, 1) = (1+x^2)/(1+x)$; then $h(0) = 1 = h(1)$ and h has a minimum of $2(\sqrt{2} - 1)$ at $x = \sqrt{2} - 1$.

EXAMPLE (iii) Consider $\chi(\underline{a}) = \mathfrak{H}_n^{[r]}(\underline{a})$ then

$$\nabla\chi(\underline{a}) = \frac{1}{\sum_{i=1}^n a_i^{r-1}}\big(r\underline{a}^{r-1} + (1-r)\underline{a}^{r-2}\chi(\underline{a})\big),$$

and so $\mathfrak{H}_n^{[r]}(\underline{a})$ is monotonic if $0 \le r \le 1$.

THEOREM 4 *If* $1 \le r \le 2$ *then*

$$\mathfrak{H}_n^{[r]}(\underline{a} + \underline{b}; \underline{w}) \le \mathfrak{H}_n^{[r]}(\underline{a}; \underline{w}) + \mathfrak{H}_n^{[r]}(\underline{b}; \underline{w}); \tag{6}$$

if $0 \le r \le 1$ *then* (~ 6) *holds.*
Inequality (6) is strict unless either $r = 1$ *or* $\underline{a} \sim \underline{b}$.

□ The case $r = 1$ is trivial so assume $r > 1$; then by Radon's inequality, 2.1 (9), with $n = 2$, $p = r$ we get that

$$\frac{\left(\left(\sum_{i=1}^n w_i a_i^r\right)^{1/r} + \left(\sum_{i=1}^n w_i b_i^r\right)^{1/r}\right)^r}{\left(\left(\sum_{i=\ s1}^n w_i a_i^{r-1}\right)^{1/r-1} + \left(\sum_{i=1}^n w_i b_i^{r-1}\right)^{1/r-1}\right)^{r-1}} \le \mathfrak{H}_n^{[r]}(\underline{a}; \underline{w}) + \mathfrak{H}_n^{[r]}(\underline{b}; \underline{w}).$$

Then (6) follows using (M) on the numerator of the left-hand side, and ($\sim$ M) on the denominator.
The case of equality follows from the cases of equality in the inequalities used in the proof.
The proof when $r < 1$ is similar. □

Theorem 4 is trivial if $r = \pm\infty$ but if $2 < r < \infty$, or $-\infty < r < 0$ then counter-examples show that the theorem may fail.

EXAMPLE (iv) If $2 < r < \infty$ take $\underline{a} = \{1, \epsilon, \epsilon, \dots, \epsilon\}$ and $\underline{b} = \{1, \epsilon^2, \epsilon^2, \dots, \epsilon^2\}$ for a suitable positive ϵ.

REMARK (iv) These means and their properties seem to be due to Beckenbach; [*Beckenbach 1950*] where an alternative proof of (6) can be found. As a result (6) is often called *Beckenbach's inequality*; another inequality with the same name is given above in 2.5.5; see also [*BB p.27*], [*Bellman 1957b*]. The above proof is

due to Danskin; [*Danskin*]. Other references are [*Bagchi & Maity; Bellman 1954; Brenner & Mays; Castellano 1948; Lehmer; Wang W L & Wang P F 1987*].

5.2 Generalizations of the Counter-harmonic Means Various authors have generalized the ratio in 5.1 (1) and while not obtaining such detailed results have extended Theorem 3.

Mitrinović & Vasić considered

$$F(x) = \frac{\mathfrak{M}_n^{[s]}(\underline{a}^x; \underline{w})}{\mathfrak{M}_n^{[r]}(\underline{a}^{x-k}; \underline{w})},$$

showing that if $r = s$ then F is increasing, or decreasing, according as $rk > 0$, or $rk < 0$; if $s < r$ then F has a unique maximum at $x = rk/(r-s)$, whereas the same point is the unique minimum if $s > r$. The methods of proof are those of elementary calculus; [*Mitrinović & Vasić 1966a*]. The equal weight case with $r = s$ had been considered earlier; see [*Izumi, Kobayashi & Takahashi; Kobayashi*].

Liu & Chen, [*Liu & Chen*], have considered the ratio

$$\mathfrak{R}(\underline{a}, \underline{w}; r, s; t) = \frac{\left(\mathfrak{M}_n^{[s]}(\underline{a}; \underline{w})\right)^t}{\left(\mathfrak{M}_n^{[r]}(\underline{a}; \underline{w})\right)^{t-1}},$$

which if $s = t = r$ and, $r - 1$ is substituted for r is just (1), in the case of r finite. They then generalize (6) as follows.

Theorem 5 *If $t \geq 1$, and $s \geq 1 \geq r$ then*

$$\mathfrak{R}(\underline{a} + \underline{b}, \underline{w}; r, s; t) \leq \mathfrak{R}(\underline{a}, \underline{w}; r, s; t) + \mathfrak{R}(\underline{b}, \underline{w}; r, s; t).$$

while if $t \leq 1$, and $s \leq 1 \leq r$ the opposite inequality holds.

Other authors have used ratios similar to those in the previous section to define new means; [*Castellano 1948; Dresher; Gini 1938; Gini & Zappa; Godunova 1967; Ku, Ku & Zhang 1999; Páles 1981; Pompilj*].

5.2.1 Gini Means Let $\underline{a}$ and $\underline{w}$ be n-tuples and for $p, q \in \mathbb{R}$ define the *Gini mean of $\underline{a}$ with weight $\underline{w}$* by

$$\mathfrak{G}_n^{p,q}(\underline{a}; \underline{w}) = \begin{cases} \left(\dfrac{\sum_{i=1}^n w_i a_i^p}{\sum_{i=1}^n w_i a_i^q}\right)^{1/p-q}, & \text{if } p \neq q, \\ \left(\prod_{i=1}^n a_i^{w_i a_i^p}\right)^{1/\sum_{i=1}^n w_i a_i^p}, & \text{if } p = q. \end{cases}$$

Convention Since $\mathfrak{G}_n^{p,q} = \mathfrak{G}_n^{q,p}$ we will assume in this section that $p \geq q$.

The Gini means include both the power means and the counter-harmonic means as special cases as the following simple identities show.

$$\mathfrak{G}_n^{p,0}(\underline{a};\underline{w}) = \mathfrak{M}_n^{[p]}(\underline{a};\underline{w});\ \mathfrak{G}_n^{p,p-1}(\underline{a};\underline{w}) = \mathfrak{H}_n^{[p]}(\underline{a};\underline{w}).$$

The means

$$\mathfrak{G}^{r+1,r}(a,b) = \mathfrak{H}^{[r+1]}(a,b) = \frac{b^{r+1}+a^{r+1}}{b^r+a^r}, r \in \mathbb{R},$$

have been called *Lehmer means*; [*Alzer 1988c; Lehmer*]. When $r=1$ the Lehmer mean is called the *contraharmonic mean*[6];

$$\mathfrak{G}^{2,1}(a,b) = \mathfrak{H}^{[2]}(a,b) = \frac{b^2+a^2}{b+a},$$

Some properties of Gini means are given in the next theorem.

THEOREM 6 *(a)*

$$\lim_{p\to q}\mathfrak{G}_n^{p,q}(\underline{a};\underline{w}) = \mathfrak{G}_n^{q,q}(\underline{a};\underline{w});$$
$$\lim_{p\to\infty}\mathfrak{G}_n^{p,q}(\underline{a};\underline{w}) = \max\underline{a};\ \lim_{q\to-\infty}\mathfrak{G}_n^{p,q}(\underline{a};\underline{w}) = \min\underline{a}.$$

(b) If $p_1 \le p_2$, $q_1 \le q_2$ then

$$\mathfrak{G}_n^{p_1,q_1}(\underline{a};\underline{w}) \le \mathfrak{G}_n^{p_2,q_2}(\underline{a};\underline{w}); \tag{7}$$

further if either $p_1 \ne p_2$ or $q_1 \ne q_2$ then inequality (7) is strict unless $\underline{a}$, is constant.
(c) If $p \ge 1 \ge q \ge 0$ then

$$\mathfrak{G}_n^{p,q}(\underline{a}+\underline{b};\underline{w}) \le \mathfrak{G}_n^{p,q}(\underline{a};\underline{w}) + \mathfrak{G}_n^{p,q}(\underline{b};\underline{w}). \tag{8}$$

□ (a) The first limit is a simple use, after taking logarithms, of l'Hôpital's Rule[7].
As to the second:

$$\begin{aligned}\log\left(\lim_{p\to\infty}\mathfrak{G}_n^{p,q}(\underline{a};\underline{w})\right) &= \lim_{p\to\infty}\frac{1}{p-q}\left(\log\left(\sum_{i=1}^{\infty} w_i a_i^p\right) - \log\left(\sum_{i=1}^{\infty} w_i a_i^q\right)\right)\\ &= \lim_{p\to\infty}\frac{1}{p}\log\left(\sum_{i=1}^{\infty} w_i a_i^p\right),\\ &= \log(\max\underline{a}), \quad \text{by 1 Theorem 2(c).}\end{aligned}$$

---

6 The contraharmonic mean is the solution x of the proportion $x-a{:}b-x{::}b{:}a$ and is a neo-Pythagorean mean; see VI 2.1.4

7 See 1 Footnote 1.

The third limit is handled in a similar manner.
(b) It is easily seen that if $p \neq q$

$$\log \circ\mathfrak{G}_n^{p,q}(\underline{a};\underline{w}) = \frac{\log \circ m(p) - \log \circ m(q)}{p-q},$$

where m is the function defined above in the proof of 5.1 Theorem 3, or in 3.1.1. This function is log-convex and so (7) in the case $p_1 \neq q_1$ and $p_2 \neq q_2$ is immediate from a basic property of convex functions, I 4.1 Remark (v), and the fact that the logarithmic function is strictly increasing; see [*PPT pp.119–120*]. The remaining cases follow by taking limits.
(c) Assume that $p > q > 0$ and write the right-hand side of (8) as,

$$\mathfrak{G}_n^{p,q}(\underline{a};\underline{w}) + \mathfrak{G}_n^{p,q}(\underline{b};\underline{w}) = \frac{\big(\mathfrak{M}_n^{[p]}(\underline{a};\underline{w})\big)^{p/(p-q)}}{\big(\mathfrak{M}_n^{[q]}(\underline{a};\underline{w})\big)^{q/(p-q)}} + \frac{\big(\mathfrak{M}_n^{[p]}(\underline{b};\underline{w})\big)^{p/(p-q)}}{\big(\mathfrak{M}_n^{[q]}(\underline{b};\underline{w})\big)^{q/(p-q)}}.$$

Now apply Radon's inequality, 2.1 (9), in the case $n = 2$, and p in that reference taken as $p/(p-q)$ to get

$$\mathfrak{G}_n^{p,q}(\underline{a};\underline{w}) + \mathfrak{G}_n^{p,q}(\underline{b};\underline{w}) \ge \frac{\big(\mathfrak{M}_n^{[p]}(\underline{a};\underline{w}) + \mathfrak{M}_n^{[p]}(\underline{b};\underline{w})\big)^{p/(p-q)}}{\big(\mathfrak{M}_n^{[q]}(\underline{a};\underline{w}) + \mathfrak{M}_n^{[q]}(\underline{b};\underline{w})\big)^{q/(p-q)}}.$$

Now by (M), $p \ge 1$, and ($\sim$M), $0 \le q \le 1$,

$$\frac{\mathfrak{M}_n^{[p]}(\underline{a};\underline{w}) + \mathfrak{M}_n^{[p]}(\underline{b};\underline{w})}{\mathfrak{M}_n^{[q]}(\underline{a};\underline{w}) + \mathfrak{M}_n^{[q]}(\underline{b};\underline{w})} \ge \frac{\mathfrak{M}_n^{[p]}(\underline{a}+\underline{b};\underline{w})}{\mathfrak{M}_n^{[q]}(\underline{a}+\underline{b};\underline{w})}.$$

Hence

$$\mathfrak{G}_n^{p,q}(\underline{a};\underline{w}) + \mathfrak{G}_n^{p,q}(\underline{b};\underline{w}) \ge \frac{\big(\mathfrak{M}_n^{[p]}(\underline{a}+\underline{b};\underline{w})\big)^{p/(p-q)}}{\big(\mathfrak{M}_n^{[q]}(\underline{a}+\underline{b};\underline{w})\big)^{q/(p-q)}} = \mathfrak{G}_n^{p,q}(\underline{a}+\underline{b};\underline{w}).$$

The other cases of (8) are easily proved. □

REMARK (i) The proof of (b) is due to Pečarić & Beesack; [*PPT p.119*], [*Pečarić & Beesack 1986*].

REMARK (ii) The case $p_1 = 1, q_1 = 0$ of (7) has already been proved; see 4.1 Remark (v).

REMARK (iii) Inequality (8) was first proved by Dresher and is sometimes referred to as *Dresher's inequality*. The proof, which generalizes that of 5.1 (6), is due to Danskin; [*PPT pp.120–121*], [*Danskin; Dresher*]. A very detailed examination of

the case $n = 2$ has been made in [*Losonczi & Páles 1996*]. See also [B^2 *pp.233, 264*], [*Guljaš, Pearce & Pečarić*]. An extension of (8) is given in IV 7.1 Corollary 5.

REMARK (iv) Note that Liapunov's inequality, 2.1 (8), is just

$$\mathfrak{G}_n^{s,t}(\underline{a};\underline{w}) \le \mathfrak{G}_n^{r,t}(\underline{a};\underline{w}),\ 0 < t < s < r.$$

REMARK (v) These means have been generalized to allow for complex conjugate p and q, [*Páles 1989*]. For another generalization see V 7.3.
Gini means have been studied in detail by many authors; see for instance [*Aczél & Daróczy; Allasia 1974-1975; Allasia & Sapelli; Brenner 1978; Clausing 1981; Daróczy & Losonczi; Daróczy & Páles 1980; Farnsworth & Orr 1986; Jecklin 1948b; Jecklin & Eisenring; Losonczi 1971a,c 1977; Páles 1981, 1983a, 1988c; Persson & Sjöstrand; Stolarsky 1996*].

5.2.2 BONFERRONI MEANS Another kind of generalization has been suggested by Bonferroni, the *Bonferroni means*; [*Bonferroni 1926, 1950*].
Let $\underline{a}$ be an n-tuple and define

$$\mathfrak{B}_n^{p,q}(\underline{a}) = \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\ne j}} a_i^p a_j^q\right)^{1/(p+q)},$$

with obvious extensions to $\mathfrak{B}_n^{p,q,r}(\underline{a})$ etc.

THEOREM 6 *With the above notation if $h > 0, q < p-h < p$ then*

$$\mathfrak{B}_n^{p,q}(\underline{a}) \ge \mathfrak{B}_n^{p-h,q+h}(\underline{a}).$$

□ For a proof the reader is referred to the references. □

5.2.3 GENERALISED POWER MEANS In a very interesting paper Ku, Ku & Zhang have considered a very general extension of power means ; [*Ku, Ku & Zhang 1999*]. Let k_i, $1 \le i \le m$, be distinct positive functions defined on the n-tuples $\underline{a}$, and write $\underline{k} = (k_1, \ldots, k_m)$. If $\underline{w}$, $\underline{x}$ are m-tuples, $\underline{x}$ allowed to be non-negative, and such that $\sum_{i=1}^m w_i x_i = 1$. If $r \in \mathbb{R}$ the *r-th generalized power mean of $\underline{a}$* is :

$$\mathfrak{K}_{m,n}^{[r]}(\underline{a};\underline{k};\underline{w},\underline{x}) = \begin{cases} \left(\sum_{i=1}^m w_i x_i \big(k_i(\underline{a})\big)^r\right)^{1/r}, & \text{if } r \ne 0,\\ \prod_{i=1}^m \big(k_i(\underline{a})\big)^{w_i x_i}, & \text{if } r = 0.\end{cases}$$

The generalized power means include power means, certain Gini means, in particular the counter-harmonic means.

EXAMPLE (i) Take $k_i(\underline{a}) = a_i$ and $x_i = W_n^{-1}$, $1 \le i \le n$, when $\mathfrak{K}^{[r]}_{m,n}(\underline{a};\underline{k};\underline{w},\underline{x}) = \mathfrak{M}^{[r]}_n(\underline{a};\underline{w})$. In fact, as we easily see,

$$\mathfrak{K}^{[r]}_{m,n}(\underline{a};\underline{k};\underline{w},\underline{x}) = \mathfrak{M}^{[r]}_m(k_1(\underline{a}),\ldots,k_m(\underline{a});\underline{w}\,\underline{x}). \tag{9}$$

EXAMPLE (ii) More generally take $x_i = \dfrac{a_i^q}{\sum_{i=1}^n w_i a_i^q}$, $k_i(\underline{a}) = a_i$, $1 \le i \le n$, when we find that $\mathfrak{K}^{[p-q]}_{n,n}(\underline{a};\underline{k};\underline{w},\underline{x}) = \mathfrak{G}^{[p,q]}_n(\underline{a},\underline{w})$, $p \ne q$.

To obtain an extension of (r;s) here we need two further concepts.

Firstly we define an *order on the set of* $\underline{x}$; $\underline{x} \gg \underline{x}' \iff$ there is a i_0, $1 \le i_0 < m$, such that

$$x_i \ge x_i',\ 1 \le i \le i_0, \quad x_{i_0+1} < x'_{i_0+1},$$
$$x_i \le x_i',\ i_0+2 \le i \le m,\ \text{if}\ i_0+2 \le m.$$

Secondly we define *a condition of equality for the* $\underline{k}$: $EQ(\underline{k},\underline{x} \gg \underline{x}',\underline{a})$ holds if and only if

$$\sum_{\substack{i=1\\ i\ne i_0, x_i \ne x'_{i_0}}}^{m} \big(k_i(\underline{a}) - k_{i_0}(\underline{a})\big)^2 = 0 \iff \underline{a} \text{ is constant.}$$

EXAMPLE (iii) If for some distinct i,j we have $k_i(\underline{a}) = k_j(\underline{a}) \iff \underline{a}$ is constant then $EQ(\underline{k},\underline{x} \gg \underline{x}',\underline{a})$ holds for all distinct $\underline{x},\underline{x}'$, with $\underline{x} \gg \underline{x}'$.

EXAMPLE (iv) Particular cases of Example (iii) are given by: $k_i(\underline{a}) = \mathfrak{M}^{[r_i]}_n(\underline{a};\underline{w})$, and $k_i(\underline{a}) = \big(\mathfrak{A}_n(\underline{a})\big)^{(n-r_i)/r_i(n-1)}\big(\mathfrak{G}_n(\underline{a})\big)^{n(r_i-1)/r_i(n-1)}$, for $1 \le i \le m$, and where $0 \le r_1 < \cdots < r_m \le n$.

THEOREM 7 *(a) If* $r,s \in \mathbb{R}$, $r < s$, *then*

$$\mathfrak{K}^{[r]}_{m,n}(\underline{a};\underline{k};\underline{w},\underline{x}) \le \mathfrak{K}^{[s]}_{m,n}(\underline{a};\underline{k};\underline{w},\underline{x}),$$

with equality if and only if $\underline{a}$ *is constant.*

(b) If $\underline{k}$ *is decreasing, that is* $k_i(\underline{a}) \ge k_{i+1}(\underline{a})$, $1 \le i < m$, $r \ge 0$, *and* $\underline{x} \gg \underline{x}'$ *then*

$$\mathfrak{K}^{[r]}_{m,n}(\underline{a};\underline{k};\underline{w},\underline{x}) \ge \mathfrak{K}^{[r]}_{m,n}(\underline{a};\underline{k};\underline{w},\underline{x}'). \tag{10}$$

If $\underline{x} \ne \underline{x}'$ *and if* $EQ(\underline{k},\underline{x} \gg \underline{x}',\underline{a})$ *is satisfied then (10) is strict unless* $\underline{a}$ *is constant.*

□ (a) This is immediate from (r;s) and (9).

(b) (i) $r=0$. Choose a β so that $\beta k_m \geq 1$. Let i_0 be the suffix given in $\underline{x} \gg \underline{x}'$, then since $\underline{k}$ is decreasing,

$$(\beta k_i)^{w_i x_i} = (\beta k_i)^{w_i x_i'}(\beta k_i)^{w_i(x_i - x_i')} \geq (\beta k_i)^{w_i x_i'}(\beta k_{i_0})^{w_i(x_i - x_i')} \quad 1 \leq i \leq i_0.$$

Further $\sum_{i=1}^m w_i x_i = \sum_{i=1}^m w_i x_i' = 1$ and so $\sum_{i=1}^{i_0} w_i(x_i - x_i') = \sum_{i=i_0+1}^m w_i(x_i' - x_i)$. Using these facts we get that

$$\begin{aligned}
\beta\mathfrak{K}_{m,n}^{[0]}(\underline{a};\underline{k};\underline{w},\underline{x}) &= \prod_{i=1}^{i_0}(\beta k_i)^{w_i x_i}(\underline{a}) \prod_{i=i_0+1}^{m} (\beta k_i)^{w_i x_i}(\underline{a}) \\
&\geq \big(\beta k_{i_0}(\underline{a})\big)^{\sum_{i=0}^{i_0} w_i(x_i - x_i')} \prod_{i=1}^{i_0}(\beta k_i)^{w_i x_i'}(\underline{a}) \prod_{i=i_0+1}^{m} (\beta k_i)^{w_i x_i}(\underline{a}) \\
&= (\beta k_{i_0})^{\sum_{i_0+1}^{m} w_i(x_i' - x_i)}(\underline{a}) \prod_{i=1}^{i_0}(\beta k_i)^{w_i x_i'}(\underline{a}) \prod_{i=i_0+1}^{m} (\beta k_i)^{w_i x_i}(\underline{a}) \\
&\geq \prod_{i=i_0+1}^{m} (\beta k_i)^{w_i(x_i' - x_i)}(\underline{a}) \prod_{i=1}^{i_0}(\beta k_i)^{w_i x_i'}(\underline{a}) \prod_{i=i_0+1}^{m} (\beta k_i)^{w_i x_i}(\underline{a}) \\
&= \beta \prod_{i=1}^{m} k_i^{w_1 x_i'}(\underline{a}) = \beta\mathfrak{K}_{m,n}^{[0]}(\underline{a};\underline{k};\underline{w},\underline{x}').
\end{aligned}$$

Equality holds if for $i \neq i_0$, $1 \leq i \leq m$, $(\beta k_i)^{w_i(x_i - x_i')}(\underline{a}) = (\beta k_{i_0})^{w_i(x_i - x_i')}$. Since $\underline{x} \gg \underline{x}'$ there is some i such that $x_i \neq x_i'$ and so the condition in the last sentence implies that if $i \neq i_0$ then $k_i(\underline{a}) = k_{i_0}(\underline{a})$, and so by the condition $EQ(\underline{k}, \underline{x} \gg \underline{x}', \underline{a})$, we must have that $\underline{a}$ is constant.

(ii) $r > 0$

As the proof of this case is analogous to the one above it will be omitted. The reader is referred to the reference for more details. □

REMARK (i) Particular cases of inequality (10) are inequalities 5.1(2),(5) and inequality 5.2.1 (7).

EXAMPLE (v) Let $m=3$ and $k_1(\underline{a}) = \mathfrak{A}_n(\underline{a}), k_2(\underline{a}) = \mathfrak{G}_n(\underline{a}), k_3(\underline{a}) = \mathfrak{H}_n(\underline{a})$ and take $r \neq 0$ in (10). If $\underline{x} \gg \underline{x}'$ then

$$\begin{aligned}
&\big(w_1 x_1 \mathfrak{A}_n^r(\underline{a}) + w_2 x_2 \mathfrak{G}_n^r(\underline{a}) + w_3 x_3 \mathfrak{H}_n^r(\underline{a})\big)^{1/r} \\
&\qquad\qquad \geq \big(w_1 x_1' \mathfrak{A}_n^r(\underline{a}) + w_2 x_2' \mathfrak{G}_n^r(\underline{a}) + w_3 x_3' \mathfrak{H}_n^r(\underline{a})\big)^{1/r}.
\end{aligned}$$

The quantity here, $\mathfrak{K}_{3,n}^{[r]}[\underline{a};\underline{k};\underline{w},\underline{x}]$, can be considered as a two parameter family of means lying between the extremes $\mathfrak{A}_n(\underline{a})$ and $\mathfrak{H}_n(\underline{a})$.

5.3 MIXED MEANS Let $\underline{a}$ be an n-tuple, k and integer, $1 \leq k \leq n$, and denote by $\underline{a}_1^{(k)}, \ldots, \underline{a}_\kappa^{(k)}$ the $\kappa, \kappa = \binom{n}{k}$, k-tuples that can be formed from the elements of $\underline{a}$.

Definition 8 *If* $s, t \in \overline{\mathbb{R}}$ *then the* mixed mean of order s and t of $\underline{a}$ taken k at a time *is*

$$\mathfrak{M}_n(s,t;k;\underline{a}) = \mathfrak{M}_\kappa^{[s]}\big(\mathfrak{M}_k^{[t]}(\underline{a}_i^{(k)}), 1 \le i \le \kappa\big). \tag{11}$$

Example (i) The following special cases are immediate:

$$\mathfrak{M}_n(s,t;1;\underline{a}) = \mathfrak{M}_n(s,s;k;\underline{a}) = \mathfrak{M}_n^{[s]}(\underline{a}); \quad \mathfrak{M}_n(s,t;n;\underline{a}) = \mathfrak{M}_n^{[t]}(\underline{a}).$$

Remark (i) An immediate consequence of (r;s) is that $\mathfrak{M}_n(s,t;k;\underline{a})$ is an increasing function of both s and t.

The main results of this section are due to Carlson, Meany and Nelson; [*Carlson 1970a,b; Carlson, Meany & Nelson 1971a,b*].

Theorem 9 *If* $-\infty \le s < t \le \infty$ *then*

$$\mathfrak{M}_n(s,t;k-1;\underline{a}) \le \mathfrak{M}_n(s,t;k;\underline{a}). \tag{12}$$

If $s > t$ *then* (~ 12) *holds, and there is equality in both cases if and only if* $\underline{a}$ *is constant.*

□ Denote by $\underline{a}_{ij}^{(k)}, 1 \le j \le k$, the collection of $(k-1)$-tuples formed from the elements of $\underline{a}_i^{(k)}$. Then each $\underline{a}_h^{(k-1)}, 1 \le h \le \kappa' = \binom{n}{k-1}$, occurs $(n-k+1)$ times in the collection of $\underline{a}_{ij}^{(k)}, 1 \le j \le k, 1 \le i \le \kappa$; note that $\kappa k = \kappa'(n-k+1)$.
Firstly

$$\mathfrak{M}_k^{[t]}(\underline{a}_i^{(k)}) = \mathfrak{M}_k^{[t]}\big(\mathfrak{M}_{k-1}^{[t]}(\underline{a}_{ij}^{(k)}), 1 \le j \le k\big);$$

and so if $s < t$ we have by (r;s) that

$$\mathfrak{M}_k^{[t]}(\underline{a}_i^k) \ge \mathfrak{M}_k^{[s]}\big(\mathfrak{M}_{k-1}^{[t]}(\underline{a}_{ij}^k), 1 \le j \le k\big). \tag{13}$$

Hence, by (11), and using the above notation,

$$\begin{aligned}\mathfrak{M}_n(s,t;k;\underline{a}) \ge& \mathfrak{M}_k^{[s]}\Big(\mathfrak{M}_k^{[s]}\big(\mathfrak{M}_{k-1}^{[t]}(\underline{a}_{ij}^k), 1 \le j \le k\big), 1 \le i \le \kappa\Big)\\ =& \mathfrak{M}_k^{[s]}\big(\mathfrak{M}_{k-1}^{[t]}(\underline{a}_h^{k-1}), 1 \le h \le \kappa'\big)\\ =& \mathfrak{M}_n(s,t;k-1;\underline{a}).\end{aligned}$$

If $s > t$ the inequality (13) is reversed. The cases of equality are immediate. □

We now wish to obtain an inequality between different mixed means.

Suppose that $k + \ell > n$; then $\underline{a}_i^{(k)}$ and $\underline{a}_j^{(\ell)}$ have $m, m = m(i,j,k,\ell)$, elements in common, $m \neq 0$, $1 \le i \le j \le \lambda = \binom{n}{\ell}$. For convenience we introduce the following notations:

$$\begin{aligned} \sigma_i &= \mathfrak{M}_k^{[s]}(\underline{a}_i^{(k)}); \quad \tau_j = \mathfrak{M}_\ell^{[t]}(\underline{a}_j^{(\ell)}); \\ \sigma_{ij} &= \mathfrak{M}_k^{[s]}(\underline{a}_i^{(k)} \cap \underline{a}_j^{(\ell)}); \quad \tau_{ij} = \mathfrak{M}_\ell^{[t]}(\underline{a}_i^{(k)} \cap \underline{a}_j^{(\ell)}); \end{aligned} \qquad 1 \le i \le \kappa, \quad 1 \le j \le \lambda.$$

LEMMA 10 *If $k + \ell > n$ then*

$$\tau_j = \mathfrak{M}_\kappa^{[t]}(\tau_{ij}, 1 \le i \le \kappa), \qquad 1 \le j \le \lambda; \tag{14}$$

$$\sigma_i = \mathfrak{M}_\lambda^{[s]}(\sigma_{ij}, 1 \le j \le \lambda), \qquad 1 \le i \le \kappa. \tag{15}$$

□ We only give a proof of (14) as that for (15) is similar. Further if $t = \pm\infty$ the result is immediate and so we will assume that t is finite. Let us also assume that $t \neq 0$.
Under these assumptions the sum on the right-hand side of (14) is a linear combination of t-th powers of the elements of $\underline{a}_j^{(\ell)}$. Since the summation extends over all the k-tuples $a_i^{(k)}$ the sum is unchanged if the elements of $\underline{a}_j^{(\ell)}$ are permuted. Hence the right-hand side is a multiple of τ_j; and by taking $\underline{a}$ constant it is seen to be actually equal to τ_j.
The case $t = 0$ can be handled in the same way. □

THEOREM 11 *If $-\infty \le s < t \le \infty$ and $k + \ell > n$ then*

$$\mathfrak{M}_n(t,s;k;\underline{a}) \le \mathfrak{M}_n(s,t;\ell;\underline{a}), \tag{16}$$

with equality if and only if $\underline{a}$ is constant.

□ We only consider the case where s, t are finite and non-zero as the other cases are either trivial or similar. Using (11), Lemma 10, 3.1.3(9) and (r;s) the following can be established.

$$\begin{aligned} \mathfrak{M}_n(t,s;k;\underline{a}) = \mathfrak{M}_k^{[t]}(\sigma_i, 1 \le i \le k) &= \mathfrak{M}_\kappa^{[t]}\big(\mathfrak{M}_\lambda^{[s]}(\sigma_{ij}, 1 \le j \le \lambda), 1 \le i \le \kappa\big) \\ &\le \mathfrak{M}_\lambda^{[s]}\big(\mathfrak{M}_\kappa^{[t]}(\tau_{ij}, 1 \le j \le \lambda), 1 \le i \le \kappa\big) \\ &= \mathfrak{M}_\lambda^{[s]}(\tau_j, 1 \le j \le \lambda) = \mathfrak{M}_n(s,t;\ell;\underline{a}). \end{aligned}$$

This gives (16) and the case of equality follows from that of 3.1.3(9) and (r;s). □

REMARK (ii) Taking $k = \ell = n$ we see, from Example (i), that both (12) and (16) include (r;s) as a special case.

Consider the following $2 \times n$ matrix

$$\mathbb{M} = \begin{pmatrix} \mathfrak{M}_n(s,t;1;\underline{a}) & \mathfrak{M}_n(s,t;2;\underline{a}) & \ldots & \mathfrak{M}_n(s,t;n-;1;\underline{a}) & \mathfrak{M}_n(s,t;n;\underline{a}) \\ \mathfrak{M}_n(t,s;n;\underline{a}) & \mathfrak{M}_n(t,s;n-1;\underline{a}) & \ldots & \mathfrak{M}_n(t,s;2;\underline{a}) & \mathfrak{M}_n(t,s;1;\underline{a}) \end{pmatrix}$$

where we assume that $s < t$ and $\underline{a}$ is not constant. The inequalities (12) and (16) can be summarized as saying: (i) the rows of $\mathbb{M}$ are strictly increasing to the right; (ii) the columns, except the first and the last, strictly increase downwards; and of course the entries in the first column both equal $\mathfrak{M}_n^{[s]}(\underline{a})$, while those in the last column both equal $\mathfrak{M}_n^{[t]}(\underline{a})$.

Example (ii) Taking $n = 3$, $t = 1$, $s = 0$, $k = 2$, $\underline{a} = (a,b,c)$ in (16) gives the inequality

$$\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{3} \le \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}; \tag{17}$$

a further discussion of this inequality, that can be compared with V 5.4(35), is in [*Carlson 1970a*].

Example (iii) Another particular case of (16) is given when $t = 1$, $s = 0$, $k = n-1$,

$$\mathfrak{A}_n\big(\mathfrak{G}_{n-1}(\underline{a}'_k); 1 \le k \le n\big) \le \mathfrak{G}_n\big(\mathfrak{A}_{n-1}(\underline{a}'_k); 1 \le k \le n\big);$$

see [*AI p.379*], [*Carlson 1970b*].

Remark (iii) Further results can be found in II 3.5 Remark(iv), V 5.4 and other mixed means are discussed in II 3.4, and 3.2.6 above; see also 6.2 Remark (i) and [*Acu; Mitrinović & Pečarić 1988b; Ozeki 1973*].

6 Some Other Results

In this section are a few results on power means that do not fit into the above discussion. Still more results can be found in the references: [*Jecklin 1963*].

6.1 Means on the Move Hoehn and Niven proved a very interesting result, called *means on the move*, that was generalized by Brenner and others; [*Aczél & Páles; Brenner 1985; Bullen 1990; Hoehn & Niven*].

If $\underline{a}, \underline{w}$ are n-tuples, and $r \in \mathbb{R}$ then the result of Hoehn and Niven considers the behaviour of $\mathfrak{M}_n^{[r]}(\underline{a}+t\underline{e};\underline{w})$ as $t \to \infty$.

Theorem 1 *With the above notation,*

$$\lim_{t\to\infty}\big(\mathfrak{M}_n^{[r]}(\underline{a}+t\underline{e};\underline{w}) - \mathfrak{A}_n(\underline{a}+t\underline{e};\underline{w})\big) = 0; \tag{1}$$

equivalently,

$$\lim_{t\to\infty}\big(\mathfrak{M}_n^{[r]}(\underline{a}+t\underline{e};\underline{w})-t\big)=\mathfrak{A}_n(\underline{a};\underline{w}).$$

□ Let us first note that both of these limits exist.
If we put $f(t)=\mathfrak{M}_n^{[r]}(\underline{a}+t\underline{e};\underline{w})-\mathfrak{A}_n(\underline{a}+t\underline{e};\underline{w})$ then a simple calculation and 1(2) give $\min\underline{a}-\mathfrak{A}_n(\underline{a};\underline{w})\le f(t)\le\max\underline{a}-\mathfrak{A}_n(\underline{a};\underline{w})$, so f is bounded.
Simple calculations show that if $M_r(t)=\mathfrak{M}_n^{[r]}(\underline{a}+t\underline{e};\underline{w})-\mathfrak{A}_n(\underline{a}+t\underline{e};\underline{w})$ then $M_r'=\big(M_{r-1}/M_r\big)^{r-1}$ and so $f'=\big(M_{r-1}/M_r\big)^{r-1}-1$. Using this we have by (r;s) that $f'\le 0$.
So f is a bounded decreasing function, $\lim_{t\to\infty}f(t)$ exists and is in the interval $[\min\underline{a}-\mathfrak{A}_n(\underline{a};\underline{w}),\max\underline{a}-\mathfrak{A}_n(\underline{a};\underline{w})]$.
If $r\ne 0$ then

$$M_r(t)-M_1(t)=\Big(t\big(W_n^{-1}\sum_{i=1}^n w_i(1+a_i/t)^r\big)^{1/r}-1-\mathfrak{A}_n(\underline{a};\underline{w})\Big)=O(1/t),\ t\to\infty.$$

by Taylor's theorem[8]. This completes the proof in this case.
If $r=0$ then by (GA), $0\ge M_0(t)-M_1(t)\ge M_{-1}(t)-M_1(t)$, so the result follows from the $r\ne 0$ case. □

Remark (i) It follows from the above argument that if $r<1<s$ and if $\underline{a}$ is not constant then $M_r'>M_s'$. This is not in general true if r,s do not straddle 1; see [*Hoehn & Niven*].

Remark (ii) Generalizations of this result can be found in IV 4.5 and VI 4.3.

Another interpretation of this result is that translating the data values to larger values pushes the power mean of the translated values closer to their arithmetic mean. Alternatively the power mean of the translated values translated back moves closer to the arithmetic mean of the original values; [*Farnsworth & Orr 1988; Wang C L 1989b*].
The idea in the last remark has been used to introduce a further generalization of the power means; [*Persson & Sjöstrand*].

Definition 2 *If $\underline{a}$ and $\underline{w}$ be n-tuples with $W_n=1$, and $t>0$ define*

$$\mathfrak{M}_n^{[p,q];t}(\underline{a};\underline{w})=\begin{cases}\Big(\big(\sum_{i=1}^n w_i(a_i^q+t)^{p/q}\big)^{q/p}-t\Big)^{1/q}, & \text{if } pq\ne 0,\\ \big(\prod_{i=1}^n(a_i^q+t)^{w_i}-t\big)^{1/q}, & \text{if } p=0,\ q\ne 0,\\ \big(\sum_{i=1}^n w_ia_i^{tp}\big)^{1/tp}, & \text{if } p\ne 0,\ q=0.\end{cases}$$

[8] See I 2.2 Footnote 2.

THEOREM 3 *With the above notations, and $q \neq 0$*

$$(a)\qquad \mathfrak{M}_n^{[p]}(\underline{a};\underline{w}) = \mathfrak{M}_n^{[p,q];0}(\underline{a};\underline{w});$$

$$(b)\qquad \lim_{t\to\infty} \mathfrak{M}_n^{[p,q];t}(\underline{a};\underline{w}) = \mathfrak{M}_n^{[q]}(\underline{a};\underline{w});$$

$$(c)\qquad \mathfrak{M}_n^{[p,q];s}(\underline{a};\underline{w}) < \mathfrak{M}_n^{[p,q];t}(\underline{a};\underline{w}),\ s<t.$$

□

(a) This is immediate.

(b) (i) $p \neq 0$

$$\begin{aligned}
\mathfrak{M}_n^{[p,q];t}(\underline{a};\underline{w}) =& t^{1/q}\Big(\Big(\sum_{i=1}^n w_i(1+\frac{a_i^q}{t})^{p/q}\Big)^{q/p} - 1\Big)^{1/q}\\
=& t^{1/q}\Big(\Big(\sum_{i=1}^n w_i(1+\frac{p}{q}\frac{a_i^q}{t} + O(t^{-2}))\Big)^{q/p} - 1\Big)^{1/q}, \text{ by I 2.1(5)},\\
=& t^{1/q}\Big(\Big(1+\frac{p}{q}\frac{\sum_{i=1}^n w_i a_i^q}{t} + O(t^{-2})\Big)^{q/p} - 1\Big)^{1/q}\\
=& t^{1/q}\Big((1/t)\big(\sum_{i=1}^n w_i a_i^q\big) + +O(t^{-2})\Big)^{1/q}, \text{ by I 2.1(5)},\\
=& \mathfrak{M}_n^{[q]}(\underline{a};\underline{w}) + O(t^{-1}),\ t\to\infty.
\end{aligned}$$

(ii) $p=0$

$$\begin{aligned}
\mathfrak{M}_n^{[0,q];t}(\underline{a};\underline{w}) =& t^{1/q}\Big(\prod_{i=1}^n (1+\frac{a_i^q}{t})^{w_i} - 1\Big)^{1/q}\\
=& t^{1/q}\Big(1+\frac{\sum_{i=1}^n w_1 a_i^q}{t} + O(t^{-2}) - 1\Big)^{1/q}, \text{ by I 2.1(5)},\\
=& \mathfrak{M}_n^{[q]}(\underline{a};\underline{w}) + O(t^{-1}),\ t\to\infty.
\end{aligned}$$

(c) We will only consider the case $0<q<p$ as the other cases are similar. The proof of Theorem 1 shows that if $s<t$ then $\mathfrak{M}_n^{[p/q]}(\underline{a}^q+s;\underline{w}) < \mathfrak{M}_n^{[p/q]}(\underline{a}^q+t;\underline{w})$. Raising both sides of this inequality to the power $1/q$ gives the result. □

6.2 HLAWKA-TYPE INEQUALITIES One theme in this book has been the study of sub-additivity or super-additivity of the difference between two sides of various inequalities, the difference being regarded as a function on the index set; see I 4.2 Theorem 15, II 3.2.2 Theorem 7, above 2.5.2 Theorem 12, 3.2.4 Corollary 18, and below IV 3.1 Theorem 1, Corollary 2. In an interesting paper Burkill, [*Burkill*], considered replacing the super-additivity of a function σ on the index sets by an

inequality related to convexity, called *H-positivity*; [*PPT p.176–178*]. Let I, J and K be disjoint index sets the set function σ is H-positive if:

$$\sigma(I \cup J \cup K) + \sigma(I) + \sigma(J) + \sigma(K) \geq \sigma(I \cup J) + \sigma(J \cup K) + \sigma(K \cup I); \qquad (2)$$

An inequality of type (2) was called by Burkill a *Hlawka inequality*, because of the basic result for triples of n-tuples due to Hlawka:

$$|\underline{a} + \underline{b} + \underline{c}| + |\underline{a}| + |\underline{b}| + |\underline{c}| \geq |\underline{a} + \underline{b}| + |\underline{b} + \underline{c}| + |\underline{c} + \underline{a}| :$$

see [*DI pp.125–127*]. The general result for n-tuples $\underline{a}_i$, $1 \leq i \leq m$,

$$\sum_{i=1}^{m} |\underline{a}_i| - \sum_{1 \leq i < j \leq m} |\underline{a}_i + a_j| + \sum_{1 \leq i < j < k \leq m} |\underline{a}_i + a_j + a_k| - \cdots + (-1)^{m-1}|\underline{a}_1 + \cdots a_m| \geq 0$$

holds in general only if $m = 1$, trivial, $m = 2$, (M), or $m = 3$, Hlawka's inequality; [*Jiang & Cheng*].

In his paper Burkill proved several results of this type by considering :
(a) $\sigma(I) = W_I \mathfrak{G}_I(\underline{a}; \underline{w})$, and (b) $\sigma(I) = W_I \phi\big(\mathfrak{M}_I^{[r]}(\underline{a}; \underline{w})\big)$, where ϕ is a function with $\phi'' - (r-1)\phi' > 0$; in all cases $I \subseteq \{1, 2, 3\}$.
In particular he obtained the following result.

THEOREM 4 *If $\underline{a}$ and $\underline{w}$ are 3-tuples, with $W_3 = 1$ then:*

$$\begin{aligned}
&\mathfrak{G}_3(\underline{a}; \underline{w}) + \mathfrak{A}_3(\underline{a}; \underline{w}) \\
&\qquad \geq (w_1 + w_2) a_1^{w_1/(w_1+w_2)} a_2^{w_2/(w_1+w_2)} \\
&\qquad\quad + (w_2 + w_3) a_2^{w_2/(w_1+w_2)} a_3^{w_3/(w_1+w_2)} + (w_3 + w_1) a_3^{w_3/(w_1+w_2)} a_1^{w_1/(w_1+w_2)}.
\end{aligned}$$

If in addition $r > 0$ and $\phi : \mathbb{R}_+ \mapsto \mathbb{R}$, with $\phi'' - (r-1)\phi' > 0$ then:

$$\begin{aligned}
&\phi\big(\mathfrak{M}_3^{[r]}(\underline{a}; \underline{w})\big) + \mathfrak{A}_3\big(\phi(\underline{a}); \underline{w}\big) \\
&\qquad \geq (w_1 + w_2)\phi\big(\mathfrak{M}_2^{[r]}(a_1, a_2; w_1, w_2)\big) \qquad (3) \\
&\qquad\quad + (w_2 + w_3)\phi\big(\mathfrak{M}_2^{[r]}(a_2, a_3; w_2, w_3)\big) + (w_3 + w_1)\phi\big(\mathfrak{M}_2^{[r]}(a_3, a_1; w_3, w_1)\big).
\end{aligned}$$

The methods of proof were elementary and the paper caused much interest. In particular if in the second case $r = 1$, when the mean is the arithmetic mean, the condition on ϕ implies that it is convex. This simpler condition has been used to obtain the same result, see [*Baston 1976; Vasić & Stanković 1976*], and [*Pečarić 1986*] for a different proof. It was Pečaric who noted that (3), with $r = 1$, implies the following more general result; [*PPT pp.171–180*].

THEOREM 5 *If $\phi : [a,b] \mapsto \mathbb{R}$ is convex, $\underline{w}$ an n-tuple, $\underline{a}$ a real n-tuple with $a \le a_i \le b$, $1 \le i \le n$, and if $2 \le k \le n$ then,*

$$W_n\left(\binom{n-2}{k-2}\phi(\mathfrak{A}_n(\underline{a};\underline{w}))+\binom{n-k}{n-1}\mathfrak{A}_n(\phi(\underline{a});\underline{w})\right) \tag{4}$$
$$\ge \frac{1}{k!}\sum_k !(w_{i_1}+\cdots+w_{i_k})\phi\left(\frac{a_{i_1}w_{i_1}+\cdots+a_{i_k}w_{i_k}}{w_{i_1}+\cdots+w_{i_k}}\right).$$

Inequality (3) with $r = 1$ is the case $n = 3, k = 2$ of (4); further in the case $\phi(x) = |x|$ (4) has been called *Djoković's inequality* and is equivalent to the Hlawka inequality; [*Djoković 1963; Takahasi, Takahashi & Honda; Takahasi, Takahashi & Wada*].

REMARK (i) The above results imply certain inequalities for mixed means due to Kober, II 3.5 Remark (iv), [*Kober; Mesihović; Ozeki 1973*].

6.3 p-MEAN CONVEXITY The main results of this chapter can be, and have been, derived from the properties of convex functions. It is possible to reverse this process.

If $-\infty \le p \le \infty$, $a, b \ge 0$ define

$$\mathfrak{m}_p(a,b) = \begin{cases} \mathfrak{M}^{[p]}(a,b) & \text{if } ab \ne 0,\\ 0 & \text{if either } a = 0 \text{ or } b = 0.\end{cases}$$

We then say that $f : \mathbb{R}^n \mapsto [0,\infty[$ is *p-mean convex* if:

$$f\big(\lambda\underline{x}+(1-\lambda)\underline{y}\big) \le \mathfrak{m}_p^{\lambda}\big(f(\underline{x}), f(\underline{y})\big),$$

for all $\underline{x}, \underline{y}$, and λ, $0 \le \lambda \le 1$. There are obvious definitions of *p-mean concavity, p-mean log-convexity* etc.; [*Das Gupta; Uhrin 1975,1982,1984*].

In particular Uhrin has proved the following theorem.

THEOREM 6 *If $p+q \ge 0$, if f is p-mean concave, and g is q-mean concave then the convolution $f\star g$, $f\star g(\underline{u}) = \int_{\mathbb{R}^n} f(\underline{u}-\underline{v})g(\underline{v})\mathrm{d}\underline{v}$, is $(p^{-1}+q^{-1}+r)^{-1}$-mean concave if $-\dfrac{1}{r} \le \dfrac{pq}{p+q} \le \infty$.*

Following the comments in II 1.2 Remark (ii) we can make a completely different definition and say that a positive function is *p-th mean convex, $p \in \mathbb{R}$, on $[a,b]$*[9] if for all $x, y, a \le x, y \le b$ and $\lambda, 0 \le \lambda \le 1$,

$$f(\lambda x+\overline{1-\lambda}y) \le \mathfrak{M}^{[p]}\big(f(x), f(y); \lambda, 1-\lambda\big);$$

equivalently if $p \ne 0$ this is just the statement that f^p is convex; if the inequality is reversed we say that f is *p-th mean concave on $[a,b]$*

6.4 VARIOUS RESULTS

[9] Here the terminology *p-convex* is more usual but when p is an integer this is confusing; see I 4.7.

THEOREM 7 *(a) If $\underline{a}$ is an n-tuple and the n-tuple $\underline{b}$ is defined by $b_1 = a_1, b_i = (a_i + a_{i-1})/2$, $2 \le i \le n$, then, with the notation of 4.1,*

$$\mathbb{Q}_n^{r,s}(\underline{b}) \le 2\mathbb{Q}_n^{r,s}(\underline{a}).$$

(b) If $n \ge 2$ and $\underline{a}$ is an n-tuple of distinct entries then

$$\mathfrak{M}_n^{[p]}(\underline{a}) \ge C_{p,n} \min_{\substack{1\le i,j\le n \\ i\ne j}} |a_i - a_j|,$$

where

$$C_{p,n} = \begin{cases} 2^{\frac{1}{p}-1}\sum_{k=1}^{(n-1)/2}(2k)^p, & \text{if } n \text{ is odd},\\ \min\{2^{\frac{1}{p}-1}, 1\}\sum_{k=1}^{n/2}(2k-1)^p, & \text{if } n \text{ is even.}\end{cases}$$

(c) Let $f : [0, a] \mapsto \mathbb{R}$ be $(k+1)$-convex, with $f^{(m)}(0) = 0$, $1 \le m \le k-1$, $\underline{a}$ a real n-tuple with entries in $[0, a]$, $\underline{w}$ an n-tuple then

$$f\big(\mathfrak{M}_n^{[k]}(\underline{a};\underline{w})\big) \le \mathfrak{A}_n\big(f(\underline{a});\underline{w}\big).$$

(d) Let $\underline{a}$ be an m-tuple, $r \in \mathbb{R}$, and define the sequence recursively by

$$a_n = \mathfrak{M}_m^{[r]}(a_{n-m}, \ldots, a_{n-1}), \quad n > m.$$

Then

$$\lim_{n\to\infty} a_n = \begin{cases} \left(\dfrac{m}{\binom{m+1}{2}}\right)^{1/r} \mathfrak{M}_m^{[r]}(i^{1/r}a_i, 1 \le i \le m), & \text{if } r \ne 0,\\ \big(\mathfrak{G}_m(a_i^i, 1 \le i \le m)\big)^{m/\binom{m+1}{2}}, & \text{if } r = 0.\end{cases}$$

□ (a) See [*Math. Lapok, Problem F 1930, 50 (1975), 11–12*].
(b) The case $n = 2$ is in [*Mitrinović, Newman & Lehmer*]; see also [*AI p.340*], The case $p > 1$ was stated without proof in [*Ozeki 1968*]; a proof was given in [*Mitrinović & Kalajdžić*]. A proof for $p < 1$ is given in [*Russell*].
(c) This is a generalization, by Vasić, of a result of Marković. A different proof has been given by Pečarić who also shows that the same inequality holds if $f(x^{1/k})$ is convex on $[0, a^k]$; [*Kečkić & Lacković; Marković; Pečarić 1980a; Vasić 1968*].
(d) See [*Elem. Math. 29 (1974), 15–16*]. □

THEOREM 8 *If $n \ge 2$ and $\underline{w}$ is an n-tuple, $W_n = 1$, $\underline{a}$ a real n-tuple with $m \le \underline{a} \le M$, and if $r < s$, $t \ne 0$, then*

$$\frac{1}{M^{(s-r)}}\left|\frac{r(r-t))}{s(r-t)}\right| \le \frac{\big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\big)^r - \big(\mathfrak{M}_n^{[t]}(\underline{a};\underline{w})\big)^r}{\big(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})\big)^s - \big(\mathfrak{M}_n^{[t]}(\underline{a};\underline{w})\big)^s} \le \frac{1}{m^{(s-r)}}\left|\frac{r(r-t))}{s(r-t)}\right|$$

REMARK (i) This result is due to Mercer and the proof uses his mean-value theorem, II 4.1 Lemma 3. If $s = 2, r = 1$ and $t \to 0$ the above inequalities reduce to those of II 4.1 Theorem 2(b); [*Mercer A 1999*]. Compare this result with the first part of 3.1.1 Theorem 2.

The following is an extension of Sierpiński's inequality, II 3.8 (57); [*Alzer, Ando & Nakamura*].

THEOREM 9 *If $\underline{a}, \underline{w}$ are a decreasing n-tuples, $n \ge 3$, then if $r > 0$,*

$$\frac{\left(\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\right)^{W_{n-1}}\left(\mathfrak{M}_n^{[-r]}(\underline{a};\underline{w})\right)^{w_n}}{\mathfrak{G}_n^{W_n}(\underline{a};\underline{w})} \ge 1, \tag{5}$$

with equality if and only if $\underline{a}$ is constant.

□ This follows from II 3.8 Theorem 27 by taking $x = 0$ and $x = r$. □

REMARK (ii) II 3.7 Corollary 28 is the case $t = 1$ of this theorem. As in that case we note that it is sufficient to assume that the n-tuples $\underline{a}, \underline{w}$ be similarly ordered, see II 3.8 Remark (viii).

REMARK (iii) Further interesting properties of the left-hand side of (5) can be found in [*Pečarić 1988; Pečarić & Raşa 1993*].

In the case of the special sequence $\underline{\nu} = \{1, 2, \ldots\}$ certain special results are obtainable of which the most interesting is perhaps the following.

THEOREM 10 *If $r > 0$ and $n \ge 1$ then*

$$\frac{n}{n+1} < \frac{\mathfrak{M}_n^{[r]}(\underline{\nu})}{\mathfrak{M}_{n+1}^{[r]}(\underline{\nu})} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{6}$$

□ The left-hand inequality is obtained by using the function $f(x) = x^r$ in I 4.1 Lemma 8. The same lemma with $f(x) = \log x$ gives the outer inequality; it remains to prove the right-hand inequality.

The right-hand inequality is obtained if we show that $v_n = \left(\sum_{i=1}^n i^r\right)/n(n!)^{r/n}$, $n = 1, 2, \ldots$ is an increasing sequence. Now define the two decreasing $n(n-1)$-tuples, by:

for $0 \le k \le n-1$, $a_{k(n-1)+1} = a_{k(n-1)+2} = \cdots = a_{k(n-1)+n-1} = \dfrac{(n-k)}{(n!)^{1/n}}$,

and for $0 \le k \le n-2$, $b_{kn+1} = b_{kn+2} = \cdots = b_{kn+n} = \dfrac{n-1-k}{\left((n-1)\right)^{1/(n-1)}}$.

Then $v_n \ge v_{n-1}$ is equivalent to

$$\sum_{i=1}^{n(n-1)} b_i^r \le \sum_{i=1}^{n(n-1)} a_i^r.$$

To see this it suffices, from I 4.1 Corollary 11, to prove that

$$\prod_{i=1}^{m} b_i \leq \prod_{i=1}^{m} a_i, \ 1 \leq m \leq n(n-1).$$

Now when $m = n(n-1)$ both sides of this last inequality are equal, and equal to 1.

For some unique k, j, $0 \leq k \leq n-1$, $1 \leq j \leq n-1$, $m = k(n-1)+j$ and then

$$\prod_{i=1}^{m} a_i = \frac{(n!)^{n-1}}{\big((n-k-1)!\big)^{n-1}} \frac{(n-k)^j}{\big(n!^{[k(n-1)+1]/n}\big)}.$$

Note that $m = k(n-1)+j = kn+(j-k)$; so there are two possibilities: (i) $j-k \geq 0$. (ii) $j-k < 0$, when we write $m = (k-1)n+n+j-k$.

Evaluating $\prod_{i=1}^{m} b_i$ in these two cases and comparing with the above value of $\prod_{i=1}^{m} a_i$ gives the result; for details see either the paper of Martins or that of Alzer. □

REMARK (iv) The outer inequality in (6) is called the *Minc-Sathre inequality*, [*DI p.85*], [*Minc &Sathre*]; the right-hand inequality is due to Martins, [*Martins*]. The final form above is due to Alzer, [*Alzer 1993d; Chan, Gao & Qi; Kuang; Qi 1999a, 2000a; Qi & Debnath 2000a; Sándor 1995b*].

REMARK (v) Considered as bounds on the central ratio the extreme terms are best possible as was pointed out by Alzer.

REMARK (vi) Martins has pointed out that in the case of $a_i = i^r$, $i \geq 1$, his inequality improves the equal weight case of (P), being just

$$\frac{\mathfrak{A}_{n+1}(\underline{a})}{\mathfrak{G}_{n+1}(\underline{a})} > \frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{G}_n(\underline{a})}.$$

This lead Alzer to further observe that the following improvement of (R) holds in this special case:

$$\mathfrak{A}_{n+1}(\underline{a}) - \mathfrak{G}_{n+1}(\underline{a}) > \mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a}).$$

The following result concerning the factorial function due to Alzer; [*Alzer 2000a*].

THEOREM 11 *There are two positive numbers α, β, $0.01 < \alpha < \beta < 11.3$, such that inequality*

$$\big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\big)! \leq \mathfrak{M}_n^{[r]}(\underline{a}!;\underline{w})$$

holds for all positive n-tuples $\underline{a}$, $\underline{w}$, with $W_n = 1$, if and only if $\alpha \le r \le \beta$. Whenever the inequality holds it is strict unless $\underline{a}$ in constant.

REMARK (vii) For the exact definition of the two constants α, β the reader should consult the reference.

REMARK (viii) The case $r = 1$ is just (J) applied to the strictly convex factorial function; I 4.1 Example(ii).

REMARK (ix) In the case of harmonic and geometric means, when r is outside the above interval $[\alpha, \beta]$, it is known that the inequality only holds for certain n-tuples $\underline{a}$; see the above reference.

A related factorial function inequality, also by Alzer, says that

$$\mathfrak{H}\big((x!)^2, (\frac{1}{x!})^2\big) \ge 1.$$

This generalizes a well-known inequality of Gautschi where the harmonic mean is replaced by the geometric mean, see [*DI p.84*], [*Alzer 1997c, 2000b*].

The following result is due to Mercer, [*Mercer A 2002*], generalizes II 5.9 (15).

THEOREM 12 *If $\underline{a}$ and $\underline{w}$ are n tuples with $\underline{a}$ not constant, $m \le \underline{a} \le M$, and $W_n = 1$ then, defining $M^{[t]} = M^{[t]}(M, m; \underline{a}) = \Big(m^t + M^t - \big(\mathfrak{M}_n^{[t]}(\underline{a}; \underline{w})\big)^t\Big)^{1/t}$, $t \in \mathbb{R}$, if $r, s \in \mathbb{R}$ with $r < s$,*

$$m < M^{[r]}(M, m; \underline{a}) < M^{[s]}(M, m; \underline{a}) < M$$

□ Using the substitutions in 1(3) and 1(4) it suffices to consider the inequality $M^{[1]}(M, m; \underline{a}) < M^{[t]}(M, m; \underline{a})$, $t > 1$. Further we can, without loss in generality, assume that $\underline{a}$ is increasing.

Let $V = M^{[t]}(M, m; \underline{a}) - M^{[1]}(M, m; \underline{a})$ then simple calculations show that: $\partial V/\partial a_i = w_i\Big(1 - \big(a_i/M^{[t]}\big)^{t-1}\Big)$, and $\partial |M^{[t]} - a_i|/\partial a_i = \pm\big(w_i a_i^{t-1}(M^{[t]})^{(1-t)/t} + 1\big)$, according as $a_i > M^{[t]}, a_i < M^{[t]}$.

Using these relations we can let the $a_i \le M^{[t]}$ tend to m and those $a_i \ge M^{[t]}$ tend to M, to get for some positive $W_1, W_2, W_1 + W_2 = 1$,

$$V > \big((1 - W_1)m^t + (1 - W_2)M^t\big)^{1/t} - \big((1 - W_1)m + (1 - W_2)M\big);$$

and so $V > 0$ by (r;s). □

REMARK (x) II 5.9 (15) is just $M^{[-1]}(M, m; \underline{a}) < M^{[0]}(M, m; \underline{a}) < M^{[1]}(M, m; \underline{a})$.

REMARK (xi) It is readily checked that $\lim_{t\to-\infty} M^{[t]}(M, m; \underline{a}) = m$, and that $\lim M^{[t]}(M, m; \underline{a}) = M$.

Finally we remark that if $r \in \mathbb{R}$ then $\mathfrak{M}_n^{[r]}(\underline{a})$, has the property of m-associativity, $1 \le m \le n$, see II 1.1. It has been proved that if two means, one defined on n-tuples and the other on m-tuples means, have this property and if both homogeneous and symmetric and one at least is an analytic function then the means are r-th power power means for some $r \in \mathbb{R}$; [*Kuczma 1993*].

IV QUASI-ARITHMETIC MEANS

The power means are defined using the convex, or concave, power, logarithmic and exponential functions. In this chapter means are defined using arbitrary convex and concave functions by a natural extension of the classical definitions and analogues of the basic results of the earlier chapters are investigated. First however we take up the problem of different convex functions defining the same means; the case of equivalent means. The generalizations (GA) and (r;s), their converses and the Rado-Popoviciu type extensions are studied under the topic of comparable means. The definition can be further extended although this leads to the topics of functional equations and functional inequalities so is not followed in detail.

1 Definitions and Basic Properties

1.1 THE DEFINITION AND EXAMPLES The power means $\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})$, $r \in \mathbb{R}$, defined in the previous chapter can be looked at in the following way.
If $r \in \mathbb{R}$ and $x > 0$ define

$$\phi(x) = \begin{cases} x^r, & \text{if } r \neq 0, \\ \log x, & \text{if } r = 0; \end{cases}$$

then

$$\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) = \phi^{-1}\Big(\frac{1}{W_n}\sum_{i=1}^n w_i\phi(a_i)\Big). \tag{1}$$

This suggests the following definition; [*DI pp.217-218; HLP pp.65–70*].

DEFINITION 1 *Let $[m, M]$ be a closed interval in $\overline{\mathbb{R}}$, and $\mathcal{M} : [m, M] \mapsto \overline{\mathbb{R}}$ be a continuous, strictly monotonic function; let $\underline{a}$ be a real n-tuple with $m < \underline{a} < M$, $\underline{w}$ be a non-negative n-tuple with $W_n \neq 0$; then the* quasi-arithmetic $\mathcal{M}$-mean of $\underline{a}$ with weight $\underline{w}$ *is*

$$\mathfrak{M}_n(\underline{a};\underline{w}) = \mathcal{M}^{-1}\left(\frac{1}{W_n}\sum_{i=1}^n w_i\mathcal{M}(a_i)\right). \tag{2}$$

The function M is said to generate, *to be a* generator *or to be a* generating function of the mean $\mathfrak{M}_n$.

In addition we will sometimes speak of the quasi-arithmetic mean, *or just the* $\mathcal{M}$-mean, *of* $\underline{a}$ *with weight* $\underline{w}$.

In the case of $n = 2$ the above definition has a simple geometric interpretation.

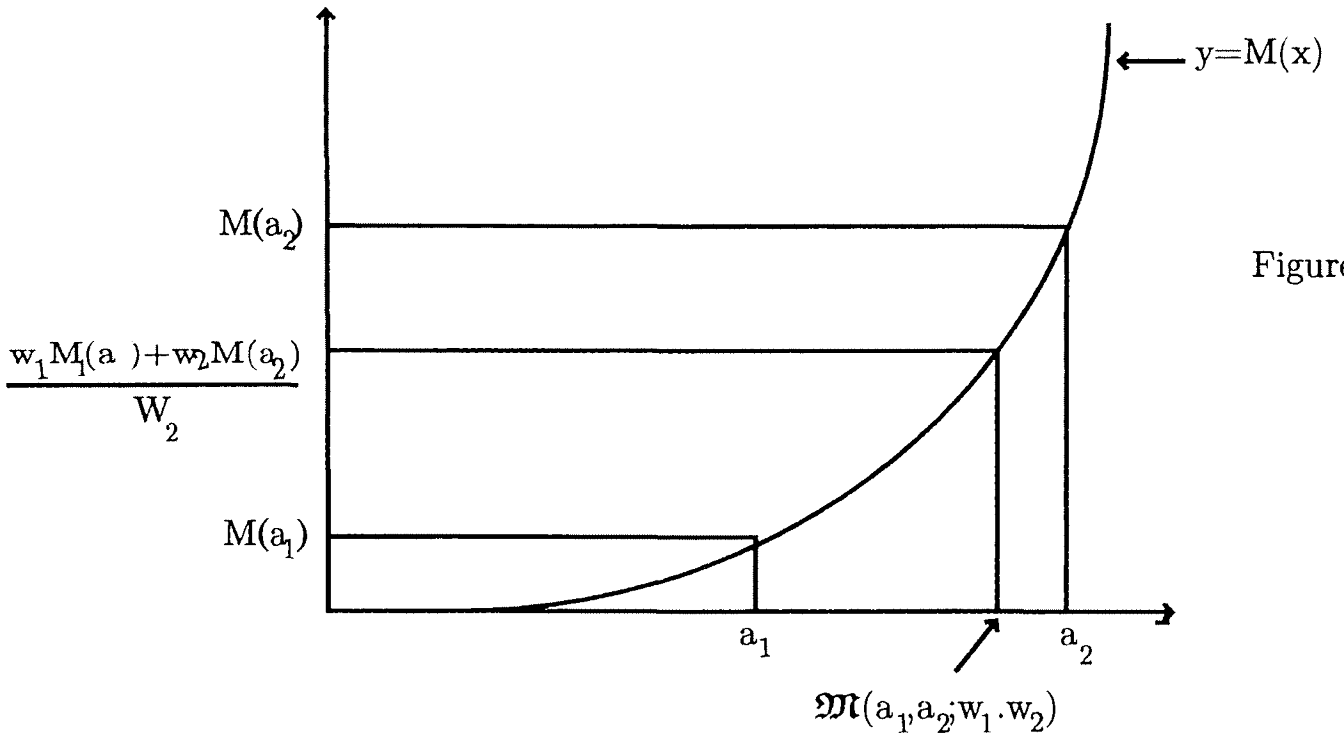

Figure 1

REMARK (i) The functions used to define the quasi-arithmetic means will be written $\mathcal{M}, \mathcal{N}, \mathcal{F}$ etc[1], and the corresponding means will then be $\mathfrak{M}, \mathfrak{N}, \mathfrak{F}$ etc. Letters such as $\mathfrak{A}, \mathfrak{G}, \mathfrak{H}, \mathfrak{Q}$ that already denote particular means will be avoided if there is any possibility of ambiguity. These functions are finite except perhaps at the end-points of their domains.

REMARK (ii) The function $\mathcal{M}^{-1}$ is continuous, strictly monotonic and defined on the closed interval $\Big[\min\{\mathcal{M}(m), \mathcal{M}(M)\}, \max\{\mathcal{M}(m), \mathcal{M}(M)\}\Big]$. Since we can write $\mathfrak{M}_n(\underline{a}; \underline{w}) = \mathcal{M}^{-1}\Big(\mathfrak{A}_n\big(\mathcal{M}(\underline{a}); \underline{w}\big)\Big)$ the internality of the arithmetic mean, II 1.2 Theorem 6, implies that $\big(\sum_{i=1}^{n} w_i \mathcal{M}(a_i)/W_n\big)$ is in the domain of $\mathcal{M}^{-1}$, and so the right-hand side of (2) is meaningful.

REMARK (iii) As usual $\mathfrak{M}_n(\underline{a})$ will indicate the mean with equal weights, and $\underline{a}$, and, or $\underline{w}$, may be omitted when there is no ambiguity; further $\mathfrak{M}_I(\underline{a}; \underline{w})$ will have the meaning expected when I is an index set; see Notations 6(xi), II 3.2.2, III 2.5.2.

REMARK (iv) As will be seen below, 1.2 Remark (ii), there is no loss in generality in assuming that $\mathcal{M}$ is strictly increasing, although in particular examples this is not always the case.

REMARK (v) Note that in this chapter we do not assume that the n-tuple $\underline{a}$ is

[1] Except in Figure 1 where M replaces $\mathcal{M}$.

positive[2], just that $m < \underline{a} < M$; $\underline{a}$ is always finite even if either m, or M, is not .

REMARK (vi) We also allow $\underline{w}$ to have zero elements. In general this allows the statement of inequalities to include the case of all $k, 1 \le k \le n$, in the statement of the $k = n$; then case of general k arises when $\underline{w}$ has $n - k$ zero elements. So there is a need to restate the cases of equality as the conditions need only apply to those elements of $\underline{a}$ with non-zero weights, that is to the essential elements of $\underline{a}$; see I 4.3, II 3.7.

In the following discussions all the n-tuples $\underline{a}, \underline{w}$, and the functions will be assumed to satisfy the various conditions in Definition 1. However in applications to particular cases and in the discussion of Rado-Popoviciu type results we will for simplicity take $\underline{w}$ to be positive. In addition in the case where several means are involved it is clear that the n-tuples $\underline{a}$ must be restricted to the intersection of the various intervals $[m, M]$ involved.

The following lemma states some simple properties of quasi-arithmetic means.

LEMMA 2 *(a) The quasi-arithmetic $\mathcal{M}$- mean has the properties: (Sy*), (Sy) in the essentially equal weight case, essentially internal, essentially reflexive, (Co) and, if $\mathcal{M}$ is increasing, (Mo).*
(b) There is a unique $\alpha, \min \underline{a} \le \alpha \le \max \underline{a}$ such that $\mathcal{M}(\alpha) = \frac{1}{W_n}\big(\sum_{i=1}^{n} w_i \mathcal{M}(a_i)\big)$. Furthermore unless $\underline{a}$ is essentially constant, some a_i is less than α, and some a_i is greater than α.
(c) If $n \ge 2$, $\underline{a}$ a fixed n-tuple and α is given with $\min \underline{a} \le \alpha \le \max \underline{a}$, then there is a non-negative n-tuple $\underline{w}$ with $W_n = 1$ such that $\mathfrak{M}_n(\underline{a}; \underline{w}) = \alpha$; further this $\underline{w}$ is unique if and only if $n = 2$.
(d) If $\underline{u}, \underline{v}$ are non-negative n-tuples with $U_n \ne 0, V_n \ne 0$ and if $\underline{u}/U_n \le \underline{v}/V_n$ then $\mathfrak{M}_n(\underline{a}; \underline{u}) \le \mathfrak{M}_n(\underline{a}; \underline{v})$.

□ All of (a) is trivial; the terminology in (a) is explained in II 3.7.
(b) This is immediate from Definition 1 and by noting that $\sum_{i=1}^{n} w_i\big(\mathcal{M}(\alpha) - \mathcal{M}(a_i)\big) = 0$, which implies that unless all the terms of this expression are zero some must be positive and some negative.
(c) This is an immediate consequence of the fact that, given $\underline{a}$, $\mathfrak{M}_n(\underline{a}; \underline{w})$ is a continuous function from the compact set of $\underline{w}$ to the closed interval $\big[\min \underline{a}, \max \underline{a}\big]$.
(d) This is trivial. □

[2] That is, Convention 3 of II 1.1 will not apply in this chapter.

REMARK (vii) This lemma justifies the use of the word mean and generalizes parts of II 1.1 Theorem 2, II 1.2 Theorem 6 and III 1 Theorem 2.

REMARK (viii) These means are discussed in detail in [*HLP pp.65–101*], [*Mitrinović & Vasić pp.104–113*], [*Aczél 1948a,1956a,1961a; Chisini 1929; de Finetti 1930,1931; Jecklin 1949a,b*].

From the remarks preceding Definition 1 the power means form an example of a scale of quasi-arithmetic means. Further examples of quasi-arithmetic means, and scales of quasi-arithmetic means, are given in the following examples.

EXAMPLE (i) [*Burrows & Talbot; Rennie 1991*] If $\gamma > 0$ and if $[m, M] = \overline{\mathbb{R}}$ let

$$\mathcal{M}(x) = \begin{cases} \gamma^x, & \text{if } \gamma \neq 1, \\ x, & \text{if } \gamma = 1. \end{cases}$$

Then $\mathfrak{M}_n(\underline{a}; \underline{w})$, written in this case $\mathfrak{M}_{\gamma,n}(\underline{a}; \underline{w})$, is

$$\mathfrak{M}_{\gamma,n}(\underline{a}; \underline{w}) = \begin{cases} \log_\gamma\Big(\dfrac{1}{W_n}\displaystyle\sum_{i=1}^n w_i\gamma^{a_i}\Big), & \text{if } \gamma \neq 1, \\ \mathfrak{A}_n(\underline{a}; \underline{w}), & \text{if } \gamma = 1. \end{cases}$$

Properties of these means are given below; see 2 Example(vi), 5 Example (iv).

EXAMPLE (ii) If $\gamma > 0$, $\gamma \neq 1$, taking $\mathcal{M}(x) = \gamma^{1/x}$ gives another family of means, called the *radical means*:

$$\mathfrak{M}_n(\underline{a}; \underline{w}) = \left(\log_\gamma\Big(\frac{1}{W_n}\sum_{i=1}^n w_i\gamma^{1/a_i}\Big)\right)^{-1}.$$

EXAMPLE (iii) If $\mathcal{M}(x) = x^x$, $x \geq e^{-1}$, we get the so-called *basis-exponential mean*; if its value is μ then

$$\mu^\mu = \frac{1}{W_n}\sum_{i=1}^n w_i a_i^{a_i}.$$

EXAMPLE (iv) If $\mathcal{M}(x) = x^{1/x}$, $x \geq e^{-1}$, then in a similar manner we get the *basis-radical mean*; if its value is ν then

$$\nu^{1/\nu} = \frac{1}{W_n}\sum_{i=1}^n w_i a_i^{1/a_i}.$$

REMARK (ix) The above means are some of the many introduced and used by the Italian school of statisticians; see [*Bonferroni 1923-4,1924-5,1927; Gini 1926; Pizzetti 1939; Ricci*].

The following results can be obtained.

THEOREM 3 *If $\underline{a}$ is a positive n-tuple then*

$$\sum_{i=1}^{n}(\mathfrak{A}_n(\underline{a}))^{a_i} \le \sum_{i=1}^{n} a_i^{a_i};\ \sum_{i=1}^{n} a_i^{1/a_i} \le \sum_{i=1}^{n}(\mathfrak{A}_n(\underline{a}))^{1/a_i};\ \mathfrak{M}_n^{[\mathfrak{A}_n(\underline{a})]}(\underline{a}) \le \frac{1}{n}\sum_{i=1}^{n} a_i^{a_i};$$

$$\sum_{i=1}^{n}(\mathfrak{M}_n^{[r]}(\underline{a}))^{a_i} \le \sum_{i=1}^{n} a_i, \text{ where } r < \max\{a_i;\ a_i < \mathfrak{M}_n^{[r]}(\underline{a})\}.$$

with equality if and only if $\underline{a}$ is constant.

□ See [*Pizetti 1939*]. □

EXAMPLE (v) Various *trigonometric means* have been studied; see in particular [*Bonferroni 1927; Jecklin 1953; Pratelli*].

If $0 \le \underline{a} \le \pi/2$ $\quad \mathfrak{S}_n(\underline{a};\underline{w}) = \arcsin\Big(\frac{1}{W_n}\sum_{i=1}^{n} w_i \sin a_i\Big)$,

$\quad \mathfrak{C}_n(\underline{a};\underline{w}) = \arccos\Big(\frac{1}{W_n}\sum_{i=1}^{n} w_i \cos a_i\Big)$;

if $0 \le \underline{a} < \pi/2$, $\quad \mathfrak{T}_n(\underline{a};\underline{w}) = \arctan\Big(\frac{1}{W_n}\sum_{i=1}^{n} w_i \tan a_i\Big)$;

and if $0 < \underline{a} \le \pi/2$, $\quad \mathfrak{CT}_n(\underline{a};\underline{w}) = \operatorname{arccot}\Big(\frac{1}{W_n}\sum_{i=1}^{n} w_i \cot a_i\Big)$.

The following result is due to Jecklin; [*Jecklin 1953*]; see 5 Example(xi); see also 2 Example(iii). No proof will be given as the theorem follows from more general results given later.

THEOREM 4 *Let $\underline{a}$ be a positive non-constant n-tuple and write $\alpha, \beta, \gamma, \delta$ for the solutions in $[0, \pi/2]$ of $\tan x = 1/\sqrt{2}$, $\cot x = 2x$, $\cot x = x$, $\tan x = 2x$, respectively and ϵ, ϕ for the solutions in $\mathbb{R}_+^*$ of $\cot x = 1/\sqrt{2}$, $\tan x = 2/x$ respectively. Then:*

$$\mathfrak{H}_n(\underline{a}) < \mathfrak{CT}_n(\underline{a}) < \mathfrak{G}_n(\underline{a}) < \mathfrak{S}_n(\underline{a}) < \mathfrak{T}_n(\underline{a}) < \mathfrak{C}_n(\underline{a}) < \mathfrak{Q}_n(\underline{a}), \qquad 0 < \underline{a} < \alpha;$$
$$\mathfrak{H}_n(\underline{a}) < \mathfrak{CT}_n(\underline{a}) < \mathfrak{G}_n(\underline{a}) < \mathfrak{S}_n(\underline{a}) < \mathfrak{C}_n(\underline{a}) < \mathfrak{T}_n(\underline{a}) < \mathfrak{Q}_n(\underline{a}), \qquad \alpha \le \underline{a} \le \beta;$$
$$\mathfrak{H}_n(\underline{a}) < \mathfrak{CT}_n(\underline{a}) < \mathfrak{G}_n(\underline{a}) < \mathfrak{S}_n(\underline{a}) < \mathfrak{C}_n(\underline{a}) < \mathfrak{Q}_n(\underline{a}) < \mathfrak{T}_n(\underline{a}), \qquad \beta \le \underline{a} \le \gamma;$$
$$\mathfrak{H}_n(\underline{a}) < \mathfrak{CT}_n(\underline{a}) < \mathfrak{S}_n(\underline{a}) < \mathfrak{G}_n(\underline{a}) < \mathfrak{C}_n(\underline{a}) < \mathfrak{Q}_n(\underline{a}) < \mathfrak{T}_n(\underline{a}), \qquad \gamma \le \underline{a} \le \epsilon;$$
$$\mathfrak{H}_n(\underline{a}) < \mathfrak{S}_n(\underline{a}) < \mathfrak{CT}_n(\underline{a}) < \mathfrak{G}_n(\underline{a}) < \mathfrak{C}_n(\underline{a}) < \mathfrak{Q}_n(\underline{a}) < \mathfrak{T}_n(\underline{a}), \qquad \gamma \le \underline{a} \le \epsilon;$$
$$\mathfrak{S}_n(\underline{a}) < \mathfrak{H}_n(\underline{a}) < \mathfrak{CT}_n(\underline{a}) < \mathfrak{G}_n(\underline{a}) < \mathfrak{C}_n(\underline{a}) < \mathfrak{Q}_n(\underline{a}) < \mathfrak{T}_n(\underline{a}), \qquad \phi \le \underline{a} \le \delta;$$
$$\mathfrak{S}_n(\underline{a}) < \mathfrak{H}_n(\underline{a}) < \mathfrak{G}_n(\underline{a}) < \mathfrak{CT}_n(\underline{a}) < \mathfrak{C}_n(\underline{a}) < \mathfrak{Q}_n(\underline{a}) < \mathfrak{T}_n(\underline{a}), \qquad \delta \le \underline{a} \le \pi/2.$$

REMARK (x) A somewhat related mean is the following defined in [*Ginalska*]. Let $\underline{a} \ge 1$ then the *exponential mean of $\underline{a}$* is: $\mathfrak{E}_n(\underline{a}) = \exp\big((\prod_{i=1}^{n}\log a_i)^{1/n}\big)$.
While it is not difficult to show that $\mathfrak{E}_n(\underline{a}) \le \mathfrak{G}_n(\underline{a})$, the exponential and harmonic means are not comparable.

1.2 EQUIVALENT QUASI-ARITHMETIC MEANS It is natural to ask how far the knowledge of the quasi-arithmetic $\mathcal{M}$-means determines the function $\mathcal{M}$; or, when do two functions generate the same mean? Equivalently: if for two functions $\mathcal{M}$ and $\mathcal{N}$ the quasi-arithmetic $\mathcal{M}$-mean of $\underline{a}$ with weight $\underline{w}$ is always equal to the quasi-arithmetic $\mathcal{N}$-mean of $\underline{a}$ with weight $\underline{w}$, is $\mathcal{M} = \mathcal{N}$?

This is answered by the following result due to Knopp and Jessen; see [*HLP pp.66–68*], [*Jessen 1931b; Knopp 1929*]; a more general result can be found in [*Bajraktarević 1958*].

THEOREM 5 *In order that $\mathfrak{M}_n(\underline{a};\underline{w}) = \mathfrak{N}_n(\underline{a};\underline{w})$, for all n and all appropriate n-tuples $\underline{a}$ in the common domain of $\mathcal{M}$ and $\mathcal{N}$ and all non-negative $\underline{w}$, $W_n \neq 0$, it is necessary and sufficient that $\mathcal{M} = \alpha\mathcal{N} + \beta$ for some $\alpha, \beta \in \mathbb{R}, \alpha \neq 0$.*

□ Suppose first that the condition holds, then:

$$\begin{aligned}\mathcal{M}\big(\mathfrak{M}_n(\underline{a};\underline{w})\big) &= \frac{1}{W_n}\Big(\sum_{i=1}^{n} w_i\mathcal{M}(a_i)\Big)\\ &= \frac{\alpha}{W_n}\Big(\sum_{i=1}^{n} w_i\mathcal{N}(a_i)\Big) + \beta\\ &= \alpha\mathcal{N}\big(\mathfrak{N}_n(\underline{a};\underline{w})\big) + \beta = \mathcal{M}\big(\mathfrak{N}_n(\underline{a};\underline{w})\big).\end{aligned}$$

So, by 1.1 Lemma 2(b), $\mathfrak{M}_n(\underline{a};\underline{w}) = \mathfrak{N}_n(\underline{a};\underline{w})$.

Now suppose that $\mathfrak{M}_2(\underline{a};\underline{w}) = \mathfrak{N}_2(\underline{a};\underline{w})$ for all appropriate 2-tuples $\underline{a}$ and $\underline{w}$, assuming, without loss in generality, that $w_1 + w_2 = 1$. That is

$$\mathcal{M}^{-1}\big(w_1\mathcal{M}(a_1) + w_2\mathcal{M}(a_2)\big) = \mathcal{N}^{-1}\big(w_1\mathcal{N}(a_1) + w_2\mathcal{N}(a_2)\big).$$

Putting $\phi = \mathcal{M}\circ\mathcal{N}^{-1}, x_i = \mathcal{N}(a_i), i = 1, 2$, this is equivalent to saying that for all (x_1, x_2),

$$\phi\big(w_1x_1 + w_2x_2\big) = w_1\phi(x_1) + w_2\phi(x_2).$$

Hence by a well known result, see [*Aczél 1966 p.49*], $\phi(x) = \alpha x + \beta$ for some $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$. Thus $\mathcal{M}\circ\mathcal{N}^{-1}(x) = \alpha x + \beta$, or $\mathcal{M} = \alpha\mathcal{N} + \beta$. □

REMARK (i) This actually proves more: *$\mathcal{M} = \alpha\mathcal{N} + \beta$ once $\mathfrak{M}_2(\underline{a};\underline{w}) = \mathfrak{N}_2(\underline{a};\underline{w})$ for all appropriate 2-tuples $\underline{a}$ and $\underline{w}$.*

REMARK (ii) An immediate consequence of this result is that $\mathcal{M}$ and $-\mathcal{M}$ define the same mean. Hence in 1.1 Definition 1 we can, without loss in generality, require that $\mathcal{M}$ be strictly increasing.

EXAMPLE (i) Using this result it is possible to fit the geometric mean into the family of power means in a neat manner. If $r \in \mathbb{R}$, define $\mathcal{F}^{[r]}(x) = \int_1^x t^{r-1}\,\mathrm{d}t$; that is

$$\mathcal{F}^{[r]}(x) = \begin{cases} \dfrac{x^r - 1}{r}, & \text{if } r \neq 0 \\ \log x, & \text{if } r = 0. \end{cases}$$

Then for $r = 0$ the quasi-arithmetic $\mathcal{F}^{[r]}$-mean is the geometric mean, and by Theorem 5, for all other values of r it is just the r-th power mean.

A similar procedure can be used to fit the mean $\mathfrak{M}_{1,n}$ into the $\mathfrak{M}_{\gamma,n}$ family of means; see 1.1 Example (i).

As a result of Theorem 5 it is possible to define a metric on the class of all continuous strictly increasing functions $\mathcal{M}$ on $[m, M]$ normalized by $\mathcal{M}(m) = m, \mathcal{M}(M) = M$. If $\mathcal{M}$ and $\mathcal{N}$ are two such normalized functions let $n \in \mathbb{N}^*$ be fixed and define

$$\rho(\mathcal{M}, \mathcal{N}) = \sup\{t;\ t = |\mathfrak{M}_n(\underline{a}; \underline{w}) - \mathfrak{N}_n(\underline{a}; \underline{w})|,\ \underline{w} > 0, m \le \underline{a} \le M\}.$$

This metric defines a metric space that is homeomorphic to the one using the sup norm; it is bounded but not totally bounded; see [*Cargo & Shisha 1969*].

As we saw above, the power means are the basic quasi-arithmetic means; we now show that they are the only homogeneous quasi-arithmetic means.

THEOREM 6 *Let $\mathcal{M} :]0, \infty[\mapsto \mathbb{R}$ be continuous and strictly monotonic and suppose that for all $k > 0$, and all positive n-tuples $\underline{a}$ and $\underline{w}$*

$$\mathfrak{M}_n(k\underline{a}; \underline{w}) = k\mathfrak{M}_n(\underline{a}; \underline{w}). \tag{3}$$

Then for some $r \in \mathbb{R}$ and all positive n-tuples $\underline{a}$ and $\underline{w}$, $\mathfrak{M}_n(\underline{a}; \underline{w}) = \mathfrak{M}_n^{[r]}(\underline{a}; \underline{w})$.

□ Clearly the power means satisfy (3).

By Theorem 5 we can assume that $\mathcal{M}(1) = 0$. By (3)

$$\mathfrak{M}_n(\underline{a}; \underline{w}) = k^{-1}\mathfrak{M}_n(k\underline{a}; \underline{w}) = \mathfrak{K}_n(\underline{a}; \underline{w}),$$

where $\mathcal{K}(x) = \mathcal{M}(kx)$.

By Theorem 5 there are $\alpha(k), \beta(k) \in \mathbb{R}$, $\alpha(k) \neq 0$, such that $\mathcal{K} = \alpha(k)\mathcal{M} + \beta(k)$.

Further since $\mathcal{M}(1) = 0$ we have that $\beta(k) = \mathcal{K}(1) = \mathcal{M}(k)$.

Hence for all positive x, y

$$\mathcal{M}(xy) = \alpha(y)\mathcal{M}(x) + \mathcal{M}(y). \tag{4}$$

Interchanging x, y and subtracting gives $\dfrac{\alpha(x)-1}{\mathcal{M}(x)} = \dfrac{\alpha(y)-1}{\mathcal{M}(y)}$, or $(\alpha - 1)/\mathcal{M}$ is a constant, c say.
Substituting this in (4) gives the following functional equation for $\mathcal{M}$:

$$\mathcal{M}(xy) = c\mathcal{M}(x)\mathcal{M}(y) + \mathcal{M}(x) + \mathcal{M}(y). \tag{5}$$

There are two cases to consider: $c = 0, c \neq 0$.
Case(i) $c = 0$
Then (5) reduces to

$$\mathcal{M}(xy) = \mathcal{M}(x) + \mathcal{M}(y), \tag{6}$$

of which the most general continuous solution is $\mathcal{M}(x) = A \log x$; see [*Aczél 1966 p.48*]. So, by Theorem 5 $\mathfrak{M}_n(\underline{a}; \underline{w}) = \mathfrak{G}_n(\underline{a}; \underline{w})$.
Case(ii) $c \neq 0$
Putting $\mathcal{N} = 1 + c\mathcal{M}$, (5) is equivalent to $\mathcal{N}(xy) = \mathcal{N}(x)\mathcal{N}(y)$, of which the most general continuous solution is $\mathcal{N}(x) = x^r$; see [*Aczél 1966 p. 48*].
So $\mathcal{M}(x) = (x^r - 1)/c$ and by Theorem 5 $\mathfrak{M}_n(\underline{a}; \underline{w}) = \mathfrak{M}_n^{[r]}(\underline{a}; \underline{w})$. □

Remark (iii) The above proof is a simplification of one due to Jessen, and can be found in [*HLP pp.68–69*]; [*Ballantine; de Finetti 1930,1931; Jessen 1931a; Nagumo 1930; Poveda*].

2 Comparable Means and Functions

We now turn our attention to generalizations of (r;s) that are valid for quasi-arithmetic means. The result, Theorem 5 below, provides yet another proof of (GA), as well as of (r;s); it seems to have first been proved by Jessen; [*HLP pp.69–70,75–76*], [*Dinghas 1968; Jecklin 1949a; Jessen 1931b*].

Definition 1 *The quasi-arithmetic $\mathcal{M}$-mean and the quasi-arithmetic $\mathcal{N}$-mean are said to be* comparable *when:*
(a) the functions $\mathcal{M}$ and $\mathcal{N}$ have the same domain;
(b) for all n and all appropriate n-tuples $\underline{a}$ and non-negative $\underline{w}$, $W_n \neq 0$, either

$$\mathfrak{M}_n(\underline{a}; \underline{w}) \leq \mathfrak{N}_n(\underline{a}; \underline{w}), \tag{1}$$

or for all n and all appropriate n-tuples $\underline{a}$ and non-negative $\underline{w}$, $W_n \neq 0$, (~1) holds.

Remark (i) Thus (GA) says that the arithmetic and geometric means are comparable, and (r;s) says that any two power means are comparable.

LEMMA 2 *If $\mathcal{M}$ and $\mathcal{N}$ are continuous functions on $[m, M]$, and if $\mathcal{N}$ is strictly monotonic and convex with respect to $\mathcal{M}$ then*

$$\mathcal{N}\big(\mathfrak{M}_n(\underline{a}; \underline{w})\big) \le \mathfrak{A}_n\big(\mathcal{N}(\underline{a}); \underline{w}\big) \tag{2}$$

for all n-tuples $\underline{a}$ such that $m < \underline{a} < M$, and all non-negative n-tuples $\underline{w}$ with $W_n \neq 0$. If $\mathcal{N}$ is strictly convex with respect to $\mathcal{M}$ there is equality in (2) if and only if $\underline{a}$ is essentially constant. If $\mathcal{N}$ is concave with respect to $\mathcal{M}$ ($\sim$2) holds.

□ Putting $\underline{b} = \mathcal{M}(\underline{a})$, $\phi = \mathcal{N} \circ \mathcal{M}^{-1}$ then ϕ is convex, see I 4.5.3, and (2) reduces to $\phi\big(\mathfrak{A}_n(\underline{b}; \underline{w})\big) \le \mathfrak{A}_n\big(\phi(\underline{b}); \underline{w}\big)$, which is just (J), I 4.2 Theorem 12. The rest of the theorem follows from the cases of equality in (J), and from the fact that if ϕ is concave then $-\phi$ is convex. □

REMARK (ii) In particular cases some care is needed in choosing the domain $[m, M]$. The function $\mathcal{N} \circ \mathcal{M}^{-1}$ has to be convex on $[\mathcal{M}(m), \mathcal{M}(M)]$, so assuming that $\mathcal{M}$ is increasing we must choose m and M so that $\mathcal{M}(m)$ and $\mathcal{M}(M)$ lie in the domain of convexity of $\mathcal{N} \circ \mathcal{M}^{-1}$.

REMARK (iii) This simple result has been rediscovered many times; see [*Bonferroni 1927; Bullen 1970b,1971b; Cargo 1965; Chimenti; Jecklin 1947; Lob; Shisha & Cargo; Watanabe*].

The hypotheses Lemma 2 are weaker than those of Theorem 5 below so this result can be used to deduce inequalities between means not obtainable from that theorem.

COROLLARY 3 *If $0 \le r < s < \infty$, $\underline{a}$ an n-tuple with $\underline{a} \ge \big((s-r)/(s+r)\big)^{r/s}$, $\underline{w}$ a positive n-tuple, then*

$$\frac{W_n}{1 + \big(\mathfrak{M}_n^{[r]}(\underline{a}; \underline{w})\big)^s} \le \sum_{i=1}^n \frac{w_i}{1 + a_i^s}, \tag{3}$$

with equality if and only if $\underline{a}$ is constant.

□ If $r \neq 0$ take $\mathcal{M}(x) = x^r$, $\mathcal{N}(x) = 1/(1+x^s)$ then $\mathcal{N} \circ \mathcal{M}^{-1}$ is strictly convex provided $0 < r < s$ and $x^{s/r} \ge (s-r)/(s+r)$. The result is then immediate from Lemma 2. If $r = 0$ the same argument applies taking $M(x) = \log x$. □

EXAMPLE (i) When $s = 1$ and $r = 0$ (3) becomes

$$\frac{W_n}{1 + \mathfrak{G}_n(\underline{a}; \underline{w})} \le \sum_{i=1}^n \frac{w_i}{1 + a_i}, \quad \underline{a} \ge 1; \tag{4}$$

and if $\underline{a} \leq 1$ ($\sim$4) holds. The equal weight case of (4) is due to Henrici and so (4) is known as *Henrici's inequality*.

In general inequality (4) cannot be improved by replacing the geometric mean by an r-th power mean, $r < 0$. However the following related result has been proved by Brenner & Alzer:

$$\sum_{i=1}^{n} \frac{w_i}{1-a_i} \begin{cases} \leq \dfrac{W_n}{1-\mathfrak{H}_n(\underline{a};\underline{w})} & \text{if } \underline{a} < 1, \\ \geq \dfrac{W_n}{1-\mathfrak{A}_n(\underline{a};\underline{w})} & \text{if } \underline{a} > 1; \end{cases}$$

also in this case the inequalities cannot be improved by using other power means in the denominators .

See [*AI p.212; DI pp.121–122*], [*Brenner & Alzer; Bullen 1969a; Kalajdžić 1973; Mitrinović & Vasić 1968c*].

The proof of Corollary 3 depends on the strict convexity of the special function $\mathcal{N} \circ \mathcal{M}^{-1}$ defined there. If then we consider any function f such that $1/(1+f)$ is strictly convex on $[m, M]$, (3) can be generalized as follows.

COROLLARY 4 *If the function f is such that $1/(1+f)$ is strictly convex on $[m, M]$, then*

$$\frac{W_n}{1+f\big(\mathfrak{A}_n(\underline{a};\underline{w})\big)} \leq \sum_{i=1}^{n} \frac{w_i}{1+f(a_i)}, \tag{5}$$

provided $m < \underline{a} < M$, and there is equality if and only if $\underline{a}$ is constant.

EXAMPLE (ii) Both Bonferroni and Giaccardi have used the strictly convex function $\mathcal{M}(x) = x \log x$, see I 4.1 Example (i), to deduce from Lemma 2 that

$$\mathfrak{A}_n(\underline{a};\underline{w}) \log \mathfrak{A}_n(\underline{a};\underline{w}) \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i a_i \log a_i;$$

[*Bonferroni 1927; Giaccardi 1955*].

EXAMPLE (iii) Using a definition in 1.1 Example (v) Lemma 2 can be used to prove that

$$\mathfrak{A}_n(\underline{a};\underline{w}) \leq \mathfrak{S}_n(\underline{a};\underline{w}),$$

provided that $0 \leq \underline{a} \leq \pi/2$; [*Lob*].

We now give the main result of this section.

THEOREM 5 *Let $\mathcal{M}$ and $\mathcal{N}$ be strictly monotonic functions on $[m, M]$, $\mathcal{N}$ strictly increasing, [respectively, decreasing], then for all n and all n-tuples $\underline{a}$, $m < \underline{a} < M$, and all non-negative n-tuples $\underline{w}$, $W_n \neq 0$,*

$$\mathfrak{M}_n(\underline{a}; \underline{w}) \leq \mathfrak{N}_n(\underline{a}; \underline{w}) \tag{6}$$

if and only if $\mathcal{N}$ is convex, [respectively, concave], with respect to $\mathcal{M}$
If $\mathcal{N}$ is strictly convex,[respectively, concave], with respect to $\mathcal{M}$ there is equality in (6) if and only if $\underline{a}$ is essentially constant.
If $\mathcal{N}$ is decreasing, [respectively, increasing], and $\mathcal{N}$ is convex, [respectively, concave], with respect to $\mathcal{M}$ then ($\sim$6) holds.

□ This is an immediate consequence of Lemma 2.

□

REMARK (iv) The cases of equality in Theorem 5, as well as conditions for ($\sim$6), can be obtained by a similar discussion.

COROLLARY 6 *The quasi-arithmetic $\mathcal{M}$-mean and quasi-arithmetic $\mathcal{N}$-mean, with $\mathcal{M}$ and $\mathcal{N}$ having a common domain, are comparable if and only if either $\mathcal{M}$ is convex with respect to $\mathcal{N}$, or $\mathcal{N}$ is convex with respect to $\mathcal{M}$.*

EXAMPLE (iv) In particular if $\mathcal{M}^{-1}$ is convex then $\mathfrak{M}_n(\underline{a}; \underline{w}) \leq \mathfrak{A}_n(\underline{a}; \underline{w})$, while if $\mathcal{M}$ is convex then $\mathfrak{M}_n(\underline{a}; \underline{w}) \geq \mathfrak{A}_n(\underline{a}; \underline{w})$.

EXAMPLE (v) If $\mathcal{M}(x) = x^r$, $\mathcal{N}(x) = x^s$, $rs \neq 0$, then $\mathcal{N} \circ \mathcal{M}^{-1}(x) = x^{s/r}$, which is strictly convex if $r < s$, so (6) is a generalization of these cases of (r;s). The cases when either $r = 0$ or $s = 0$ can be discussed similarly by taking $\mathcal{M}$ or $\mathcal{N}$ to be log.

EXAMPLE (vi) Similarly, for the means in 1.1 Example (i), taking $\mathcal{M}(x) = \mu^x$, $\mathcal{N}(x) = \nu^x$, $\mu, \nu \geq 0$, then $\mathcal{N} \circ \mathcal{M}^{-1}(x) = \nu^{\log_\mu x}$ which is strictly convex if $\nu > \mu$. So under this condition we have that

$$\mathfrak{M}_{\mu,n}(\underline{a}; \underline{w}) \leq \mathfrak{M}_{\nu,n}(\underline{a}; \underline{w}),$$

with equality if and only if $\underline{a}$ is constant; this result is in [*Bonferroni 1927; Pizzetti 1939*].

THEOREM 7 *Let $\mathcal{M}$ and $\mathcal{N}$ be strictly increasing functions on $[m, M]$, and suppose that for all 2-tuples $\underline{a}$, $m < \underline{a} < M$, and all non-negative 2-tuples $\underline{w}$, $W_2 \neq 0$,*

$\mathfrak{M}_2(\underline{a};\underline{w}) \le \mathfrak{N}_2(\underline{a};\underline{w})$ *then for all* n, $n \ge 2$, *and all* n*-tuples* $\underline{a}$, $m < \underline{a} < M$, *and all non-negative* n*-tuples* $\underline{w}$, $W_n \ne 0$, *inequality (6) holds.*

□ Suppose that (6) has been established for all n, $2 \le n \le k$.
We can write $\mathfrak{M}_{k+1}(\underline{a};\underline{w}) = \mathfrak{M}_k(\underline{\tilde{a}};\underline{\tilde{w}})$, where

$$\tilde{a}_i = \begin{cases} a_i, & \text{if } 1 \le i \le k-1, \\ \mathfrak{M}_2\Big(a_k, a_{k+1}; \dfrac{w_k}{w_k + w_{k+1}}, \dfrac{w_{k+1}}{w_k + w_{k+1}}\Big), & \text{if } i = k; \end{cases}$$

and

$$\tilde{w}_i = \begin{cases} w_i, & \text{if } 1 \le i \le k-1, \\ w_k + w_{k+1}, & \text{if } i = k; \end{cases}$$

The hypothesis of the theorem implies that $\underline{\tilde{a}} \le \underline{a}^*$ where

$$a_i^* = \begin{cases} \tilde{a}_i, & \text{if } 1 \le i \le k-1, \\ \mathfrak{N}_2\Big(a_k, a_{k+1}; \dfrac{w_k}{w_k + w_{k+1}}, \dfrac{w_{k+1}}{w_k + w_{k+1}}\Big), & \text{if } i = k; \end{cases}$$

So by monotonicity, 1 Lemma 2(a), and the induction hypothesis,

$$\mathfrak{M}_{k+1}(\underline{a};\underline{w}) = \mathfrak{M}_k(\underline{\tilde{a}};\underline{\tilde{w}}) \le \mathfrak{M}_k(\underline{a}^*;\underline{\tilde{w}}) \le \mathfrak{N}_k(\underline{a}^*;\underline{\tilde{w}}) = \mathfrak{N}_{k+1}(\underline{a};\underline{w}).$$

□

COROLLARY 8 *Let* $\mathcal{M}, \mathcal{N}$ *be strictly monotonic functions defined on* $[m, M]$ *and let* $\tilde{\mathcal{M}} = \alpha\mathcal{M} + \beta, \tilde{\mathcal{N}} = \gamma\mathcal{N} + \delta$, *where* $\alpha, \beta, \gamma, \delta$ *are real numbers chosen so that* $\tilde{\mathcal{M}}(m) = \tilde{\mathcal{N}}(m) = 0, \tilde{\mathcal{M}}(M) = \tilde{\mathcal{N}}(M) = 1$. *Then inequality (6) holds if and only if the graph of* $\tilde{\mathcal{M}}$ *lies above the graph of* $\tilde{\mathcal{N}}$; *equivalently, if and only if for all* a_1, a_2, $m < a_1, a_2 < M$, *and positive* w_1, w_2 *with* $w_1 + w_2 = 1$,

$$\frac{w_1\mathcal{M}(a_1) + w_2\mathcal{M}(a_2) - \mathcal{M}(w_1a_1+w_2a_2)}{\mathcal{M}(a_2) - \mathcal{M}(a_1)} \le \frac{w_1\mathcal{N}(a_1) + w_2\mathcal{N}(a_2) - \mathcal{N}(w_1a_1+w_2a_2)}{\mathcal{N}(a_2) - \mathcal{N}(a_1)}.$$

□ By 1.2 Theorem 5 the means in (6) are given by $\tilde{\mathcal{M}}, \tilde{\mathcal{N}}$, as well as by $\mathcal{M}, \mathcal{N}$; and by Theorem 7 it is sufficient to consider the case of $n = 2$. However in this case the result is immediate using the geometric interpretation in Figure 1. □

Other necessary and sufficient conditions for (6) have been given in [*Mikusiński*]. The first is that $\mathcal{N}$ *be less log-convex than* $\mathcal{M}$ in the sense that for all distinct a_1, a_2

$$\frac{\Delta^2\mathcal{M}(a_1, a_2)}{\Delta\mathcal{M}(a_1, a_2)} \ge \frac{\Delta^2\mathcal{N}\mathcal{M}(a_1, a_2)}{\Delta\mathcal{N}(a_1, a_2)},$$

where:
$\Delta\mathcal{M}F(a_1,a_2)=\mathcal{M}(a_2)-\mathcal{M}(a_1)$, $\Delta^2\mathcal{M}(a_1,a_2)=\dfrac{\mathcal{M}(a_1)+\mathcal{M}(a_2)}{2}-\mathcal{M}(\dfrac{a_1+a_2}{2})$.
The second condition is given in I 4.5.3 Theorem 34 : $\dfrac{\mathcal{M}''}{\mathcal{M}'}\geq\dfrac{\mathcal{N}''}{\mathcal{N}'}$; see also [*Cargo 1965*].

A simple generalization of Theorem 5 is the following result.

THEOREM 9 *If N is strictly increasing then*

$$f\big(\mathfrak{M}_n(\underline{a};\underline{w})\big)\leq\mathfrak{N}_n\big(f(\underline{a});\underline{w}\big)$$

if and only if $N\circ f\circ M^{-1}$ is convex; further if this last function is strictly convex this inequality is strict unless $\underline{a}$ is essentially constant.
In particular

$$f\big(\mathfrak{A}_n(\underline{a};\underline{w})\big)\leq\mathfrak{G}_n\big(f(\underline{a});\underline{w}\big)$$

if and only if f is log-convex.

□ The first part is immediate by applying (J) to the n-tuple $\mathcal{M}(\underline{a})$ using the function $N\circ f\circ M^{-1}$. The second part then follows on taking $\mathfrak{N}_n=\mathfrak{G}_n$, and $\mathfrak{M}_n=\mathfrak{A}_n$. □

COROLLARY 10 *(a) If $\underline{a}$ and $\underline{w}$ are positive n-tuples then*

$$\big(\mathfrak{A}_n(\underline{a};\underline{w})\big)^{\mathfrak{A}_n(\underline{a};\underline{w})}\leq\mathfrak{G}_n(\underline{a}^{\underline{a}};\underline{w}),$$

with equality if and only if $\underline{a}$ is constant.
(b) [SHANNON'S INEQUALITY] *If $\underline{p}$ and $\underline{q}$ are positive n-tuples with $P_n=Q_n=1$ then*

$$\sum_{i=1}^{n}q_i\log p_i\leq\sum_{i=1}^{n}q_i\log q_i.$$

□ (a) Since $x\log x=\log x^x$ is strictly convex, see above Example (ii), this result follows from the second part of Theorem 9.
(b) Take $\underline{w}=\underline{q},\underline{a}=\underline{p}/\underline{q}$ in (a). A direct proof can also be given by noting that the inequality is just

$$0\leq\sum_{i=1}^{n}p_i(q_i/p_i)\log(q_i/p_i),$$

which is an immediate consequence of (J) and the convexity of $x\log x$. □

REMARK (v) It is easy to see that Shannon's inequality holds with log replaced by $\log_b$ for any base $b>1$. The inequality is fundamental to the notion of entropy; [*MPF pp.635–650*], [*Rassias pp.127–164*], [*Mond & Pečarić 2001*].

The concept of comparability can easily be extended to functions; see [*HLP pp.5–6*].

DEFINITION 11 *If A is a subset of $\mathbb{R}^n$ then the functions $f, g : A \mapsto \overline{\mathbb{R}}$ are said to be* comparable *if either for all $\underline{a} \in A$, $f(\underline{a}) \le g(\underline{a})$, or for all $\underline{a} \in A$, $f(\underline{a}) \ge g(\underline{a})$.*

EXAMPLE (vii) The quasi-arithmetic $\mathcal{M}$-mean and the quasi-arithmetic $\mathcal{N}$-mean are comparable if and only if the two functions of $2n$ variables

$$m(\underline{a}, \underline{w}) = \mathcal{M}^{-1}\big(\sum_{i=1}^{n} w_i\mathcal{M}(a_i)\big), \quad n(\underline{a}, \underline{w}) = \mathcal{N}^{-1}\big(\sum_{i=1}^{n} w_i\mathcal{N}(a_i)\big),$$

are comparable.

EXAMPLE (viii) The basic result III 3.1.3 Theorem 7(a) just says that for all $r \in \overline{\mathbb{R}}$ the function $\mathfrak{M}_n^{[r]}(\underline{a} + \underline{b}; \underline{w})$ is comparable to the function $\mathfrak{M}_n^{[r]}(\underline{a}; \underline{w}) + \mathfrak{M}_n^{[r]}(\underline{b}; \underline{w})$. On the other hand III 3.1.4 Theorem 9 shows that $\mathfrak{M}_n^{[r]}(\underline{a}\,\underline{b}; \underline{w})$ is not in general comparable to $\mathfrak{M}_n^{[r]}(\underline{a}; \underline{w})\mathfrak{M}_n^{[r]}(\underline{b}; \underline{w})$.

We can rephrase III 3.1.2 Corollary 5 as a theorem concerning comparability.

THEOREM 12 *If $r_i > 0$, $0 \le i \le m$, then the functions $\mathfrak{M}_n^{[r_0]}\big(\prod_{i=1}^{m} \underline{a}^{(i)}; \underline{w}\big)$ and $\prod_{k=1}^{m} \mathfrak{M}_n^{[r_i]}(\underline{a}^{(i)}; \underline{w})$, defined on the non-negative non-zero n-tuples $\underline{a}, \underline{w}$, are comparable if and only if $1/r_0 \ge \sum_{i=1}^{m} 1/r_i$, when*

$$\mathfrak{M}_n^{[r_0]}\big(\prod_{i=1}^{m} \underline{a}^{(i)}; \underline{w}\big) \le \prod_{i=1}^{m} \mathfrak{M}_n^{[r_i]}(\underline{a}^{(i)}; \underline{w}). \tag{7}$$

□ Taking $\underline{a}^{(i)} = (1, 0, \ldots, 0)$, $1 \le i \le m$, in (7) implies that $1/r_0 \ge \sum_{i=1}^{m} 1/r_i$. The converse is immediate from III 3.1.2 Corollary 5 and (r;s). □

Properties of analogous generalizations of the weighted power sums defined in III 2.3 Remark (iii) have been considered; see [*Vasić & Pečarić 1980a*].

If $\underline{a}, \underline{w}$ and $\mathcal{M}$ are as in 1.1 Definition 1 define

$$\mathcal{S}_{\mathcal{M}}(\underline{a}; \underline{w}) = \mathcal{M}^{-1}\big(\sum_{i=1}^{n} w_i\mathcal{M}(a_i)\big), \tag{8}$$

under the conditions: (i) $\underline{w} \ge 1$,(ii) $\mathcal{M} :]0, \infty[\mapsto]0, \infty[$, (iii) either $\lim_{x\to\infty} \mathcal{M}(x) = \infty$ or $\lim_{x\to 0} \mathcal{M}(x) = \infty$.

THEOREM 13 *$\mathcal{S}_{\mathcal{M}}$ and $\mathcal{S}_{\mathcal{N}}$ are comparable if either (a) $\mathcal{M}$ and $\mathcal{N}$ are strictly monotonic in opposite senses, or (b) $\mathcal{M}$ and $\mathcal{N}$ are strictly monotonic in the same sense, and $\mathcal{M}/\mathcal{N}$ is monotonic. If in case (a) $\mathcal{M}$ is increasing, or in case (b) $\mathcal{M}/\mathcal{N}$ is decreasing then*

$$\mathcal{S}_{\mathcal{N}}(\underline{a}; \underline{w}) \le \mathcal{S}_{\mathcal{M}}(\underline{a}; \underline{w})$$

□ The proof is along the lines of [*HLP Theorem 105, pp.84-85*]. □

REMARK (vi) Further properties of these sums are given below in 5 Theorems 9 and 10.

3 Results of Rado-Popoviciu Type

Inequality 2(6) is, in a certain sense, an ultimate generalization of (GA), and it is natural to ask if certain refinements of (GA), such as were discussed in II 3, have extensions to quasi-arithmetic means. If so they should include as special cases not only the results in II 3 but also those in III 3.2.
In this section such extensions are considered, and we also discuss certain particular cases that could have been proved using the methods of the earlier chapters.

3.1 SOME GENERAL INEQUALITIES

THEOREM 1 *Let $[m, M]$ be a closed interval in $\overline{\mathbb{R}}$, and $\mathcal{M} : [m, M] \mapsto \overline{\mathbb{R}}$ be a continuous, strictly monotonic function, and $\mathcal{N} : [m, M] \mapsto \overline{\mathbb{R}}$ be continuous and convex with respect to $\mathcal{M}$; let $\underline{a}$ be a sequence, $m < \underline{a} < M$, and $\underline{w}$ a positive sequence. For all index sets I define*

$$\alpha(\mathcal{N}, \mathcal{M}; \underline{w}; I) = \alpha(I) = W_I \mathcal{N}\big(\mathfrak{M}_I(\underline{a}; \underline{w})\big).$$

Then α is sub-additive, that is if I, J are two non-intersecting index sets then

$$\alpha(I \cup J) \le \alpha(I) + \alpha(J). \tag{1}$$

Further if $\mathcal{N}$ is strictly convex with respect to $\mathcal{M}$ inequality (1) is strict unless $\mathcal{N}\big(\mathfrak{M}_I(\underline{a}; \underline{w})\big) = \mathcal{N}\big(\mathfrak{M}_J(\underline{a}; \underline{w})\big)$. If $\mathcal{N}$ is concave with respect to M then (~1) holds.

□ This is an immediate consequence of (J), and the convexity of $\mathcal{N} \circ \mathcal{M}^{-1}$.□

COROLLARY 2 *Let $[m, M]$ be a closed interval in $\overline{\mathbb{R}}$, and $\mathcal{M}, \mathcal{N}, \mathcal{K}, \mathcal{L} : [m, M] \mapsto \overline{\mathbb{R}}$ be a continuous, with $\mathcal{M}$ and $\mathcal{K}$ strictly monotonic, $\mathcal{N}$ convex with respect to $\mathcal{M}$, $\mathcal{L}$ concave with respect to $\mathcal{K}$; let $\underline{a}$ be a sequence, $m < \underline{a} < M$, and $\underline{v}, \underline{w}$ positive sequences. For all index sets I define*

$$\beta(I) = \alpha(\mathcal{N}, \mathcal{M}; \underline{v}; I) - \alpha(\mathcal{L}, \mathcal{K}; \underline{w}; I),$$

where α is as in Theorem 1. Then β is sub-additive, that is If I, J are two non-intersecting index sets

$$\beta(I \cup J) \le \beta(I) + \beta(J). \tag{2}$$

If $\mathcal{N}$ is strictly convex with respect to $\mathcal{M}$, and $\mathcal{L}$ is strictly concave with respect to $\mathcal{K}$ then (2) is strict unless (a) $\mathcal{N}\big(\mathfrak{M}_I(\underline{a};\underline{v})\big) = \mathcal{N}\big(\mathfrak{M}_J(\underline{a};\underline{v})\big)$ and (b) $\mathcal{L}\big(\mathfrak{K}_I(\underline{a};\underline{w})\big) = \mathcal{L}\big(\mathfrak{K}_J(\underline{a};\underline{w})\big)$. If $\mathcal{N}$ is strictly convex with respect to $\mathcal{M}$, and if for some $a, b \in \mathbb{R}$, $\mathcal{L} = a\mathcal{K} + b$ then equality occurs in (2) if and only if (a) holds; and if $\mathcal{N} = a\mathcal{M} + B$ and $\mathcal{L}$ is strictly concave with respect to $\mathcal{K}$ then equality occurs if and only if (b) holds.

□ This is a trivial consequence of Theorem 1. □

REMARK (i) These results, that are almost trivial consequences of (J), include as special cases many of the complicated inequalities discussed in II 3 and III 3.2; [*Bullen 1965,1970b,1971b; Mitrinović & Vasić 1968d*].

REMARK (ii) If in Corollary 2 we take $\mathcal{N}$ concave with respect to $\mathcal{M}$ and $\mathcal{L}$ convex with respect to $\mathcal{K}$ then ($\sim$2) holds and the cases of equality are easily stated.
As with (R) and (P) the simplest cases of the above result occurs when $I = \{1, \ldots, n\}$ and $J = \{n+1, \ldots, n+m\}$, $n, m \geq 1$. Then if as in II 3.2.2 we put $\underline{a} = (a_1, \ldots, a_{n+m})$ and $\tilde{\underline{a}} = (a_{n+1}, \ldots, a_{n+m})$, and use a natural extension of the notation of II 3.2.2(14) we have the following deduction from Corollary 2.

COROLLARY 3 *With the assumptions of Corollary 2, and the above notations,*

$$\begin{aligned} &V_{n+m}\mathcal{N}\big(\mathfrak{M}_{n+m}(\underline{a};\underline{w})\big) - W_{n+m}\mathcal{L}\big(\mathfrak{K}_{n+m}(\underline{a};\underline{w})\big) \\ &\leq \Big(V_n\mathcal{N}\big(\mathfrak{M}_n(\underline{a};\underline{w})\big) - W_n\mathcal{L}\big(\mathfrak{K}_n(\underline{a};\underline{w})\big)\Big) + \Big(\tilde{V}_m\mathcal{N}\big(\tilde{\mathfrak{M}}_m(\underline{a};\underline{w}) - \tilde{W}_m\mathcal{L}\big(\tilde{\mathfrak{K}}_m(\underline{a};\underline{w})\big)\Big). \end{aligned} \tag{3}$$

In particular

$$\begin{aligned} &V_n\mathcal{N}\big(\mathfrak{M}_n(\underline{a};\underline{w})\big) - W_n\mathcal{L}\big(\mathfrak{K}_n(\underline{a};\underline{w})\big) \\ &\quad \leq \Big(V_{n-1}\mathcal{N}\big(\mathfrak{M}_{n-1}(\underline{a};\underline{w}) - W_{n-1}\mathcal{L}\big(\mathfrak{K}_{n-1}(\underline{a};\underline{w})\Big) + \big(v_n\mathcal{N}(a_n) - w_n\mathcal{L}(a_n)\big). \end{aligned} \tag{4}$$

REMARK (iii) The cases of equality in Corollary 3 are easy to state as conditions (a) and (b) of Corollary 2 reduce to:
(a)′ $\mathcal{N}\big(\mathfrak{M}_n(\underline{a};\underline{v})\big) = \mathcal{N}\big(\tilde{\mathfrak{M}}_m(\underline{a};\underline{v})\big)$, (b)′ $\mathcal{L}\big(\mathfrak{K}_n(\underline{a};\underline{w})\big) = \mathcal{L}\big(\tilde{\mathfrak{K}}_m(\underline{a};\underline{w})\big)$.
The following general inequality extends the Mitrinović & Vasić result that contains extra parameters, II 3.2.1 Theorem 5; see [*Bullen 1970b,1971b*].

THEOREM 4 *Let $\underline{a}, \underline{v}, \underline{w}, \mathcal{M}, \mathcal{N}, \mathcal{K}, \mathcal{L}$ be as in Corollary 2, and let $\mu, \nu \in \mathbb{R}$, then*

$$\begin{aligned} &W_n\Big(\mathcal{N} \circ \mathcal{M}^{-1}\Big(\lambda\mathfrak{A}_n\big(\mathcal{M}(\underline{a});\underline{v}\big) + \mu\Big) - \mathcal{L} \circ \mathcal{K}^{-1}\Big(\mathfrak{A}_n\big(\mathcal{K}(\underline{a});\underline{w}\big) + \nu\Big)\Big) \\ &\quad \leq W_{n-1}\Big(\mathcal{N} \circ \mathcal{M}^{-1}\Big(\lambda'\mathfrak{A}_{n-1}\big(\mathcal{K}(\underline{a});\underline{v}\big) + \mu'\Big) - L \circ \mathcal{K}^{-1}\Big(\mathfrak{A}_{n-1}\big(\mathcal{K}(\underline{a});\underline{w}\big) + \nu'\Big)\Big) \end{aligned} \tag{5}$$

where

$$\lambda = \frac{V_n w_n}{v_n W_n} \frac{\mathcal{M} \circ \mathcal{N}^{-1} \circ \mathcal{L}(a_n)}{\mathcal{M}(a_n)}, \quad \lambda' = \frac{W_n V_{n-1}}{V_n W_{n-1}} \lambda, \quad \mu' = \frac{W_n}{W_{n-1}} \mu, \quad \nu' = \frac{W_n}{W_{n-1}} \nu.$$

If $\mathcal{N}$ is strictly convex with respect to $\mathcal{M}$, and $\mathcal{L}$ strictly concave with respect to $\mathcal{K}$ then (5) is strict unless (a) $\mathcal{M}(a_n) = \dfrac{W_n V_{n-1}}{V_n W_{n-1}} \mathfrak{A}_{n-1}(\mathcal{M}(\underline{a}); \underline{v}) + \mu'$, and (b) $\mathcal{K}(a_n) = \mathfrak{A}_{n-1}(\mathcal{K}(\underline{a}); \underline{w}) + \nu'$.

□ Since $\mathcal{L} \circ \mathcal{K}^{-1}$ is concave,

$$\begin{aligned} W_n \mathcal{L} \circ \mathcal{K}^{-1} &\Big(\mathfrak{A}_n(\mathcal{K}(\underline{a}); \underline{w}) + \nu \Big) \\ &= W_n \mathcal{L} \circ \mathcal{K}^{-1} \left(\frac{W_{n-1}}{W_n} \Big(\mathfrak{A}_{n-1}(\mathcal{K}(\underline{a}); \underline{w}) + \nu' \Big) + \frac{w_n}{W_n} \mathcal{K}(a_n) \right) \\ &\geq W_{n-1} \mathcal{L} \circ \mathcal{K}^{-1} \Big(\mathfrak{A}_{n-1}(\mathcal{K}(\underline{a}); \underline{w}) + \nu' \Big) + w_n \mathcal{L}(a_n). \end{aligned}$$

Since $\mathcal{N} \circ \mathcal{M}^{-1}$ is convex

$$\begin{aligned} W_n \mathcal{N} \circ \mathcal{M}^{-1} &\Big(\lambda \mathfrak{A}_n(\mathcal{M}(\underline{a}); \underline{v}) + \mu \Big) \\ &= W_n \mathcal{N} \circ \mathcal{M}^{-1} \left(\frac{W_{n-1}}{W_n} \Big(\lambda' \mathfrak{A}_{n-1}(\mathcal{M}(\underline{a}); \underline{v}) + \mu' \Big) + \frac{w_n}{W_n} \mathcal{M} \circ \mathcal{N}^{-1} \circ \mathcal{L}(a_n) \right) \\ &\leq W_{n-1} \mathcal{N} \circ \mathcal{M}^{-1} \Big(\lambda' \mathfrak{A}_{n-1}(\mathcal{M}(\underline{a}); \underline{v}) + \mu' \Big) + w_n \mathcal{L}(a_n). \end{aligned}$$

Inequality (5) is now immediate, as are the cases of equality. □

REMARK (iv) If $\lambda = 1$, $\mu = \nu = 0$, $\underline{v} = \underline{w}$, $\mathcal{N} = \mathcal{L}$ then (5) reduces to a special case of (4)—the simple case in which the last two terms on the right-hand side are absent.

3.2 SOME APPLICATIONS OF THE GENERAL INEQUALITIES We first state some results of the type considered in II 3.2.2 and III 3.2.4 .

THEOREM 5 *(a) If $r/t \leq 1$ then $\mu(I) = W_I\big(\mathfrak{M}_I^{[r]}(\underline{a}; \underline{w})\big)^t$ is a sub-additive function on the index sets.*
(b) If $r/t \leq 1 \leq s/u$ then $\nu(I) = V_I\big(\mathfrak{M}_I^{[r]}(\underline{a}; \underline{v})\big)^t - W_I\big(\mathfrak{M}_I^{[s]}(\underline{a}; \underline{w})\big)^u$ is a sub-additive function on the index sets.
(c) If $r \leq 0 \leq s$ then $\rho(I) = \big(\mathfrak{M}_I^{[r]}(\underline{a}; \underline{v})\big)^{V_I} \Big/ \big(\mathfrak{M}_I^{[s]}(\underline{a}; \underline{w})\big)^{W_I}$ is a log-subadditive function on the index sets.

□ (a) In 3.1 Theorem 1 take $\mathcal{N}(x) = x^t$, $\mathcal{M}(x) = x^r, r \neq 0, \mathcal{M}(x) = \log x$, $r = 0$.
(b) In 3.1 Corollary 2 take $\mathcal{M}$, $\mathcal{N}$ as in (a), and $\mathcal{L}(x) = x^u$, $\mathcal{K}(x) = x^s$.

(c) In 3.1 Corollary 2 take $\mathcal{M}(x) = x^r$, $\mathcal{K}(x) = x^s$ and $\mathcal{N}(x) = \mathcal{L}(x) = \log x$. □

Using 3.1 Corollary 3 we now give generalizations of II 3.2.2 Theorem 8 and III 3.2.4 Theorem 19; the notation is defined in those references and is also used in 3.1 Corollary 3 above.

THEOREM 6 *(a) If $r/t \le 1 \le s/u$ then*

$$\begin{aligned}&W_{n+m}\big(\mathfrak{M}_{n+m}^{[s]}(\underline{a};\underline{w})\big)^u - V_{n+m}\big(\mathfrak{M}_{n+m}^{[r]}(\underline{a};\underline{v})\big)^t\\ &\quad\ge W_n\big(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})\big)^u - V_n\big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{v})\big)^t + \tilde{W}_m\big(\tilde{\mathfrak{M}}_m^{[s]}(\underline{a};\underline{w})\big)^u - \tilde{V}_m\big(\tilde{\mathfrak{M}}_m^{[r]}(\underline{a};\underline{v})\big)^t,\end{aligned}$$

with equality if and only if: either (i) $r/t < 1 < s/u$ and $\mathfrak{M}_n^{[r]}(\underline{a};\underline{v}) = \tilde{\mathfrak{M}}_m^{[r]}(\underline{a};\underline{v})$, and $\mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) = \tilde{\mathfrak{M}}_m^{[s]}(\underline{a};\underline{w})$; or (ii) $r/t < 1 = s/u$ and $\mathfrak{M}_n^{[r]}(\underline{a};\underline{v}) = \tilde{\mathfrak{M}}_m^{[r]}(\underline{a};\underline{v})$; or (iii) $r/t = 1 < s/u$ and $\mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) = \tilde{\mathfrak{M}}_m^{[s]}(\underline{a};\underline{w})$; or (iv) $r/t = 1 = s/u$.
In particular

$$\begin{aligned}&W_n\big(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})\big)^u - V_n\big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{v})\big)^t\\ &\quad\ge W_{n-1}\big(\mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})\big)^u - V_{n-1}\big(\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{v})\big)^t + (w_n a_n^u - v_n a_n^t),\end{aligned} \tag{6}$$

with equality if and only if: either (i′) $r/t<1<s/t$ and $\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{v}) = \mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w}) = a_n$; or (ii′) $r/t < 1 = s/t$ and $\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{v}) = a_n$; or (iii′) $r/t = 1 < s/t$ and $\mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w}) = a_n$; or (iv′) $r/t = 1 = s/t$.
(b) If $r \le 0 \le s$ then

$$\frac{\big(\mathfrak{M}_{n+m}^{[s]}(\underline{a};\underline{w})\big)^{W_{n+m}}}{\big(\mathfrak{M}_{n+m}^{[r]}(\underline{a};\underline{v})\big)^{V_{n+m}}} \ge \frac{\big(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})\big)^{W_n}}{\big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{v})\big)^{V_n}} \frac{\big(\tilde{\mathfrak{M}}_m^{[s]}(\underline{a};\underline{w})\big)^{\tilde{W}_m}}{\big(\tilde{\mathfrak{M}}_m^{[r]}(\underline{a};\underline{v})\big)^{\tilde{V}_m}},$$

with equality if and only if: either (i) $r < 0 < s$ and the conditions in (a)(i) hold; or (ii) $r < 0 = s$ and the condition in (a)(ii) hold; or (iii) $r = 0 < s$ and the condition in (a)(iii) hold ; or (iv) $r = 0 = s$.
In particular

$$\frac{\big(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})\big)^{W_n}}{\big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{v})\big)^{V_n}} \ge a_n^{w_n - v_n} \frac{\big(\mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})\big)^{W_{n-1}}}{\big(\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{v})\big)^{V_{n-1}}}, \tag{7}$$

with equality if and only if: either (i′) $r < 0 < s$ and the conditions in (a)(i′) hold; or (ii′) $r < 0 = s$ and the condition in (a)(ii′) holds; or (iii′) $r = 0 < s$ and the condition in (a)(iii′) holds; or (iv′) $r = 0 = s$.

□ These follow immediately from 3.1 Corollary 3 using the method in 3.2 Theorem 5. □

REMARK (i) If $\underline{v} = \underline{w}$, $r = 0$, $s = u = t = 1$ then (6) reduces to (R), and if $\underline{v} = \underline{w}$, $r = 0$, $s = 1$ then (7) reduces to (P).

THEOREM 7 *Let $r \in \mathbb{R}$, $f : \mathbb{R}_+^* \mapsto \mathbb{R}_+^*$ be such that $1/(1+f)$ is strictly monotonic, and either $1/\big(1+f(x^{1/r})\big)$ is strictly convex when $r \neq 0$, or $1/\big(1+f(e^x)\big)$ is strictly convex when $r = 0$, and if for all index sets I, we define*

$$\chi(I) = \frac{V_I}{1+f\big(\mathfrak{M}_I^{[r]}(\underline{a};\underline{v})\big)} - \sum_{i\in I} \frac{w_i}{1+f(a_i)}$$

then: (a) χ is sub-additive; (b) in particular, using the notation of Theorem 6,

$$\begin{aligned}\frac{V_{n+m}}{1+f\big(\mathfrak{M}_{n+m}^{[r]}(\underline{a};\underline{v})\big)} - \sum_{i=1}^{n+m} \frac{w_i}{1+f(a_i)} &\\ \le \left(\frac{V_n}{1+f\big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{v})\big)} - \sum_{i=1}^{n} \frac{w_i}{1+f(a_i)}\right)&\\ + \left(\frac{\tilde{V}_m}{1+f\big(\tilde{\mathfrak{M}}_m^{[r]}(\underline{a};\underline{v})\big)} - \sum_{i=n+1}^{n+m} \frac{w_i}{1+f(a_i)}\right)&,\end{aligned}$$

with equality if and only if $\mathfrak{M}_n^{[r]}(\underline{a};\underline{v}) = \tilde{\mathfrak{M}}_m^{[r]}(\underline{a};\underline{w})$; and,

$$\begin{aligned}\frac{V_n}{1+f\big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{v})\big)} - \sum_{i=1}^{n} \frac{w_i}{1+f(a_i)} &\qquad (8)\\ \le \frac{v_n - w_n}{1+f(a_n)} + \left(\frac{V_{n-1}}{1+\big(f\big(\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{v})\big)} - \sum_{i=1}^{n-1} \frac{w_i}{1+f(a_i)}\right)&,\end{aligned}$$

with equality if and only if $a_n = \tilde{\mathfrak{M}}_{n-1}^{[r]}(\underline{a};\underline{v})$.

□ These results follow from 3.1 Corollaries 2 and 3 using $\mathcal{M}(x) = x^r$, $r \neq 0$, $\mathcal{M}(x) = \log x$, $r = 0$, $\mathcal{N}(x) = \mathcal{K}(x) = \mathcal{L}^{-1}(x) = \dfrac{1}{\big(1+f(x)\big)}$. □

REMARK (ii) It should be remarked that in particular cases the exact intervals in which the functions $1/\big(1+f(x^{1/r})\big)$ and $1/\big(1+f(e^x)\big)$ are strictly convex need not be $\mathbb{R}_+^*$ and this puts some restrictions on the n-tuple $\underline{a}$.

REMARK (iii) Repeated application of (8) leads to

$$\sum_{i=1}^{n} \frac{w_i}{1+f(a_i)} \ge \frac{V_n}{1+f\big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{v})\big)} + \sum_{i=1}^{n} \frac{w_i - v_i}{1+f(a_i)}, \qquad (9)$$

with equality if and only if $\underline{a}$ is constant. If $\underline{v} = \underline{w}$, and $f(x) = x^s$, $0 \le r < s$, (9) reduces to Henrici's inequality, 2 Corollary 3. So (8) is a Rado type extension of Henrici's inequality.

REMARK (iv) If $\underline{v} = \underline{w}$ various inequalities above are much neater, and the corresponding functions of index sets are not only sub-additive but also decreasing.

We now consider some special cases of 3.1 Theorem 4.

THEOREM 8 *(a) If $-\infty < r/t \le 1 \le s/u < \infty$, $r \ne 0$, then*

$$
\begin{aligned}
W_n\Big(\big(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})\big)^u - \big(\frac{V_n w_n}{v_n W_n}\big)^{t/r} a_n^{u-t}\big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{v})\big)^t\Big) \\
\ge W_{n-1}\Big(\big(\mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})\big)^u - \big(\frac{V_{n-1} w_n}{v_n W_{n-1}}\big)^{t/r} a_n^{u-t}\big(\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{v})\big)^t\Big),
\end{aligned} \tag{10}
$$

with equality if and only if either: (i) $r/t < 1 < s/u$ and $a_n = \mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w}) = \big(\frac{V_{n-1}w_n}{W_{n-1}v_n}\big)^{1/r}\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{v})$; or (ii) $r/t = 1 < s/u$ and $a_n = \mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})$; or (iii) $r/t < 1 = s/u$ and $a_n = \big(\frac{V_{n-1}w_n}{W_{n-1}v_n}\big)^{1/r}\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{v})$; or (iv) $r/t = 1 = s/u$.
(b)

$$
\begin{aligned}
W_n\Big(\mathfrak{A}_n(\underline{a};\underline{w}) - \big(\mathfrak{G}_n(\underline{a};\underline{v})\big)^{V_n w_n/v_n W_n}\Big) \\
\ge W_{n-1}\Big(\mathfrak{A}_{n-1}(\underline{a};\underline{w}) - \big(\mathfrak{G}_{n-1}(\underline{a};\underline{v})\big)^{V_{n-1} w_n/v_n W_{n-1}}\Big),
\end{aligned} \tag{11}
$$

with equality if and only if $a_n = \big(\mathfrak{G}_n(\underline{a};\underline{v})\big)^{v_n W_{n-1}/V_{n-1} w_n}$.
(c) If $\lambda > 0$ then

$$
\begin{aligned}
W_n\big(\mathfrak{A}_n(\underline{a};\underline{w}) - \lambda\frac{V_n w_n}{v_n W_n}\mathfrak{G}_n(\underline{a};\underline{v})\big) \\
\ge W_{n-1}\big(\mathfrak{A}_{n-1}(\underline{a};\underline{w}) - \lambda^{V_n/V_{n-1}}\frac{V_{n-1} w_n}{W_{n-1} v_n}\mathfrak{G}_{n-1}(\underline{a};\underline{v})\big),
\end{aligned} \tag{12}
$$

with equality if and only if $a_n = \lambda^{V_n/V_{n-1}}\mathfrak{G}_{n-1}(\underline{a};\underline{v})$.

□ (a) Take $\mathcal{K}(x) = x^t$, $\mathcal{L}(x) = x^u$, $\mathcal{M}(x) = x^r$, $\mathcal{N}(x) = x^s$ and $\mu = \nu = 0$ in 3.1 Theorem 4.
(b) Take $\mathcal{K}(x) = \mathcal{L}(x) = \mathcal{N}(x) = x$, $\mathcal{M}(x) = \log x$ and $\mu = \nu = 0$ in 3.1 Theorem 4.
(c) Take $\mathcal{K}(x) = \mathcal{L}(x) = \mathcal{N}(x) = -x$, $\mathcal{M}(x) = \log x$ and $\mu = \nu = \log\lambda$ in 3.1 Theorem 4. □

REMARK (v) By taking $\mathcal{M}(x) = \log x$ the result in (a) can be extended to cover the case $r = 0$; and similarly taking $\mathcal{N}(x) = \log x$ the case $s = 0$ is obtained.

REMARK (vi) Inequalities (11)—(12), as well as the extra cases mentioned in Remark (v) can all be obtained from II 3.2.1 Theorem 5 by taking special values of λ and μ.

4 Further Inequalities

Much of this section could have been placed earlier in either Chapter II or Chapter III but the use of quasi-arithmetic means simplifies many of the arguments.

4.1 ČAKALOV'S INEQUALITY The following special case of 3.1 Corollary 3 is a simple generalization of (R).

THEOREM 1 *If $\mathcal{M}$ is defined on $\mathbb{R}_+^*$, $n \geq 2$ then in order that for all positive n-tuples $\underline{a}$ and non-negative n-tuples $\underline{w}$, $W_n \neq 0$, we have the inequality*

$$W_n\big(\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{M}_n(\underline{a};\underline{w})\big) \geq W_{n-1}\big(\mathfrak{A}_{n-1}(\underline{a};\underline{w}) - \mathfrak{M}_n(\underline{a};\underline{w})\big), \tag{1}$$

it is necessary and sufficient that $\mathcal{M}^{-1}$ be convex. If $\mathcal{M}^{-1}$ is strictly convex then (1) is strict unless $a_n = \mathfrak{M}_{n-1}(\underline{a};\underline{w})$.

□ Taking $\mathcal{N}(x) = \mathcal{K}(x) = \mathcal{L}(x) = x$, and $\underline{v} = \underline{w}$ in 3.1(4) gives (1); and the case of equality follows using Remark (iii) of that section.
On the other hand with $n = 2$ in (1) we get, writing $b_i = \mathcal{M}(a_i)$, $i = 1, 2$,

$$\frac{w_1\mathcal{M}^{-1}(b_1) + w_2\mathcal{M}^{-1}(b_2)}{w_1 + w_2} \geq \mathcal{M}^{-1}\Big(\frac{w_1b_1 + w_2b_2}{w_1 + w_2}\Big).$$

So if (1) holds for all positive $\underline{a}, \underline{w}$ then $\mathcal{M}^{-1}$ is convex. □

REMARK (i) In the case of equal weights this result is due to Popoviciu, [*Popoviciu 1959b,1961a*].

REMARK (ii) If course if $\mathcal{M} = \log$ then (1) is just (R).

Suppose now that not all of $a_1, \ldots, a_{n-1}$ are equal, that is $\underline{a}'_n$ is not constant, then (1) can be written

$$\frac{\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{M}_n(\underline{a};\underline{w})}{\mathfrak{A}_{n-1}(\underline{a};\underline{w}) - \mathfrak{M}_{n-1}(\underline{a};\underline{w})} \geq \frac{W_{n-1}}{W_n}.$$

That is to say the left-hand side is bounded below, as a function of $\underline{a}$. Further this lower bound is attained if and only if $a_n = \mathfrak{M}_{n-1}(\underline{a};\underline{w})$, and this is not possible if $\max \underline{a}'_n \leq a_n$. In particular the lower bound is not attained for non-constant increasing n-tuples. As a result we could ask if the lower bound could be improved in that case. This problem was first considered by Čakalov, in the equal weight case with $\mathcal{M} = \log$, when of course $\mathfrak{M}_n(\underline{a};\underline{w}) = \mathfrak{G}_n(\underline{a})$; [*DI p.43*], [*Čakalov 1946*].

THEOREM 2 *If $\mathcal{M}$ is strictly monotonic, $\underline{w}$ a positive n-tuple and $\underline{a}$ an increasing, non-constant, positive n-tuple and if $n > 2$ then*

$$\frac{W_n^2}{W_n - w_1}\big(\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{M}_n(\underline{a};\underline{w})\big) \geq \frac{W_{n-1}^2}{W_{n-1} - w_1}\big(\mathfrak{A}_{n-1}(\underline{a};\underline{w}) - \mathfrak{M}_{n-1}(\underline{a};\underline{w})\big), \tag{2}$$

provided $\mathcal{M}^{-1}$ is convex and 3-convex, or 3-concave, according as $\mathcal{M}$ is increasing or decreasing.

□ We assume that $\mathcal{M}^{-1}$ is convex, 3-convex and increasing; the other case follows similarly.

Let $D(\underline{a})$ be the difference between the left-hand side and right-hand side of (2) and we have to show that $D(\underline{a}) \geq 0$.

For simplicity we will write $m = \mathcal{M}^{-1}$, and $D_k(x) = D(\underline{a}')$, where $a_i' = a_i$, $1 \leq i \leq k$ and $a_i' = x$, $k+1 \leq i \leq n, a_k \leq x \leq a_n$. Further let $B_i(\underline{a}') = B_i(x) = B_i = \mathfrak{A}_i(\mathcal{M}(\underline{a}'); \underline{w})) = \mathcal{M}(\mathfrak{M}_i(\underline{a}'; \underline{w}))$, $1 \leq i \leq n$. Then for $1 \leq k \leq n-1$,

$$B_n = \frac{W_k}{W_n} B_k + \big(1 - \frac{W_k}{W_n}\mathcal{M}(x)\big), \quad B_{n-1} = \frac{W_k}{W_{n-1}n} B_k + \big(1 - \frac{W_k}{W_{n-1}}\mathcal{M}(x)\big).$$

Since m is 3-convex it is differentiable, I 4.7 Theorem 44(b), and simple calculations give

$$\begin{aligned} D_k'(x) =& \mathcal{M}'(x)\Big(\frac{W_n(W_n - W_k)}{W_n - w_1}\big(m' \circ \mathcal{M}(x) - m'(B_n)\big) \\ & - \frac{W_{n-1}(W_{n-1} - W_k)}{W_{n-1} - w_1}\big(m' \circ \mathcal{M}(x) - m'(B_{n-1})\big)\Big). \end{aligned}$$

On rewriting this becomes

$$\begin{aligned} D_k'(x) =& \frac{w_n W_k \mathcal{M}'(x)\big(\mathcal{M}(x) - B_k\big)}{W_n W_{n-1}(W_n - w_1)(W_{n-1} - w_1)}\Big(W_n W_{n-1}(W_k - w_1)[\mathcal{M}(x), B_n; m'] \\ & + W_k(W_n - w_1)(W_{n-1} - W_k)\big(\mathcal{M}(x) - B_k\big)[\mathcal{M}(x), B_n, B_{n-1}; m']\Big). \end{aligned}$$

As $\mathcal{M}^{-1}$ is convex we have by I 4.1 Theorem 4 (b) that $[\mathcal{M}(x), B_n; m']_1 \geq 0$; M^{-1} is 3-convex so m' is convex, I 4.7 Theorem 44(b), and, by I 4.1 (2*) and I 4.7 Remark(ii), $[\mathcal{M}(x), B_n, B_{n-1}; m']_2 \geq 0$; $\mathcal{M}$ is increasing, by hypothesis, and differentiable since m is differentiable, so $\mathcal{M}' \geq 0$.

Hence $D_k'(x) \geq 0$, $1 \leq k \leq n-1$, $a_k \leq x \leq a_n$. This implies that

$$D(\underline{a}) = D_{n-1}(a_n) \geq D_{n-1}(a_{n-1}) = D_{n-2}(a_{n-1}) \geq D_{n-2}(a_{n-2}) \geq \ldots \geq D_1(a_1) = 0$$

□

Remark (iii) This proof is an adaption of the proof (*lxxii*) of (GA), II 2.4.6; see also proof (*xi*) of (r;s), III 3.1.1 Theorem 1.

Since the means in (2) are almost symmetric, Theorem 2 can be stated as follows:

THEOREM 3 *If $\mathcal{M}$, $\underline{w}$ are as in Theorem 2, and if $\underline{a}$ is non-constant with* $\max a'_n \le a_n$ *then*

$$\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{M}_n(\underline{a};\underline{w}) \ge \lambda_n\big(\mathfrak{A}_{n-1}(\underline{a};\underline{w}) - \mathfrak{M}_{n-1}(\underline{a};\underline{w})\big), \tag{3}$$

where $\lambda_n = \dfrac{W_{n-1}^2(W_n - w_j)}{W_n^2(W_{n-1} - w_j)}$, *and where j is defined by* $a_j = \min \underline{a}'_n$.

REMARK (iv) It is easily seen that $\lambda_n > W_{n-1}/W_n$ and so, for this restricted class of sequences, (2) is more precise than (1). In general, unlike W_{n-1}/W_n, the lower bound λ_n is not attained.

EXAMPLE (i) If λ_n is replaced by a $\lambda'_n > \lambda_n$ there are n-tuples for which (3) would then fail. To see this take $M = \log$, $a_1 = 1 - \epsilon$, $\epsilon > 0$, $a_2 = \ldots = a_n = 1$, when

$$\frac{\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})}{\mathfrak{A}_{n-1}(\underline{a};\underline{w}) - \mathfrak{G}_{n-1}(\underline{a};\underline{w})} = \frac{1 - (w_1/W_n) - (1-\epsilon)^{w_1/W_n}}{1 - (w_1/W_{n-1}) - (1-\epsilon)^{w_1/W_n}},$$

and as $\epsilon \to 0+$ the right-hand side tends to λ_n.

EXAMPLE (ii) There is no corresponding result for (P). To see this consider the following: $a_1 = \ldots = a_{n-2} = 1$, $a_{n-1} = a_n = x$ and let $\mu_n > (n-1)/n$ when

$$\lim_{n\to\infty} \frac{\mathfrak{A}_n(\underline{a})/\mathfrak{G}_n(\underline{a})}{\big(\mathfrak{A}_{n-1}(\underline{a})/\mathfrak{G}_{n-1}(\underline{a})\big)^{\mu_n}} = 0.$$

This shows that the exponent $(n-1)/n$ that occurs in (P) cannot be improved by assuming that $\underline{a}$ is increasing; [*Bullen 1969d; Diananda 1969*].

4.2 A THEOREM OF GODUNOVA The following simple theorem has many interesting corollaries; see [*Godunova 1965*].

THEOREM 4 *Let $\mathcal{M}$ $:]0,\infty[\mapsto \mathbb{R}$ be continuous, strictly increasing, with $\mathcal{M}^{-1}$ convex and* $\lim_{x\to 0}\mathcal{M}(x) = 0$, *or $-\infty$; suppose further that $\underline{a}$ and $\underline{b}$ are two positive sequences, and that for each $n, n \ge 1$, $\underline{w}^{(n)}$ is a positive n-tuple with $W_n^{(n)} = 1$, and $\sum_{n=k}^{\infty} w_k^{(n)} b_n \le C$, $k \ge 1$. Then*

$$\sum_{n=1}^{\infty} b_n \mathfrak{M}_n(\underline{a};\underline{w}^{(n)}) \le C\sum_{n=1}^{\infty} a_n. \tag{4}$$

If $C = \lim_{n\to\infty} \dfrac{1}{n}\sum_{k=1}^{n} b_n$ *then the constant in (4) is best possible.*

□ By 2 Example (iv), $\mathfrak{M}_n(\underline{a};\underline{w}^{(n)}) \le \mathfrak{A}_n(\underline{a};\underline{w}^{(n)}) = \sum_{k=1}^{n} w_k^{(n)} a_k$, $n \ge 1$. Hence

$$\begin{aligned}\sum_{n=1}^{\infty} b_n \mathfrak{M}_n(\underline{a};\underline{w}^{(n)}) &\le \sum_{n=1}^{\infty} b_n\Big(\sum_{k=1}^{n} w_k^{(n)} a_k\Big) = \sum_{k=1}^{n} a_k \sum_{n=k}^{\infty} w_k^{(n)} b_n \\ &\le C\sum_{k=1}^{\infty} a_k, \quad \text{by the above hypothesis.}\end{aligned} \tag{5}$$

Now let $a_k = 1, \le k \le m, = 0, k > m$; then from (4)

$$C \ge \frac{1}{m}\sum_{n=1}^{m} b_n + \frac{1}{m}\sum_{n=1}^{\infty} b_n \mathcal{M}^{-1}(\mathcal{M}(1)W_m^{(n)}) \ge \frac{1}{m}\sum_{n=1}^{m} b_n,$$

using the hypotheses on $\mathcal{M}$. This completes the proof. □

COROLLARY 5 *The following inequalities hold for all positive sequences $\underline{a}$ and real numbers $p, q > 1$.*

(*a*) $$\sum_{n=1}^{\infty}\Big((q-1)^p \frac{(q^n-1)^{1-p}}{q^n}(\sum_{k=1}^{n} q^{k-1}a_k)^p\Big) \le \sum_{n=1}^{\infty} a_n^p;$$

(*b*) $$\sum_{n=1}^{\infty}\Big(\frac{q^n-1}{q_n}(\prod_{k=1}^{n} a_k^{q^{k-1}})^{(q-1)/n}\Big) < \sum_{n=1}^{\infty} a_n;$$

(*c*) $$\sum_{n=1}^{\infty}\Big(\frac{n}{n+1}\tan^p(\frac{\pi}{2n})(\sum_{k=1}^{n}\sin\frac{k\pi}{n}a_k)^p\Big) < 2\pi\sum_{n=1}^{\infty} a_n^p;$$

(*d*) $$\sum_{n=1}^{\infty}\Big(\frac{n}{n+1}(\prod_{k=1}^{n} a_k^{\sin(k\pi/n)})^{\tan(\pi/2n)}\Big) < 2\pi\sum_{n=1}^{\infty} a_n;$$

(*e*) $$\sum_{n=1}^{\infty}\Big(\frac{1}{n+1}(\frac{1}{n}\sum_{k=1}^{n} a_k)^p\Big) < \sum_{n=1}^{\infty}\frac{a_n^p}{n};$$

(*f*) $$\sum_{n=1}^{\infty}\Big(\frac{1}{n+1}(n!\prod_{k=1}^{n} a_k)^{1/n}\Big) < \sum_{n=1}^{\infty} a_n.$$

□ These all result from Theorem 4. In (a), (c) and (e) $\underline{a}$ is replaced by $\underline{a}^p$, and $\mathcal{M}(x) = x^{1/p}$. Then: in (a) take $w_k^{(n)} = \dfrac{q^{k-1}(q-1)}{q^n-1}$, $b_k = \dfrac{q^k-1}{q^k}$, $n, k \ge 1$; in (c) $w_k^{(n)} = \sin\dfrac{k\pi}{n}\tan\dfrac{\pi}{2n}$, $b_k = \dfrac{k}{k+1}$, $n, k \ge 1$; in (e) $w_k^{(n)} = \dfrac{1}{n}$, $b_k = \dfrac{1}{k+1}$, $n, k \ge 1$, noticing that $\dfrac{1}{k} \le \displaystyle\sum_{n=k}^{\infty}\frac{1}{n(n+1)}$, and use (5).
In (b), (d) and (f) take $\mathcal{M}(x) = \log x$. Then: in (b) take $\underline{w}^{(n)}, \underline{b}$ as in (a); in (d) take $\underline{w}^{(n)}, \underline{b}$ as in (c); in (f) take $\underline{w}^{(n)}, \underline{b}$ as in (e).
(f) Same as (e) but with $\mathcal{M}(x) = \log x$, and the sequence just $\underline{a}$. □

REMARK (i) Since $e^{-1} < \dfrac{(n!)^{1/n}}{n+1}$, see I 2.2(a), Corollary 5(f) implies Carleman's inequality, II 3.4 Corollary 16; see [*PPT p.231*].

REMARK (ii) Taking $\mathcal{M}(x) = x^{1/p}$ or $\log x$, $w_k^{(n)} = \dfrac{\binom{n}{k}\alpha^{n-k}\beta^k}{(\alpha+\beta)^n}$, $b_k = 1$, $n, k \ge 1$, where $\alpha, \beta > 0$, $\alpha + \beta > 1$, then an inequality of Knopp, a generalization of Carleman's inequality, follows from Theorem 4; [*Knopp 1930; Levin 1937*].

REMARK (iii) Another proof of (4) that leads to many other examples has been given; [*Vasić & Pečarić 1982b*].

4.3 A PROBLEM OF OPPENHEIM In this section we consider the following interesting question. *If the quasi-arithmetic $\mathcal{M}$-means of $\underline{a}$ and $\underline{b}$ are in a certain order when can it be deduced that their quasi-arithmetic $\mathcal{N}$-means are in the same order?*
The question was first studied, and solved, by Oppenheim for the arithmetic and geometric means, in the case $n = 3$; [*Oppenheim 1965,1968*]. The extension to general n, and in particular the important condition (6) below is essentially due, to Godunova & Levin, [*Godunova & Levin 1970*], who however failed to apply their theorem, Theorem 6 below, to Oppenheim's problem. Another answer to the general case is given in [*Vasić 1972*]; the discussion below follows that in [*Bullen, Vasić & Stanković*].

THEOREM 6 *Let $\underline{a}$ and $\underline{b}$ be increasing positive n-tuples, $n \geq 3$, such that for some m, $1 < m < n$,*

$$a_i \leq b_i, \quad 1 \leq i \leq n-m; \qquad a_i \geq b_i, \quad n-m+2 \leq i \leq n. \tag{6}$$

Let $\underline{w}$ be another positive n-tuple, and $\mathcal{M} : \mathbb{R}_+ \mapsto \mathbb{R}$ an increasing concave function with $x\mathcal{M}'(x)$ also increasing. If

$$\mathfrak{A}_n(\underline{a};\underline{w}) \leq \mathfrak{A}_n(\underline{b};\underline{w}), \tag{7}$$

and if $0 \leq \alpha \leq 1 - \dfrac{W_{n-m}}{W_n}$ then

$$\mathfrak{A}_n\big(\mathcal{M}(\underline{a});\underline{w}\big) - \alpha\mathcal{M}\big(\mathfrak{A}_n(\underline{a};\underline{w})\big) \leq \mathfrak{A}_n\big(\mathcal{M}(\underline{b});\underline{w}\big) - \alpha\mathcal{M}\big(\mathfrak{A}_n(\underline{b};\underline{w})\big). \tag{8}$$

If $\alpha = 0$ we do not need to assume that $x\mathcal{M}'(x)$ is increasing. If $\mathcal{M}$ is strictly increasing, $\mathcal{M}'$ strictly decreasing, and $x\mathcal{M}'(x)$ strictly increasing, a condition that can be omitted if $\alpha = 0$, then (8) is strict unless $\underline{a} = \underline{b}$.

□ If $0 \leq \lambda \leq 1$ and $c_k = (1-\lambda)a_k + \lambda b_k$, $1 \leq k \leq n$, then $\underline{c}$, like $\underline{a}$ and $\underline{b}$, is an increasing positive n-tuple.
Further define $\phi(\lambda) = \mathfrak{A}_n\big(\mathcal{M}(\underline{c});\underline{w}\big) - \alpha\mathcal{M}\big(\mathfrak{A}_n(\underline{c};\underline{w})\big)$, when (8) is just $\phi(0) \leq \phi(1)$.
This last inequality will follow if we can prove that $\phi' \geq 0$.
Now

$$W_n\phi'(\lambda) = \sum_{i=1}^{n} w_i(b_i - a_i)\mathcal{M}'(c_k) - \alpha\Big(\sum_{i=1}^{n} w_i(b_i - a_i)\Big)\mathcal{M}'\Big(\frac{1}{W_n}\sum_{i=1}^{n} w_i c_i\Big). \tag{9}$$

If $\alpha = 0$ the right-hand side of (9) reduces to its first term and since $\mathcal{M}' \ge 0$ and, from (7), $\sum_{i=1}^n w_i(b_i - a_i) \ge 0$, we have that $\phi' \ge 0$. So let us assume that $\alpha \ne 0$. From the definition of $\underline{c}$,

$$\sum_{i=1}^n w_i c_i \ge \sum_{i=n-m+1}^n w_i c_i \ge c_{n-m+1}(W_n - W_{n-m}).$$

Consider first the last term on the right-hand side of (9). Since $\mathcal{M}$ is concave $\mathcal{M}'$ is decreasing, I 4.1 Theorem 4(b), and so, again using $\sum_{i=1}^n w_i(b_i - a_i) \ge 0$:

$$\begin{aligned} \alpha\Big(\sum_{i=1}^n w_i(b_i - a_i)\Big)\mathcal{M}'\big(\frac{1}{W_n}\sum_{i=1}^n w_i c_i\big) &\le \alpha\big(\sum_{i=1}^n w_i(b_i - a_i)\big)\mathcal{M}'\big(c_{n-m+1}(1 - \frac{W_{n-m}}{W_n})\big) \\ &\le \alpha\Big(\sum_{i=1}^n w_i(b_i - a_i)\Big)\mathcal{M}'(c_{n-m+1})\frac{1}{\Big(1 - \dfrac{W_{n-m}}{W_n}\Big)} \\ &\le \sum_{i=1}^n w_i(b_i - a_i)\mathcal{M}'\big(c_{n-m+1}\big), \text{ by the monotonicity of } x\mathcal{M}'(x) \end{aligned} \tag{10}$$

Now we turn to the first term on the right-hand side of (9).

$$\begin{aligned} \sum_{i=1}^n w_i(b_i - a_i)\mathcal{M}'(c_k) &= \sum_{i=1}^{n-m} w_i(b_i - a_i)\mathcal{M}'(c_k) - \sum_{i=n-m+1}^n w_i(a_i - b_i)\mathcal{M}'(c_k) \\ &\ge \sum_{i=1}^{n-m} w_i(b_i - a_i)\mathcal{M}'(c_{n-m+1}) - \sum_{i=n-m+1}^n w_i(a_i - b_i)\mathcal{M}'(c_{n-m+1}) \\ &= \sum_{i=1}^n w_i(b_i - a_i)\mathcal{M}'(c_{n-m+1}), \text{by the monotonicity of } \mathcal{M}'. \end{aligned} \tag{11}$$

From (10) and (11) we see that $\phi' \ge 0$. □

REMARK (i) It is easily seen that ($\sim$8) holds if $\mathcal{M}$ is convex and decreasing, with $x\mathcal{M}'(x)$ decreasing; and if $\mathcal{M}$ is convex, increasing and $x\mathcal{M}'(x)$ is increasing and if ($\sim$7) holds so does ($\sim$8).

COROLLARY 7 *Let $n \ge 3$, $1 < m < n$, let $\underline{a}, \underline{b}$ be two increasing positive n-tuples that satisfy (6), and let $\underline{w}$ another positive n-tuple. If $s \in \mathbb{R}$ and*

$$\mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) \le \mathfrak{M}_n^{[s]}(\underline{b};\underline{w}), \tag{12}$$

then if $0 \le \alpha \le 1 - \dfrac{W_{n-m}}{W_n}$ *and* $s > 0$,

$$\mathfrak{G}_n(\underline{a};\underline{w})\Big(\frac{\mathfrak{M}_n^{[s]}(\underline{b};\underline{w})}{\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})}\Big)^\alpha \le \mathfrak{G}_n(\underline{b};\underline{w}); \tag{13}$$

and if $0 < t < s$,

$$\left(\left(\mathfrak{M}_n^{[t]}(\underline{a};\underline{w})\right)^t - \alpha\left(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})\right)^t\right)^{1/t} \le \left(\left(\mathfrak{M}_n^{[t]}(\underline{b};\underline{w})\right)^t - \alpha\left(\mathfrak{M}_n^{[s]}(\underline{b};\underline{w})\right)^t\right)^{1/t}. \quad (14)$$

If ($\sim$ 12) *is assumed and if* $s < 0$ ($\sim$13) *holds , or if* $s < t < 0$ *then* ($\sim$14) *holds. There is equality in (13), (14) if and only if* $\underline{a} = \underline{b}$.

□ This is immediate from Theorem 6, and Remark (i), by taking $\mathcal{M}(x)$ variously as $x^r, \log x, e^x$. □

REMARK (ii) The case $n = 3$, $m = 2$, $w_1 = w_2 = w_3$, $s = 1$, $\alpha = 2/3$ of (13) is due to Oppenheim. By consideration of a special case Oppenheim pointed out that if $s = 0$ the analogous inequality to (13) with $\mathfrak{G}_n(\underline{a};\underline{w})$ replaced by $\mathfrak{A}_n(\underline{a};\underline{w})$ does not hold in general. The correct form is given, as we have seen, by (14).

EXAMPLE (i) In general (8) does not hold if $\alpha > 1 - W_{n-m}/W_n$. Let $n = 3$, $w_1 = w_2 = w_3$, $0 < a_1 \le a_2 \le a_3, 0 < b_1 \le b_2 \le b_3$ Then (6) implies that $a_1 \le b_1, b_3 \le a_3$ and $\alpha = 2/3$. If $s = 1$ then (13) says that $a_1+a_2+a_3 \le b_1+b_2+b_3$ implies that $A = \dfrac{a_1a_2a_3}{b_1b_2b_3}\left(\dfrac{b_1+b_2+b_3}{a_1+a_2+a_3}\right)^{3\alpha} \le 1$. However if $a_1 = b_1 = 1, a_2 = a - 1$, $a_3 = b_3 = b_2 = a, a > 2$, then

$$A = 1 + \frac{1}{a}\left(\frac{3\alpha}{2} - 1\right) + O(a^{-2}).$$

So if $\alpha > 2/3$ and a is large enough we have that $A > 1$. This gives the counterexample that is due to Oppenheim.

The weakest case of either Theorem 6, or Corollary 7, namely $\alpha = 0$, gives an answer to the problem of Oppenheim stated at the beginning of this section.

COROLLARY 8 *Let* $\mathcal{M} : \mathbb{R}_+ \mapsto \mathbb{R}$ *be strictly increasing with* $\mathcal{M}'$ *decreasing. Suppose further that* $\underline{a}, \underline{b}$ *and* $\underline{w}$ *are positive* n*-tuples,* $n \ge 3$*, with* $\underline{a}, \underline{b}$ *increasing and satisfying (6). Then*

$$\mathfrak{A}_n(\underline{a};\underline{w}) \le \mathfrak{A}_n(\underline{b};\underline{w}) \implies \mathfrak{M}_n(\underline{a};\underline{w}) \le \mathfrak{M}_n(\underline{b};\underline{w}). \quad (15)$$

If $\mathcal{M}'$ *is strictly increasing there is equality on the right-hand side of (15) if and only if* $\underline{a} = \underline{b}$.

This corollary is clearly equivalent to the following:

COROLLARY 9 *With the notations and conditions of Corollary 8, and if* $\mathcal{N} : \mathbb{R}_+ \mapsto \mathbb{R}$ *is strictly increasing then*

$$\mathfrak{N}_n(\underline{a};\underline{w}) \le \mathfrak{N}_n(\underline{b};\underline{w}) \implies \mathfrak{M}_n(\mathcal{N}(\underline{a});\underline{w}) \le \mathfrak{M}_n(\mathcal{N}(\underline{b});\underline{w}). \quad (16)$$

REMARK (iii) Using Remark (i) we see that Corollary 9 holds if we assume instead that $\mathcal{M}$ is decreasing, and $\mathcal{M}'$ increasing; if instead we assume that $\mathcal{M}$ and $\mathcal{M}'$ are increasing then the inequality right-hand side of (15) is reversed.

REMARK (iv) Bullen, Vasić & Stanković have given a more general condition than (6) for the validity of (15); but then (16) fails and so this more general result has fewer applications.

COROLLARY 10 *Let $\underline{a}, \underline{b}$ and $\underline{w}$ be as in Corollary 8, and $s, t \in \mathbb{R}$.*
(a) If $t < s$ then:

$$\mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) \le \mathfrak{M}_n^{[s]}(\underline{b};\underline{w}) \implies \mathfrak{M}_n^{[t]}(\underline{a};\underline{w}) \le \mathfrak{M}_n^{[t]}(\underline{b};\underline{w}); \tag{17}$$

(b) If $t > s$ then:

$$\mathfrak{M}_n^{[s]}(\underline{b};\underline{w}) \le \mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) \implies \mathfrak{M}_n^{[t]}(\underline{b};\underline{w}) \le \mathfrak{M}_n^{[t]}(\underline{a};\underline{w}). \tag{18}$$

There is equality on the right-hand sides of (17), and (18) if and only if $\underline{a} = \underline{b}$.

□ Immediate from Corollary 8 and Remark (iii), using $\mathcal{M}(x) = x^r, \log x$ or , e^x.

□

REMARK (v) The case $n = 3$, $s = 1$, $t = 0$, or $s = 0$, $t = 1$, $w_1 = w_2 = w_3$ of (17) and (18) are those originally discussed by Oppenheim in a very different manner.

REMARK (vi) Since the larger α the stronger is inequality (8), Corollary 7 is more precise than Corollary 10.

COROLLARY 11 *Let $\underline{a}$ and $\underline{b}$ be positive n-tuples, not being rearrangements one of the other, but such that their increasing rearrangements satisfy (6), and $\underline{w}$ be another positive n-tuple. Further suppose that*

$$\min \underline{a} < \min \underline{b} \le \max \underline{b} < \max \underline{a},$$

then there is a unique $s \in \mathbb{R}$ such that

$$\mathfrak{M}_n^{[t]}(\underline{a};\underline{w}) \begin{cases} < \mathfrak{M}_n^{[t]}(\underline{b};\underline{w}) & \text{if } t < s, \\ > \mathfrak{M}_n^{[t]}(\underline{b};\underline{w}) & \text{if } t > s, \\ = \mathfrak{M}_n^{[t]}(\underline{b};\underline{w}) & \text{if } t = s. \end{cases}$$

□ This is an immediate from Corollary 10 and the continuity of $\mathfrak{M}_n^{[r]}(\underline{c};\underline{w})$ as a function of r. □

REMARK (vii) A generalization of Theorem 6 can be found in [*Vasić & Pečarić 1982b*].

4.4 KY FAN'S INEQUALITY The following important inequality is due to Levinson; [*Bullen 1973b; Gavrea & Gurzău 1987; Gavrea & Ivan; Levinson 1964; Pečarić 1989-1990*].

THEOREM 12 [LEVINSON'S INEQUALITY] *Let I be an interval in $\mathbb{R}$ and $\mathcal{M} : I \mapsto \mathbb{R}$ be 3-convex, $\underline{w}$ a positive n-tuple, $n \geq 2$, $\underline{a}$ and $\underline{b}$ n-tuples with elements in I and satisfying*

$$\max \underline{a} \leq \min \underline{b}, \qquad a_1 + b_1 = \cdots = a_n + b_n. \tag{19}$$

Then

$$\mathfrak{A}_n\big(\mathcal{M}(\underline{a}); \underline{w}\big) - \mathcal{M}\big(\mathfrak{A}_n(\underline{a}; \underline{w})\big) \leq \mathfrak{A}_n\big(\mathcal{M}(\underline{b}); \underline{w}\big) - \mathcal{M}\big(\mathfrak{A}_n(\underline{b}; \underline{w})\big). \tag{20}$$

If $\mathcal{M}$ is strictly 3-convex then equality occurs in (20) if and only if $\underline{a}$, or $\underline{b}$, is constant.
Conversely if for a continuous $\mathcal{M} : I \mapsto \mathbb{R}$ (20) holds, holds strictly, for all positive 2-tuples $\underline{w}$, and all non-constant 2-tuples $\underline{a}$ and $\underline{b}$ with elements in I satisfying (19), with $n = 2$, then $\mathcal{M}$ is 3-convex, strictly 3-convex.

□ In proving (20) we can assume without loss in generality that $\underline{a}$ is increasing, when of course $\underline{b}$ is decreasing.
The proof is by induction on n. So first let us consider the case $n = 2$.
Note that if $x_i \in I, 1 \leq i \leq 5$, $x_i > x_{i+3}, 0 \leq i \leq 2$, the 3-convexity of $\mathcal{M}$ implies that

$$[x_0, x_1, x_2; \mathcal{M}]_2 \geq [x_3, x_4, x_5; \mathcal{M}]_2, \tag{21}$$

and (21) is strict if $\mathcal{M}$ is strictly 3-convex; see I 4.7 Lemma 45 (b);
Putting $x_0 = b_1, x_1 = \mathfrak{A}_2(\underline{b}; \underline{w}), x_2 = b_2, x_3 = a_2, x_4 = \mathfrak{A}_2(\underline{a}; \underline{w}), x_5 = a_1$, and given (19), (21) reduces to the $n = 2$ case of (20), with strict inequality if (21) is strict.

Now assume the result for all n, $2 \leq n \leq m - 1$.

$$\begin{aligned}
&\mathfrak{A}_m\big(\mathcal{M}(\underline{a}); \underline{w}\big) - \mathfrak{A}_m\big(\mathcal{M}(\underline{b}); \underline{w}\big)\\
&\quad = \frac{W_{m-1}}{W_m}\Big(\mathfrak{A}_{m-1}\big(\mathcal{M}(\underline{a}); \underline{w}\big) - \mathfrak{A}_{m-1}\big(\mathcal{M}(\underline{b}); \underline{w}\big)\Big) + \frac{w_m}{W_m}\big(\mathcal{M}(a_m) - \mathcal{M}(b_m)\big)\\
&\quad \leq \frac{W_{m-1}}{W_m}\Big(\mathcal{M}\big(\mathfrak{A}_{m-1}(\underline{a}; \underline{w})\big) - \mathcal{M}\big(\mathfrak{A}_{m-1}(\underline{b}; \underline{w})\big)\Big) + \frac{w_m}{W_m}\big(\mathcal{M}(a_m) - \mathcal{M}(b_m)\big),\\
&\qquad\qquad\qquad\qquad \text{by the induction hypothesis,}\\
&\quad \leq \mathcal{M}\big(\mathfrak{A}_m(\underline{a}; \underline{w})\big) - \mathcal{M}\big(\mathfrak{A}_m(\underline{b}; \underline{w})\big), \qquad \text{by the case } n = 2.
\end{aligned}$$

The case where $\mathcal{M}$ is strictly 3-convex, and the case of equality are easily obtained from this proof.

Now assume that $n = 2$ and that (20) holds when $a_1 = x, b_1 = x + 3h, a_2 = b_2 = x + 3h/2, w_2 = 2w_1 = 2$. Then (20)reduces to

$$6h^3[x + 3h, x + 2h, x + h, x; M]_3 \geq 0. \tag{22}$$

The hypotheses imply that (22) holds, or holds strictly, for all $x \in I$ and $h > 0$, small enough; so $\mathcal{M}$ is 3-convex, strictly 3-convex; see I 4.7 Lemma 45(e). □

REMARK (i) If $\mathcal{M}$ is 3-concave ($\sim$20) holds,

REMARK (ii) Inequality (20) is just the $\alpha = 1$ case of inequality (8).

REMARK (iii) Pečarić has shown that (19) can be replaced by $a_1 - b_1 = \cdots = a_n - b_n$; *[Pečarić 1982]*.

COROLLARY 13 *Let $n \geq 2$, and let $\underline{a}$ and $\underline{b}$ be positive n-tuples that satisfy (19), $\underline{w}$ another positive n-tuple. If $s > 0$ and $t < s$ or $t > 2s$; or $s = 0$ and $t > 0$; or $s < 0$ and $s > t > 2s$ then*
if $t \neq 0$

$$\left(\left(\mathfrak{M}_n^{[t]}(\underline{a};\underline{w})\right)^t - \left(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})\right)^t\right)^{1/t} \leq \left(\left(\mathfrak{M}_n^{[t]}(\underline{b};\underline{w})\right)^t - \left(\mathfrak{M}_n^{[s]}(\underline{b};\underline{w})\right)^t\right)^{1/t}, \tag{23}$$

while if $s > 0$

$$\frac{\mathfrak{G}_n(\underline{a};\underline{w})}{\mathfrak{G}_n(\underline{b};\underline{w})} \leq \frac{\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})}{\mathfrak{M}_n^{[s]}(\underline{b};\underline{w})}. \tag{24}$$

Equality occurs in any of these inequalities if and only if $\underline{a}$, or $\underline{b}$, is constant.

□ This follows from Theorem 12 and I 4.7 Examples (iv), (v). For (23): in (20) take $\mathcal{M}(x) = x^r$, put $r = s/t$ and replace $\underline{a}$ by $\underline{a}^s$. For (24): in (20) take $\mathcal{M}(x) = \log x$, and replace $\underline{a}$ by $\underline{a}^s$. □

REMARK (iv) Corollary 13 is analogous to 4.3 Corollary 7.

A particular case of this last corollary is a famous inequality due to Ky Fan; *[BB p.5; DI p.150]*. Several independent proofs of this interesting result will be given. The following notation is standard in the discussion of the Ky Fan: if $\underline{a}$ and $\underline{w}$ are positive n-tuples, $n \geq 2$, with $0 < \underline{a} \leq \dfrac{1}{2}$, then we write

$$\mathfrak{A}'_n(\underline{a};\underline{w}) = \mathfrak{A}_n(1-a_1,\ldots,1-a_n;\underline{w}), \quad \mathfrak{G}'_n(\underline{a};\underline{w}) = \mathfrak{G}_n(1-a_1,\ldots,1-a_n;\underline{w}), \tag{25}$$

with obvious extensions to other means.

THEOREM 14 [KY FAN'S INEQUALITY] *If $\underline{a}$ and $\underline{w}$ are positive n-tuples, $n \ge 2$, with $0 < \underline{a} \le \dfrac{1}{2}$ then*

$$\frac{\mathfrak{G}_n(\underline{a};\underline{w})}{\mathfrak{G}'_n(\underline{a};\underline{w})} \le \frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{A}'_n(\underline{a};\underline{w})}, \tag{26}$$

with equality if and only if $\underline{a}$ is constant.

□ (*i*) This is just a special case of (24); take $s = 1$ and $a_1+b_1 = \cdots = a_n+b_n = 1$.

(*ii*) Noting that $f(x) = \log\big((1-x)/x\big)$ is strictly convex on $]0, \frac{1}{2}]$; the result follows from (J); [*Chong K K 1998*].

(*iii*) A very simple proof of the equal weight case can be based on the Cauchy principle of mathematical induction, II 2.2.4 Theorem 7; [*BB p.5*], [*Dubeau 1991b*]. Writing $\overline{a} = \mathfrak{A}_n(\underline{a})$, when $0 < \overline{a} \le \frac{1}{2}$, then (26), in the equal weight case, is just

$$\Big(\prod_{i=1}^{n} \frac{a_i}{1-a_i}\Big)^{1/n} \le \frac{\overline{a}}{1-\overline{a}}. \tag{27}$$

When $n = 2$, and putting $p = a_1a_2$, (26) reduces to $p(1-2\overline{a}) \le \overline{a}^2(1-2\overline{a})$. This however is immediate by the $n = 2$ case of (GA), and the fact that $0 < \overline{a} \le \frac{1}{2}$.

Now suppose that (27) holds for a particular value of $n > 2$ we first show that it holds for $2n$.

Let $\overline{a}_1 = \mathfrak{A}_n(a_1, \ldots, a_n), \overline{a}_2 = \mathfrak{A}_n(a_{n+1}, \ldots, a_{2n}), \overline{a} = \mathfrak{A}_n(a_1, \ldots, a_{2n})$ then

$$\begin{aligned}\Big(\prod_{i=1}^{2n} \frac{a_i}{1-a_i}\Big)^{1/2n} &= \sqrt{\Big(\prod_{i=1}^{n} \frac{a_i}{1-a_i}\Big)^{1/n}\Big(\prod_{i=n+1}^{2n} \frac{a_i}{1-a_i}\Big)^{1/n}}\\ &\le \sqrt{\Big(\frac{\overline{a}_1}{1-\overline{a}_1}\Big)\Big(\frac{\overline{a}_2}{1-\overline{a}_2}\Big)}, \text{ by the induction hypothesis,}\\ &\le \frac{\overline{a}}{1-\overline{a}}, \text{ by the case n=2.}\end{aligned}$$

Now we show that if (27) holds for a value of $n > 2$ then it holds for $n - 1$.

Let $\overline{a} = \mathfrak{A}_{n-1}(a_1, \ldots a_{n-1})$, then:

$$\begin{aligned}\Big(\frac{\overline{a}}{1-\overline{a}} \prod_{i=1}^{n-1} \frac{a_i}{1-a_i}\Big)^{1/n} &\le \frac{\mathfrak{A}_n(\overline{a}, a_1, \ldots, a_{n-1})}{1-\mathfrak{A}_n(\overline{a}, a_1, \ldots, a_{n-1})}, \text{by the induction hypothesis,}\\ &= \frac{\overline{a}}{1-\overline{a}}.\end{aligned}$$

This completes the induction, by II 2.2.4 Theorem 7.

(*iii*) [*Sándor 1990b*] Inequality (26) is an easy deduction from Henrici's inequality, 2(4). In 2(4) putting $a_i = (1-b_i)/b_i$, $1 \le i \le n$, gives (25) with $\underline{a}$ replaced by $\underline{b}$.

(*iv*) Another proof, due to Segre, can be found in VI 4.5 Example(v); [*Segre*]. □

Among the many generalizations of (26) the following are worth noting.

THEOREM 15 *If $\underline{a}$ and $\underline{w}$ are positive n-tuples, $n \geq 2$, with $0 < \underline{a} \leq \dfrac{1}{2}$ then*

$$\frac{\mathfrak{H}_n(\underline{a};\underline{w})}{\mathfrak{H}'_n(\underline{a};\underline{w})} \leq \frac{\mathfrak{G}_n(\underline{a};\underline{w})}{\mathfrak{G}'_n(\underline{a};\underline{w})};$$
$$\mathfrak{G}_n(\underline{a};\underline{w}) - \mathfrak{G}'_n(\underline{a};\underline{w}) \leq \mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{A}'_n(\underline{a};\underline{w});$$
$$\frac{1}{\mathfrak{H}'_n(\underline{a})} - \frac{1}{\mathfrak{H}_n(\underline{a})} \leq \frac{1}{\mathfrak{G}'_n(\underline{a})} - \frac{1}{\mathfrak{G}_n(\underline{a})} \leq \frac{1}{\mathfrak{A}'_n(\underline{a})} - \frac{1}{\mathfrak{A}_n(\underline{a})};$$

with equality if and only if $\underline{a}$ is constant.

REMARK (v) The first inequality is due to Wang & Wang, [*Wang P F & Wang W L; Wang W L & Wang P F 1984*], and as a result the combined inequality,

$$\frac{\mathfrak{H}_n(\underline{a};\underline{w})}{\mathfrak{H}'_n(\underline{a};\underline{w})} \leq \frac{\mathfrak{G}_n(\underline{a};\underline{w})}{\mathfrak{G}'_n(\underline{a};\underline{w})} \leq \frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{A}'_n(\underline{a};\underline{w})},$$

is called the *Ky Fan-Wang-Wang inequality*.

REMARK (vi) The second inequality, an additive analogue of the Ky Fan inequality, is due to Alzer; [*Alzer 1988a,1990e,1997b; McGregor 1996; Mercer P*].

REMARK (vii) The third inequality is due to Alzer, [*Alzer 1990g*]; it generalizes the equal weight case of an earlier result of Sándor, [*Sándor 1991a*]:

$$\frac{1}{\mathfrak{H}'_n(\underline{a};\underline{w})} - \frac{1}{\mathfrak{H}_n(\underline{a};\underline{w})} \leq \frac{1}{\mathfrak{A}'_n(\underline{a};\underline{w})} - \frac{1}{\mathfrak{A}_n(\underline{a};\underline{w})}.$$

Alzer, [*Alzer 1995a*], has also proved

$$\frac{1}{\mathfrak{H}'_n(\underline{a};\underline{w})} - \frac{1}{\mathfrak{H}_n(\underline{a};\underline{w})} \leq \frac{1}{\mathfrak{G}'_n(\underline{a};\underline{w})} - \frac{1}{\mathfrak{G}_n(\underline{a};\underline{w})}.$$

Reference should be made to VI 2.1.2 Theorem 13.

Extensions to all power means have also been obtained; [*Wang Z & Chen; Wang, Li & Chen 1988*].

THEOREM 16 *If $\underline{a}$ is an n-tuple, $n \geq 2$, with $0 < \underline{a} \leq \dfrac{1}{2}$ and $r, s \in \mathbb{R}$, $r < s$, then*

$$\frac{\mathfrak{M}_n^{[r]}(\underline{a})}{\mathfrak{M}_n'^{[r]}(\underline{a})} \leq \frac{\mathfrak{M}_n^{s]}(\underline{a})}{\mathfrak{M}_n'^{[s]}(\underline{a})}$$

if and only if $|r+s| < 3$, and $\dfrac{2^r}{r} \geq \dfrac{2^s}{s}$ if $r > 0$, and $r2^r \leq s2^s$ if $s < 0$.

In addition we have various extensions of Rado-Popoviciu type due to Wang C L; [*Alzer 1990i; Kwon 1996; Wang C L 1980c,1984b*].

THEOREM 17 *If $\underline{a}$ and $\underline{w}$ are positive n-tuples, $n \ge 2$, with $0 < \underline{a} \le \dfrac{1}{2}$ then*

$$W_{n-1}\big(\mathfrak{A}_{n-1}(\underline{a};\underline{w})\mathfrak{G}'_{n-1}(\underline{a};\underline{w}) - \mathfrak{A}'_{n-1}(\underline{a};\underline{w})\mathfrak{G}_{n-1}(\underline{a};\underline{w})\big) \le W_n\big(\mathfrak{A}_n(\underline{a};\underline{w})\mathfrak{G}'_n(\underline{a};\underline{w}) - \mathfrak{A}'_n(\underline{a};\underline{w})\mathfrak{G}_n(\underline{a};\underline{w})\big);$$

$$\left(\frac{\mathfrak{A}_{n-1}(\underline{a};\underline{w})/\mathfrak{A}'_{n-1}(\underline{a};\underline{w})}{\mathfrak{G}_{n-1}(\underline{a};\underline{w})/\mathfrak{G}'_{n-1}(\underline{a};\underline{w})}\right)^{W_{n-1}} \le \left(\frac{\mathfrak{A}_n(\underline{a};\underline{w})/\mathfrak{A}'_n(\underline{a};\underline{w})}{\mathfrak{G}_n(\underline{a};\underline{w})/\mathfrak{G}'_n(\underline{a};\underline{w})}\right)^{W_n}.$$

Much research has been carried out on this inequality and many generalizations have been given by several authors. There is an important survey article by Alzer that contains interesting related results and a comprehensive bibliography, [*Alzer 1995a*]. Converse inequalities have also been proved and Alzer has given various Ky Fan inequalities for the pseudo arithmetic and geometric means, see II 5.8; [*Alzer 1990q*].

For further information see all the above references and also the following: [*Alzer 1988b,1989f,g,h,i,1990f,h,j,k,m,1991e,1993a,b,1996b,1999a,2001b; Alzer, Ruscheweyh & Salinas; Chan, Goldberg & Gonek; Chong K K 2000,2001; Dragomir 1992b; Dragomir & Ionescu 1990; El-Neweihi & Proschan; Farwig & Zwick 1985; Gao P; Jiang; Ku, Ku & Zhang 1999; Lawrence & Segalman; McGregor 1993a,b; Mercer A 1996,1998, 2000; Neuman & Pečarić; Pečarić 1980b,1981c; Pečarić & Alzer; Pečarić & Raşa 2000; Pečarić & Zwick; Popoviciu 1961b; Vasić & Janić; Wang C L 1988a; Wang C S; Wang W L 1999; Wang & Wu; Wang Z, Chen & Li; Yang & Wang ; Yang Y 1988; Zhang Z L; Zwick*].

4.5 MEANS ON THE MOVE The results of III 6.1 have been extended to a certain class of quasi-arithmetic means; see [*Boas & Brenner; Gigante 1995a*].

THEOREM 18 *If $\mathcal{M} : \mathbb{R}_+^* \mapsto \mathbb{R}$ and $\underline{a}$ and $\underline{w}$ are positive n-tuples then*

$$\lim_{t\to\infty}\big(\mathfrak{M}_n(\underline{a}+t\underline{e};\underline{w}) - t\big) = \mathfrak{A}_n(\underline{a};\underline{w})$$

provided $\mathcal{M}$ satisfies one of the two following sets of conditions.

(*a*) $\displaystyle\lim_{t\to\infty}\mathcal{M}(t) = \lim_{t\to\infty}\mathcal{M}^{-1}(t) = \infty;\ \lim_{t\to\infty}\frac{\mathcal{M}'(t)}{\mathcal{M}(t)} = \lim_{t\to\infty}\frac{(\mathcal{M}^{-1})'(t)}{\mathcal{M}^{-1}(t)} = 0,$

$\displaystyle\lim_{t\to\infty}\frac{\mathcal{M}'(t+s)}{\mathcal{M}(t)} = \lim_{t\to\infty}\frac{(\mathcal{M}^{-1})'(t+s)}{\mathcal{M}^{-1}(t)} = 1,$ *uniformly on compacts sets, of s;*

(*b*) $\displaystyle\lim_{t\to\infty}\mathcal{M}(t) = \lim_{t\to\infty}\mathcal{M}^{-1}(t) = 0;$ *$\mathcal{M}$ has the other properties listed in (a), and*

$\displaystyle\lim_{t\to\infty}\frac{(\mathcal{M}^{-1})'(t+s)}{\mathcal{M}^{-1}(t)} = 1,\ (\mathcal{M}^{-1})'\Big(\mathcal{M}(t)(1+\epsilon)\Big) = (1+\eta)\Big((\mathcal{M}^{-1})'\circ\mathcal{M}\Big)(t),$

where $\epsilon, \eta \to 0$, as $t \to \infty$.

REMARK (i) It is easily checked that if $\mathcal{M}(x) = x^r$, $r \neq 0$, then $\mathcal{M}$ has these properties.
Further extensions can be found in [*Aczél, Losconzi & Páles; Aczél & Páles; Brenner & Carlson*].

THEOREM 19 *Let $\mathcal{M}$ be a homogeneous function on positive n-tuples differentiable at $\underline{e}$, and such that if $\underline{a}$ is constant, $\underline{a} = (a, \dots, a)$, say, then $\mathcal{M}(\underline{a}) = a$. If $\partial\mathcal{M}/\partial a_k(\underline{e}) = w_k$, $1 \leq k \leq n$, then $W_n = 1$ and*

$$\lim_{t\to\infty} \big(\mathcal{M}(\underline{a} + t\underline{e}) - t\big) = \mathfrak{A}_n(\underline{a}; \underline{w}). \tag{28}$$

If $\mathcal{M}$ has continuous second order derivatives in the neighbourhood of $\underline{e}$,

$$\mathcal{M}(\underline{a} + t\underline{e}) = t + \mathfrak{A}_n(\underline{a}; \underline{w}) + O(1/t),\ t \to \infty. \tag{29}$$

□ By Euler's theorem on homogeneous functions, see I 4.6 Remark (ii), $\sum_{k=1}^{n} a_k \dfrac{\partial\mathcal{M}}{\partial a_k}(\underline{a}) = \mathcal{M}(\underline{a})$. Substituting $\underline{a} = \underline{e}$ gives $W_n = 1$. In addition

$$\begin{aligned}\lim_{t\to\infty} \big(\mathcal{M}(\underline{a} + t\underline{e}) - t\big) &= \lim_{t\to\infty} t\big(\mathcal{M}(t^{-1}\underline{a} + \underline{e}) - 1\big)\\ &= \lim_{s\to 0} \frac{\mathcal{M}(\underline{e} + s\underline{a}) - \mathcal{M}(\underline{e})}{s} = \nabla\mathcal{M}(\underline{e}).\underline{a}, \quad \text{which is (28).}\end{aligned}$$

Finally if $\mathcal{M}$ if has continuous second order derivatives in the neighbourhood of $\underline{e}$,

$$\begin{aligned}\mathcal{M}(\underline{a} + t\underline{e}) =& t\mathcal{M}(\underline{e} + \underline{a}/t), \text{ by homogeneity,}\\ =& t\mathcal{M}(\underline{e}) + \sum_{k=1}^{n} a_k \frac{\partial\mathcal{M}}{\partial a_k}(\underline{e}) + O(t^{-1}), \text{ by Taylor's theorem,}\end{aligned}$$

which is (29). □

REMARK (ii) An interesting application of this result to define scales of means is in [*Persson & Sjöstrand*], see III 6.1.

5 Generalizations of the Hölder and Minkowski Inequalities

In section 2 we discussed the comparability of means and obtained the inequality 2(6), a generalization of (r;s). It is natural to ask if the other inequalities between the power means, III 3.1.2(7) and III 3.1.3(8), that are derived from (H) and (M), have analogues for the more general means being considered here.
First we give a simple generalization of III 3.1.3 Theorem 7(b) and inequality III 3.1.3(9); [*Aumann 1934; Jessen 1931a*].

THEOREM 1 *Let $\mathcal{M}$ and $\mathcal{N}$ be continuous strictly monotonic functions defined on the same interval, then for all appropriate $\underline{a}_{(i)} = (a_{i1}, \ldots, a_{in})$, $1 \le i \le m$, $\underline{a}^{(j)} = (a_{1j}, \ldots, a_{mj})$, $1 \le j \le n$, and non-negative n-tuples $\underline{u}$ and $\underline{v}$,*

$$\mathfrak{N}_n\big(\mathfrak{M}_m(\underline{a}^{(j)}; \underline{u}); \underline{v}\big) \le \mathfrak{M}_m\big(\mathfrak{N}_n(\underline{a}_{(i)}; \underline{v}); \underline{u}\big) \tag{1}$$

if and only if the quasi-arithmetic $\mathcal{H}$-mean, where $\mathcal{H} = \mathcal{N} \circ \mathcal{M}^{-1}$, is a convex function on the set of sequences on which it is defined.

□ (1) is an easy consequence of the property of the function $\mathcal{H}$ stated in the theorem. □

REMARK (i) A simple condition that implies that the function $\mathcal{H}$ above satisfies the required condition is given in Theorem 7 below.

We now consider a more general problem and to fix ideas suppose that we have three functions $\mathcal{K} : [k_1, k_2] \mapsto \mathbb{R}$, $\mathcal{L} : [\ell_1, \ell_2] \mapsto \mathbb{R}$, $\mathcal{M} : [m_1, m_2] \mapsto \mathbb{R}$. and the associated quasi-arithmetic means: $\mathfrak{K}_n(\underline{a}; \underline{w})$, $\mathfrak{L}_n(\underline{b}; \underline{w})$, and $\mathfrak{M}_n(\underline{c}; \underline{w})$ where $\underline{w}$ is a positive n-tuple with $W_n = 1$.

We are interested in obtaining inequalities of the type

$$f\big(\mathfrak{K}_n(\underline{a}; \underline{w}), \mathfrak{L}_n(\underline{b}; \underline{w})\big) \ge \mathfrak{M}_n\big(f(\underline{a}, \underline{b}); \underline{w}\big) \tag{2}$$

or its reverse, where $f : [k_1, k_2] \times [\ell_1, \ell_2] \mapsto [m_1, m_2]$ is a continuous function.

The most important examples of f are (i) $f(x, y) = x + y$, when inequality (2) is said to be *additive*, and (ii) $f(x, y) = xy$, when inequality (2) is said to be *multiplicative*. For instance the III 3.1.2(7) is multiplicative, while III 3.1.3(8) is additive.

The extreme generality of (2) means that often different looking inequalities are in some sense equivalent and it is worthwhile clearing up this point before proceeding to the main theorem.

(A) A new inequality can be obtained from (2) by changing f as follows.

Let $\sigma : [m_1, m_2] \mapsto [j_1, j_2]$ be strictly monotonic and continuous and put $g = \sigma \circ f$, $\mathcal{J} = \mathcal{M} \circ \sigma^{-1}$, when (2) becomes

$$g\big(\mathfrak{K}_n(\underline{a}; \underline{w}), \mathfrak{L}_n(\underline{b}; \underline{w})\big) \ge \mathfrak{J}_n\big(g(\underline{a}, \underline{b}); \underline{w}\big), \tag{3}$$

if we assume σ to be increasing, or (~3) if we assume σ to be decreasing.

EXAMPLE (i) In particular taking $\sigma = \mathcal{M}$, we have that $\mathcal{J} = 1$, and (3) becomes

$$g\big(\mathfrak{K}_n(\underline{a}; \underline{w}), \mathfrak{L}_n(\underline{b}; \underline{w})\big) \ge \mathfrak{A}_n\big(g(\underline{a}, \underline{b}); \underline{w}\big),$$

where $g = \mathcal{M} \circ f$.

(B) Instead of changing f we could change $\underline{a}$ and $\underline{b}$ as follows.

Let $\kappa : [k_1, k_2] \mapsto [s_1, s_2]$, $\lambda : [\ell_1, \ell_2] \mapsto [t_1, t_2]$ be strictly monotonic and continuous, put $\underline{a}^* = \kappa(\underline{a})$, $\underline{b}^* = \lambda(\underline{b})$, and let $g : [s_1, s_2] \times [t_1, t_2] \mapsto [m_1, m_2]$ be defined by $g(s,t) = f\big(\kappa^{-1}(s), \lambda^{-1}(t)\big)$ and finally put $\mathcal{S} = \mathcal{K} \circ \kappa^{-1}$, $\mathcal{T} = \mathcal{L} \circ \lambda^{-1}$, then (2) becomes

$$g\big(\mathfrak{S}_n(\underline{a}^*; \underline{w}), \mathfrak{T}_n(\underline{b}^*; \underline{w})\big) \ge \mathfrak{M}_n\big(g(\underline{a}^*, \underline{b}^*; \underline{w}\big), \tag{4}$$

or (∼4).

Example (ii) In particular taking $\kappa = \mathcal{K}$, $\lambda = \mathcal{L}$ then $\mathcal{S} = \mathcal{T} = 1$ and (4) becomes

$$g\big(\mathfrak{A}_n(\underline{a}^*; \underline{w}), \mathfrak{A}_n(\underline{b}^*; \underline{w})\big) \ge \mathfrak{M}_n\big(g(\underline{a}^*, \underline{b}^*); \underline{w}\big),$$

where $g(s,t) = f\big(\mathcal{K}^{-1}(s), \mathcal{L}^{-1}(t)\big)$.

Definition 2 *Inequalities derived from an inequality of type (2), or (∼2), by a finite number of applications of the procedures (A), and, or (B) are said to be equivalent.*

Remark (ii) This relation is an equivalence relation[3].

Remark (iii) It is clear that both of the operations (A) and (B) take strict inequalities into strict inequalities; so if the cases of equality are known for an equality then they can be determined for the equivalent inequality.

Example (iii) It is not difficult to see that every multiplicative inequality is equivalent to some additive inequality. Suppose that $f(x,y) = xy$ and that the domains of both $\mathcal{K}$ and $\mathcal{L}$ are $\mathbb{R}_+$ when (2) is just

$$\mathfrak{K}_n(\underline{a}; \underline{w})\mathfrak{L}_n(\underline{b}; \underline{w}) \ge \mathfrak{M}_n(\underline{a}\,\underline{b}; \underline{w})$$

Using (4) with $\kappa = \lambda = \log$ this becomes

$$\mathfrak{S}_n(\underline{a}^*; \underline{w}) + \mathfrak{T}_n(\underline{b}^*; \underline{w}) \ge \mathfrak{U}_n(\underline{a}^* + \underline{b}^*; \underline{w})$$

where $\underline{a}^* = \log \underline{a}$, $\underline{b}^* = \log \underline{b}$, $\mathcal{S} = \mathcal{K} \circ \exp$, $\mathcal{T} = \mathcal{L} \circ \exp$, $\mathcal{U} = \mathcal{M} \circ \exp$.

Example (iv) Consider III 3.1.2(7) with q, r, s positive then, as in Example (iii), putting $\gamma_1 = e^q, \gamma_2 = e^r, \gamma_3 = e^s$ we get the following equivalent inequality;

$$\mathfrak{M}_{\gamma_1,n}(\underline{a}^*; \underline{w}) + \mathfrak{M}_{\gamma_2,n}(\underline{b}^*; \underline{w}) \ge \mathfrak{M}_{\gamma_3,n}(\underline{a}^* + \underline{b}^*; \underline{w}), \tag{5}$$

---

[3] That is if we write $(I)\sim(J)$ when the two inequalities (I),(J) are equivalent then: (i) $(I)\sim(I)$, (ii) $(I)\sim(J)\Rightarrow(J)\sim(I)$, (iii) $(I)\sim(J)$ and $(J)\sim(K)\Rightarrow(I)\sim(K)$; see [*CE p.559; EM3 p.402*].

provided $1/\log\gamma_1 + 1/\log\gamma_2 \le 1/\log\gamma_3$, $\gamma_1, \gamma_2, \gamma_3 \ge 1$; these means are defined in 1.1 Example (i). The case (a) of equality for III 3.1.2 (7) shows that in the case of non-constant $\underline{a}^*, \underline{b}^*$ there is equality in (5) if and only if $1/\log\gamma_1 + 1/\log\gamma_2 = 1/\log\gamma_3$, and $\gamma_1^{\underline{a}^*} = \gamma_2^{\underline{b}^*}$.

EXAMPLE (v) The reader can easily check that the additive inequality III 3.1.3(8) is equivalent to the multiplicative,

$$\Big(\exp\sum_{i=1}^n w_i(\log a_i)^r\Big)\Big(\exp\sum_{i=1}^n w_i(\log b_i)^r\Big) \ge \Big(\exp\sum_{i=1}^n w_i(\log a_ib_i)^r\Big), \quad r > 1.$$

THEOREM 3 *A necessary and sufficient condition for (2), respectively (~2), is that the function*

$$H(s,t) = \mathcal{M}\Big(f\big(\mathcal{K}^{-1}(s), \mathcal{L}^{-1}(t)\big)\Big),$$

be concave, respectively convex.

□ Use (A) as in Example (i) and then (B) as in Example (ii) to get the following inequality equivalent to (2),

$$H\big(\mathfrak{A}_n(\underline{a}^*;\underline{w}), \mathfrak{A}_n(\underline{b}^*;\underline{w})\big) \ge \mathfrak{A}_n\big(H(\underline{a}^*,\underline{b}^*);\underline{w}\big)$$

which just says that H is concave. □

EXAMPLE (vi) Let$f(x,y) = xy$ and $\mathcal{M} = 1$ then $H(s,t) = \mathcal{K}^{-1}(s)\mathcal{L}^{-1}(t)$. If H is concave then (2) gives the following generalization of III 3.1.2(7).

$$\mathfrak{A}_n(\underline{a}\,\underline{b};\underline{w}) \le \mathfrak{K}_n(\underline{a};\underline{w})\mathfrak{L}_n(\underline{b};\underline{w}). \tag{6}$$

In particular if $H(s,t) = s^{1/q}t^{1/r}$ then H is concave if $q, r > 1$ and $q^{-1} + r^{-1} \le 1$; if all these inequalities are reversed then H is convex. In either case we get III 3.1.2(7) with $s = 1$. To get the general case take $H(s,t) = s^{p/q}t^{p/r}$, using p here for the s used in III 3.1.2(7).

EXAMPLE (vii) If $H(s,t) = (s^{1/p} + t^{1/p})^p$ then H is concave if $p > 1$, see I 4.6 Theorem 40 (d), (e), and (2) reduces to (M).

EXAMPLE (viii) Finally taking $H(s,t) = \exp\Big(\dfrac{\log\gamma_3}{\log\gamma_1}\log s + \dfrac{\log\gamma_3}{\log\gamma_2}\log t\Big)$ then H is concave provided $1/\log\gamma_1 + 1/\log\gamma_2 \le 1/\log\gamma_3$, see I 4.6 Theorem 40 (d), (e), and then (2) reduces to (5).

It follows from the discussion of convex functions of two variables in I 4.6 that it is often convenient to assume the existence of continuous first and second order derivatives of functions being used, and not much generality is lost in so doing.

COROLLARY 4 *If H is defined as in Theorem 3 and if $H''_{11} < 0$, or $H''_{22} < 0$, and $H''_{11}H''_{22} - H''^2_{12} > 0$ then there is equality in (2) if and only if $\underline{a}$ and $\underline{b}$ are constant.*

□ Immediate using I 4.6 Remark (iv) and I 4.6 (21). □

REMARK (iv) If the inequalities in Corollary 4 are not strict other cases of equality are possible. More precision can be obtained but for details the reader is referred to [*Beck 1969b*].

EXAMPLE (ix) Consider the case considered in Example (vi) above, that is $H(s,t) = s^{p/q}t^{p/r}$ then:
$H''_{11}(s,t) = \frac{p}{q}(\frac{p}{q} - 1)s^{p/q-2}t^{p/r}$; $H''_{11}(s,t) = \frac{p}{r}(\frac{p}{r} - 1)s^{p/q}t^{p/r-2}$; $H''_{12}(s,t) = \frac{p^2}{qr}s^{p/q-1}t^{p/r-1}$. Hence

$$\big(H''_{11}H''_{22} - H''^2_{12}\big)(s,t) = \frac{p^2}{qr}(1 - p/q - p/r)s^{2(p/q-1)}t^{p/r-1}.$$

Clearly if $p/q \ge 0, p/r \ge 0$ and $p/q + p/r \le 1$ then H is concave, by I 4.6 Remark (iv). If these inequalities are strict then by Corollary 4 there is equality if and only if $\underline{a}$ and $\underline{b}$ are constant. Suppose however that $p/q + p/r = 1$, as in (H) when $p = 1$, then we have to consider the quadratic form $h^2H''_{11} + 2hkH''_{12} + k^2H''_{22}$, and in this case this is just $-\frac{p^2}{qr}s^{p/q}t^{p/r}\frac{p^2}{qr}(h/s - k/t)^2$, which is zero if $h/s = k/t$; so then there is equality in (2) if $\underline{a}^q \sim \underline{b}^r$.

COROLLARY 5 *If $f(x,y) = x + y$, when $H(s,t) = \mathcal{M}\big(\mathcal{K}^{-1}(s) + \mathcal{L}^{-1}(t)\big)$, and if $E = \mathcal{K}'/\mathcal{K}'', F = \mathcal{L}'/\mathcal{L}'', G = \mathcal{M}'/\mathcal{M}''$ and if all of these first and second derivatives are positive then (2) holds if and only if*

$$G(x+y) \ge E(x) + F(y). \tag{7}$$

□ In this case the conditions in I 4.6 Remark (iv) give

$$\frac{1}{G(x+y)} \le \frac{1}{E(x)}; \quad \frac{1}{G(x+y)} \le \frac{1}{F(y)}; \quad \frac{1}{G(x+y)}\Big(\frac{1}{E(x)} + \frac{1}{F(y)}\Big) \le \frac{1}{E(x)F(y)},$$

which easily leads to the result. □

REMARK (v) The functions $\mathcal{K}, \mathcal{L}, \mathcal{M}$ determine the functions E, F, G respectively, and the converse is true. For instance

$$\mathcal{K}(x) = c\int_a^x \exp\Big(\int_a^u \frac{1}{E(t)}\,\mathrm{d}t\Big)\,du, \quad c > 0,\ x \ge a.$$

EXAMPLE (x) If $E = F = G$ then (7) just says that E is super-additive, which is implied if $E(x)/x$ is increasing. For instance if $E(x) = cx$, $c > 0$ then $\mathcal{K}(x) = x^{1+1/c}$ for which the additive form of (2) becomes III 3.1.3(8), taking $c = 1/(r-1)$.

EXAMPLE (xi) If in the previous example we take $E = \tan$ then $\mathcal{K} = -\cos$ and (2) leads to the inequality $\mathfrak{C}_n(\underline{a};\underline{w}) + \mathfrak{C}_n(\underline{b};\underline{w}) \geq \mathfrak{C}_n(\underline{a}+\underline{b};\underline{w})$, if $0 \leq \underline{a},\underline{b} \leq \pi/4$; the means being defined in 1.1 Example(v).

EXAMPLE (xii) Another simple possibility is to take functions E, F, G constant; $E = \epsilon, F = \phi, G = \gamma$ say, with $\gamma \geq \epsilon + \phi$. Then $\mathcal{K}(s) = x^s, \mathcal{L}(s) = y^s, \mathcal{M}(s) = z^s$ where $x = e^{1/\epsilon}, y = e^{1/\phi}, z = e^{1/\gamma}$. In this case (2) reduces to (5).

COROLLARY 6 *If $f(x,y) = xy$, when $H(s,t) = \mathcal{M}\big(\mathcal{K}^{-1}(s)\mathcal{L}^{-1}(t)\big)$, and if*

$$A(x) = \frac{\mathcal{K}'(x)}{\mathcal{K}'(x) + x\mathcal{K}''(x)}, \quad B(x) = \frac{\mathcal{L}'(x)}{\mathcal{L}'(x) + x\mathcal{L}''(x)}, \quad C(x) = \frac{\mathcal{M}'(x)}{\mathcal{M}'(x) + x\mathcal{M}''(x)},$$

and if $\mathcal{K}', \mathcal{L}', \mathcal{M}', A, B, C$ are all positive then (2) holds if and only if

$$C(xy) \geq A(x) + B(y).$$

□ The proof is similar to that of Corollary 5; see [*Beck 1970*]. □

REMARK (vi) As in Remark (v), a knowledge of the functions A, B, C determines the functions $\mathcal{K}, \mathcal{L}, \mathcal{M}$ respectively, for instance

$$\mathcal{K}(x) = c\int_a^x \exp\Big(\int_a^u \frac{1}{tA(t)}\,\mathrm{d}t\Big)\frac{1}{u}\,\mathrm{d}u$$

EXAMPLE (xiii) If the functions A, B, C constants, $A = \alpha, B = \beta, C = \gamma$, say, with $\gamma \geq \alpha + \beta$ then the multiplicative form of (2) becomes III 3.1.2(7), with $p = 1/\alpha, q = 1/\beta, r = 1/\gamma$.

A different application of Theorem 3 is to determine conditions under which $\mathfrak{M}_n(\underline{a};\underline{w})$ is convex as a function of $\underline{a}$.

THEOREM 7 *If $\mathcal{M}$ has a continuous second order derivative and if $\mathcal{M}' > 0, \mathcal{M}'' > 0$, then $\mathfrak{M}_n(\underline{a};\underline{w})$ is convex as a function of $\underline{a}$ if and only if $\mathcal{M}'/\mathcal{M}''$ is concave.*

□ By continuity $\mathfrak{M}_n(\underline{a};\underline{w})$ is convex if and only if and only if for all $\underline{a}, \underline{b}$,

$$\mathfrak{M}_n\Big(\frac{\underline{a}+\underline{b}}{2};\underline{w}\Big) \leq \frac{\mathfrak{M}_n(\underline{a};\underline{w}) + \mathfrak{M}_n(\underline{a};\underline{w})}{2}, \tag{8}$$

see I 4.5.1, I 4.6. By Theorem 3 with $\mathcal{K} = \mathcal{L} = \mathcal{M}$ and $f(x,y) = (x+y)/2$ this holds if and only if $H(s,t) = \mathcal{M}\big(\frac{1}{2}\mathcal{M}^{-1}(s) + \frac{1}{2}\mathcal{M}^{-1}(t)\big)$ is concave.

An application of I 4.6 (21), shows that this is so if and only if

$$\frac{\mathcal{M}'(\frac{x+y}{2})}{\mathcal{M}''(\frac{x+y}{2})} \geq \frac{1}{2}\Big(\frac{\mathcal{M}'(x)}{\mathcal{M}''(x)} + \frac{\mathcal{M}'(y)}{\mathcal{M}''(y)}\Big),$$

which completes the proof. □

REMARK (vii) The main results above are due to Beck although less successful attempts to obtain such results had been made earlier by Cooper; see [*Beck 1970, 1975,1977; Cooper 1927b,1928*].

REMARK (viii) Inequality (8) is another generalization of (M), and Theorem 7 has been used by various authors to obtain other general inequalities.

EXAMPLE (xiv) If $\underline{a}_{(i)}, \underline{w}_{(i)}$ are defined using the notation in Theorem 1, each $\underline{w}_{(i)}$ a positive n-tuple with $\sum_{j=1}^{n} w_{ij} = 1$, $1 \le i \le m$, then the function

$$S(x) = \mathcal{M}\big(\sum_{i=1}^{m} u_i \mathfrak{M}_n(x\underline{a}_{(i)}; \underline{w}_{(i)})\big),$$

where $\underline{u}$ is a positive m-tuple and $\mathcal{M}$ satisfies the hypotheses of Theorem 7, is convex and

$$S(x) \ge \mathcal{M}\big(\sum_{i=1}^{m} u_i \mathfrak{A}_k(x\underline{a}_{(i)}; \underline{w}_{(i)})\big).$$

This generalizes the result of Eliezer & Daykin mentioned in III 2.5.4 Remark (i); see [*Godunova & Čebaevskaya; Godunova & Levin 1968*].

Inequality (6) generalizes (H) and this particular case of (6) is derived from two inverse functions in either of the two following ways.
Let $\alpha > 0$, and put $k(x) = x^\alpha, \ell(x) = k^{-1}(x) = x^{1/\alpha}$ and then define $\mathcal{K}, \mathcal{L}$ by either

$$\text{(i) } \mathcal{K}(x) = xk(x), \mathcal{L}(x) = x\ell(x), \quad \text{or} \quad \text{(ii) } \mathcal{K}(x) = \textstyle\int_0^x k, \mathcal{L}(x) = \int_0^x \ell.$$

In looking for more straight forward conditions for the validity of (6) than the concavity of $\mathcal{K}^{-1}(s)\mathcal{L}^{-1}(t)$ one might wonder if any other pairs of inverse functions k, ℓ could be used as above. That this is not the case follows from the next theorem.

THEOREM 8 *Let $k : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be strictly increasing, with continuous second derivative, and $k(0) = 0$; let $\ell = k^{-1}$ and define $\mathcal{K}$ and $\mathcal{L}$ by*

$$\textit{either} \quad (a)\ \mathcal{K}(x) = xk(x),\ \mathcal{L}(x) = x\ell(x). \qquad \textit{or} \quad (b)\ \mathcal{K}(x) = \int_0^x k, \mathcal{L}(x) = \int_0^x \ell.$$

If (6) holds for all $\underline{a}, \underline{b}$ then k is a power function and the inequality (6) is just (H).

□ By Theorem 3 $\mathcal{K}^{-1}(s)\mathcal{L}^{-1}(t)$ is concave so from I 4.6 Remark (iv) we have, $(\mathcal{K}^{-1})'' \le 0$, $(\mathcal{L}^{-1})'' \le 0$ and for all $x, y > 0$,

$$\Big((\mathcal{K}^{-1})'(x)(\mathcal{L}^{-1})'(y)\Big)^2 \le \mathcal{K}^{-1}(x)\mathcal{L}^{-1}(y)((\mathcal{K}^{-1})''(x)(\mathcal{L}^{-1})''(y) \tag{9}$$

Assume now that (a) holds. Then : $\mathcal{K}(\mathcal{K}^{-1}(x)) = \mathcal{K}^{-1}(x)k(\mathcal{K}^{-1}(x))$, from which $k(\mathcal{K}^{-1}(x)) = \dfrac{x}{\mathcal{K}^{-1}(x)}$. Hence $\mathcal{K}^{-1}(x) = \ell\Big(\dfrac{x}{\mathcal{K}^{-1}(x)}\Big)$, or $x = \mathcal{L}\Big(\dfrac{x}{\mathcal{K}^{-1}(x)}\Big)$ and so

$$\mathcal{K}^{-1}(x)\mathcal{L}^{-1}(x) = x. \tag{10}$$

Differentiating (10) twice gives

$$(\mathcal{K}^{-1})''\mathcal{L}^{-1} + 2(\mathcal{K}^{-1})'(\mathcal{L}^{-1})' + \mathcal{K}^{-1}(\mathcal{L}^{-1})'' = 0 \tag{11}$$

Applying (9) with $x = y$ we get from (11) that

$$\big((\mathcal{K}^{-1})''\mathcal{L}^{-1} + \mathcal{K}^{-1}(\mathcal{L}^{-1})''\big)^2 = \big(2(\mathcal{K}^{-1})'(\mathcal{L}^{-1})'\big)^2 \le 4\mathcal{K}^{-1}\mathcal{L}^{-1}(\mathcal{K}^{-1})''(\mathcal{L}^{-1})''$$

or $\big((\mathcal{K}^{-1})''\mathcal{L}^{-1} - \mathcal{K}^{-1}(\mathcal{L}^{-1})''\big)^2 \le 0$. Hence

$$(\mathcal{K}^{-1})''\mathcal{L}^{-1} = \mathcal{K}^{-1}(\mathcal{L}L^{-1})'' = -(\mathcal{K}^{-1})'(\mathcal{L}^{-1})',\ (\mathcal{K}^{-1})''\mathcal{L}^{-1} + (\mathcal{K}^{-1})'(\mathcal{L}^{-1})' = 0,$$

or $(\mathcal{K}^{-1})'\mathcal{L}^{-1}$ is constant. This by (10) implies that $\dfrac{x(\mathcal{K}^{-1})'(x)}{\mathcal{K}^{-1}(x)}$ is constant or $\mathcal{K}^{-1}$ is a power, from which the theorem follows.
If instead we assume that (b) holds then (9) becomes $\dfrac{k(\alpha)\ell(\beta)}{\alpha\beta k'(\alpha)\ell'(\beta)} \le 1$, where $x = \mathcal{K}(\alpha), y = \mathcal{L}(\beta)$. Since ℓ is the inverse of k this can be written, putting $\gamma = \mathcal{L}(\beta)$, $\dfrac{k(\alpha)}{\alpha k'(\alpha)} \le \dfrac{k(\gamma)}{\gamma k'(\gamma)}$.
Since α, γ are arbitrary this implies that $\dfrac{k(x)}{xk'(x)}$ is constant and so k is a power.□

REMARK (ix) The part (a) of this result is due to Cooper; see [*HLP pp.82–83*], [*Cooper 1928*].

Some results have been obtained for the generalized weighted sums defined in 2(8)

THEOREM 9 *If* $\mathcal{M} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *is convex, strictly monotonic, with* $\mathcal{M}(0) = 0$ *and* $\log \circ \mathcal{M} \circ \exp$ *convex then*

$$\mathcal{S}_{\mathcal{M}}(\underline{a} + \underline{b}; \underline{w}) \le \mathcal{S}_{\mathcal{M}}(\underline{a}; \underline{w}) + \mathcal{S}_{\mathcal{M}}(\underline{b}; \underline{w}).$$

This result, due to Milovanović & Milovanović, generalizes an equal weight case of Mulholland; [*AI p.57*], [*Milovanović & Milovanović 1979; Mulholland*]; see also [*Páles 1999*].

THEOREM 10 *If $\mathcal{M}$ has continuous second derivative, is strictly monotonic and convex and if $\mathcal{M}/\mathcal{M}'$ is convex then $\mathcal{S}_{\mathcal{M}}(\underline{a};\underline{w})$ is convex as a function of $\underline{a}$.*

This result is due to Vasić & Pečarić, and generalizes a result of Hardy, Littlewood & Pólya; [*HLP pp.85–88*], [*Vasić & Pečarić 1980a*].

6 Converse Inequalities

In this section the converse inequalities of III 4 are extended to quasi-arithmetic means. Several authors have results of this kind, see for instance [*Izumi*], but the discussion below gives the very general results of Beck, [*Beck 1969b*].

THEOREM 1 *Let $\mathcal{M}$ and $\mathcal{N}$ be strictly monotonic functions defined on $[m, M]$, a sub-interval of $\mathbb{R}$, $\mathcal{N}$ increasing, $\mathcal{N}$ strictly convex with respect to $\mathcal{M}$. Further suppose that $f : [m, M] \times [m, M] \mapsto \mathbb{R}$ is continuous, strictly increasing in the first variable, strictly decreasing in the second, and with $f(x, x) = C$, $m \le x \le M$. Then for all appropriate n-tuples $\underline{a}$ and all non-negative n-tuples $\underline{w}$ with $W_n = 1$,*

$$f\big(\mathfrak{N}_n(\underline{a};\underline{w}), \mathfrak{M}_n(\underline{a};\underline{w})\big) \ge C, \tag{1}$$

there is equality if and only if $\underline{a}$ is essentially constant. In addition there is at least one λ, $0 < \lambda < 1$, such that

$$f\big(\mathfrak{N}_n(\underline{a};\underline{w}), \mathfrak{M}_n(\underline{a};\underline{w})\big) \le f\big(\mathfrak{N}_2(m, M; \lambda, 1-\lambda), \mathfrak{M}_2(m, M; \lambda, 1-\lambda)\big). \tag{2}$$

If λ is unique there is equality in (2) if and only if there is an index set I such that $W_I = \lambda$, $a_i = M$, $i \in I$, $a_i = m$, $i \notin I$.

□ Inequality (1) is an immediate consequence of 2 Theorem 5 and the properties of f.

Since $m \le \underline{a} \le M$ by 1.1 Lemma 2(c) there is a $\underline{\lambda}$, $0 \le \underline{\lambda} \le 1$, such that $a_i = \mathfrak{N}_2(m, M; \lambda_i, 1-\lambda_i)$, $1 \le i \le n$. So by simple calculations

$$\mathfrak{N}_n(\underline{a};\underline{w}) = \mathfrak{N}_2(m, M; \lambda, 1-\lambda) \quad \text{where} \quad \lambda = \mathfrak{A}_n(\underline{\lambda};\underline{w}). \tag{3}$$

If $c_i = \mathfrak{M}_2(m, M; \lambda_i, 1-\lambda_i)$, $1 \le i \le n$, then $\mathfrak{M}_n(\underline{c};\underline{w}) = \mathfrak{M}_2(m, M; \lambda, 1-\lambda)$.
By 2 Theorem 5, $\underline{c} \le \underline{a}$, and so by 1.1 Lemma 2(a) $\mathfrak{M}_n(\underline{c};\underline{w}) \le \mathfrak{M}_n(\underline{a};\underline{w})$; that is

$$\mathfrak{M}_n(\underline{a};\underline{w}) \ge \mathfrak{M}_2(m, M; \lambda, 1-\lambda). \tag{4}$$

Using (3), (4) and the properties of f we get inequality (2), for some λ, $0 \le \lambda \le 1$. Now call the right-hand side of (2) $\phi(\lambda)$, $0 \le \lambda \le 1$; then this function is continuous and by the first part $\phi \ge C$, further $\phi(0) = \phi(1) = C$; so for some λ, $0 < \lambda < 1$, ϕ

takes its maximum value and this gives (2) for some λ, $0<\lambda<1$, and completes the proof except for the cases of equality.

For equality in (2) we need equality in (4), which requires that $\underline{c}=\underline{a}$, or that $\mathfrak{M}_2(m,M;\lambda_i,1-\lambda_i)=\mathfrak{N}_2(m,M;\lambda_i,1-\lambda_i)$, $1\le i\le n$. Since $m\neq M$ the hypothesis of strict convexity of $\mathcal{N}$ with respect to $\mathcal{M}$ then implies that $\lambda_i=0$, or 1, $1\le i\le n$. This gives the cases of equality. □

REMARK (i) Note the essential use of non-negative, as opposed to positive weights.

EXAMPLE (i) The basic examples of f are $x-y$ and x/y, when $C=0,1$ respectively.

EXAMPLE (ii) Taking $f(x,y)=x/y$, $\mathcal{N}(x)=x^s$, $s\neq 0$, $=\log x$, $s=0$, $\mathcal{M}(x)=x^r$, $r\neq 0$, $=\log x$, $r=0$, $r<s$, then the second part of Theorem 1 reduces to III 4.1 Theorem 3; while if we take $f(x,y)=x-y$ it reduces to III 4.2 Theorem 10.

In III 4.1 Remark (ii) an extension due to Beckenbach was mentioned. We show that such an extension exists in the present general situation and state the Beckenbach result as a corollary; [*Beckenbach 1964*]. For this extension we use the following notation: If $0<s<n$ let

$$\underline{a}=(a_1,\ldots,a_n)=(b_1,\ldots,b_s,c_1,\ldots,c_{n-s})=(\underline{b},\underline{c}),$$
$$\underline{w}=(w_1,\ldots,w_n)=(u_1,\ldots,u_s,v_1,\ldots,v_{n-s})=(\underline{u},\underline{v}).$$

THEOREM 2 *Let $\mathcal{M},\mathcal{N}$ and f be as in Theorem 1, $\underline{b}$ an s-tuple, $m\le\underline{a}\le M$, $\underline{w}=(\underline{u},\underline{v})$ a non-negative n-tuple with $W_n=1$, and put $\sigma=1-U_s=V_{n-s}$. Then for all appropriate $\underline{c}$: (a) there is a y, $m\le y\le M$, such that*

$$f\big(\mathfrak{N}_n(\underline{a};\underline{w}),\mathfrak{M}_n(\underline{a};\underline{w})\big)\ge f\big(\mathfrak{N}_{s+1}(\underline{b},y;\underline{u},\sigma),\mathfrak{M}_{s+1}(\underline{b},y;\underline{u},\sigma)\big);\tag{5}$$

if y is unique then there is equality in (5) if and only if for all c_i with $u_i>0$ we have that $c_i=y$;
(b) there is a λ, $0<\lambda<\sigma$ such that

$$f\big(\mathfrak{N}_n(\underline{a};\underline{w}),\mathfrak{M}_n(\underline{a};\underline{w})\big)\\ \le f\big(\mathfrak{N}_{s+2}(\underline{b},m,M;\underline{u},\lambda,1-\lambda),\mathfrak{M}_{s+2}(\underline{b},m,M;\underline{u},\lambda,1-\lambda)\big);\tag{6}$$

if λ is unique there is equality in (6) if there is an index set I such that $W_I=\lambda$ and $a_i=M$, $i\in I, a_i=m$, $i\notin I$.

□ (a) Put $x=\mathfrak{N}_{n-s}(\underline{c};w_{s+1}/\sigma,\ldots,w_n/\sigma)$; then $\mathfrak{N}_n(\underline{a};\underline{w})=\mathfrak{N}_n(\underline{b},\underline{c};\underline{u},\underline{v})=\mathfrak{N}_{s+1}(\underline{b},x;\underline{u},\sigma)$, a trivial identity.

If $z = \mathfrak{M}_{n-s}\big(\underline{c}; w_{s+1}/\sigma, \ldots, w_n/\sigma\big)$ then trivially, $\mathfrak{M}_n(\underline{a};\underline{w}) = \mathfrak{M}_{s+1}(\underline{b}, z; \underline{u}, \sigma)$. From the hypotheses and 2 Theorem 5 $z \le x$, and so by 1 Lemma 2(b)

$$\mathfrak{M}_n(\underline{a};\underline{w}) = \mathfrak{M}_{s+1}(\underline{b}, z; \underline{u}, \sigma) \le \mathfrak{M}_{s+1}(\underline{b}, x; \underline{u}, \sigma). \tag{7}$$

Hence using the above identity , (7) and the hypotheses

$$f\big(\mathfrak{N}_n(\underline{a};\underline{w}), \mathfrak{M}_n(\underline{a};\underline{w})\big) \ge f\big(\mathfrak{N}_{s+1}(\underline{b}, x; \underline{u}, \sigma), \mathfrak{M}_{s+1}(\underline{b}, x; \underline{u}, \sigma)\big),$$

and taking as y the values of x for which the right-hand side of this last inequality is a maximum, (5) follows.

The cases of equality are immediate.

(b) The proof is similar to the corresponding proof of Theorem 1. □

REMARK (ii) Theorem 1, in particular (2), shows that in general the right-hand side of (5) is an improvement on the obvious lower bound C.

COROLLARY 3 *Let $\underline{w}$ be a positive n-tuple, $\underline{b}$ a positive m-tuple, $1 \le m < n$; then for all positive $(n-m)$-tuples $\underline{c} = (c_{m+1}, \ldots, c_n)$, and any r, s, $-\infty \le r < s \le \infty$,*

$$\frac{\mathfrak{M}_n^{[s]}(\underline{b}, \underline{c}; \underline{w})}{\mathfrak{M}_n^{[r]}(\underline{b}, \underline{c}; \underline{w})} \ge \frac{\mathfrak{M}_n^{[s]}(\underline{b}, \underline{\alpha}; \underline{w})}{\mathfrak{M}_n^{[r]}(\underline{b}, \underline{\alpha}; \underline{w})} \tag{8}$$

where, except when $r = -s = -\infty$, $\underline{\alpha}$ is constant with each term equal to α,

$$\alpha = \begin{cases} \min \underline{b}, & \text{if } r = -\infty < s < \infty,\\[1ex] \left(\dfrac{\sum_{i=1}^n w_i b_i^s}{\sum_{i=1}^n w_i b_i^r}\right)^{1/(s-r)}, & \text{if } -\infty < r < s < \infty,\\[1ex] \max \underline{b}, & \text{if } -\infty < r < s = \infty; \end{cases}$$

if $r = -s = -\infty$ then $\underline{\alpha}$ is arbitrary, subject to $\min \underline{b} \le \underline{\alpha} \le \max \underline{b}$. There is equality on (9) if and only if $c_i = \alpha$, $m+1 \le i \le n$, except in the case $r = -s = -\infty$, when there is equality if and only if $\min \underline{b} \le \underline{c} \le \max \underline{b}$.

□ Except in the cases that involve $r = -\infty$, and, or $s = \infty$, this is an immediate consequence of Theorem 2(a) with $f(x, y) = x/y$, and $\mathcal{M}, \mathcal{N}$ appropriate power or logarithmic functions.

Consider as typical of the cases when one of r, s is infinite, the case r finite and not zero, $s = \infty$. Look at the function $\phi(\underline{c}) = \dfrac{\max\{\underline{b}, \underline{c}\}}{\mathfrak{M}_n^{[r]}(\underline{b}, \underline{c}; \underline{w})}$. Simple computations show that if $c_j < \max\{\underline{b}, \underline{c}\}$ then $\phi'_j < 0$, while if $c_j = \max\{\underline{b}, \underline{c}\}$ then $\phi'_j > 0$. So (8) holds in this case with equality as stated.

If $r = -s = -\infty$ consider instead the function $\phi(\underline{c}) = \max\{\underline{b}, \underline{c}\}/\min\{\underline{b}, \underline{c}\}$. From the definition of $\underline{\alpha}$ in this case $\min\{\underline{b}, \underline{\alpha}\} = \min\{\underline{b}, \max\{\underline{b}, \underline{\alpha}\}\} = \max \underline{b}$. Since in any case $\min\{\underline{b}, \underline{c}\} \le \min \underline{b}$ and $\max\{\underline{b}, \underline{c}\} \ge \max \underline{b}$, the result holds in this case with equality as stated. □

COROLLARY 4 *Let* $0 < m_1 < m_2$, $\underline{w}$ *be a positive* n*-tuple with* $W_n = 1$, $\underline{b}$ *a positive* m*-tuple,* $0 < m < n$*. Then for any positive* $n - m$*-tuple* $\underline{c}$, $m_1 \le \underline{c} \le m_2$*, and any* r, s, $-\infty \le r < s \le \infty$*, we have*

$$\frac{\mathfrak{M}_n^{[s]}(\underline{b}, \underline{c}; \underline{w})}{\mathfrak{M}_n^{[r]}(\underline{b}, \underline{c}; \underline{w})} \le \frac{\mathfrak{M}_{m+2}^{[s]}(\underline{b}, m_1, m_2; \underline{v}, \sigma - \alpha, \alpha)}{\mathfrak{M}_{m+2}^{[r]}(\underline{b}, m_1, m_2; \underline{v}, \sigma - \alpha, \alpha)}, \tag{9}$$

where $\underline{v} = (w_1, \ldots, w_m)$, $\sigma = 1 - W_m$*, and* $\alpha = 0$ *if* $\theta \le 0$, $\alpha = \theta$ *if* $0 \le \theta \le \sigma$*, and* $\alpha = \sigma$ *if* $\theta \ge \sigma$*, where: (i) if* r, s *are finite and* $rs \ne 0$ *then,*

$$\theta = \frac{1}{s - r}\left(\frac{r\left(\sum_{i=1}^m w_i b_i^r + \sigma m_1^r\right)}{m_2^r - m_1^r} - \frac{s\left(\sum_{i=1}^m w_i b_i^s + \sigma m_1^s\right)}{m_2^s - m_1^s}\right); \tag{10}$$

(ii) if r, s *are finite and* $r = 0$*, or* $s = 0$, θ *is given by the appropriate limit of (10); (iii) if* r *is finite, and* $s = \infty$, $\theta = 0$*; (iv) if* $r = -\infty$ *and* s *is finite,* $\theta = \sigma$*; (v) if* $r = -s = \infty$, $\theta = \sigma/2$*. Equality holds in (9) for finite* r, s *if and only if there is an index set* I *with* $W_I = \sigma$, $c_i = m_2, i \in I$, $c_i = m_1$, $i \notin i$*. If* $r = -\infty$ *, and* s *is finite, [*r *finite and* $s = \infty$*], there is equality if and only if the previous condition holds and* $\min \underline{b} = m_1$*,* $[\max \underline{b} = m_2]$*. If* $r = -s = \infty$ *there is equality if and only if* $\min\{\underline{b}, \underline{c}\} = m_1$ *and* $\max\{\underline{b}, \underline{c}\} = m_2$.

□ If r and s are finite this is a consequence of Theorem 2; the other cases can be checked by inspection. □

7 Generalizations of the Quasi-arithmetic Means

As general as the definition of a quasi-arithmetic is, even more general means have been defined and we consider some of these generalizations in this section. In the end the extreme generality leads to topics in functional inequalities which are beyond the scope of this book; see [*Aczél*].

7.1 A MEAN OF BAJRAKTAREVIĆ Let I be a closed interval in $\mathbb{R}$, $a_i \in I$, $1 \le i \le n$, $\underline{\omega} = (\omega_1, \ldots, \omega_n)$ *an* n*-tuple of weight functions, where* $\omega_i : I \mapsto \mathbb{R}_+^*$, $1 \le i \le n$, and $\mathcal{M} : I \mapsto \mathbb{R}$ is strictly monotonic; then define

$$\mathfrak{M}_n(\underline{a}; \underline{\omega}) = \mathcal{M}^{-1}\left(\frac{\sum_{i=1}^n \omega_i(a_i)\mathcal{M}(a_i)}{\sum_{i=1}^n \omega_i(a_i)}\right); \tag{1}$$

as with 1(2) the function $\mathcal{M}$ is said to *generate, to be a generator of or to be a generating function of this mean, the Bajraktarević mean.*

EXAMPLE (i) If each ω_i is constant, w_i say, $1 \le i \le n$, then (1) reduces to the quasi-arithmetic $\mathcal{M}$-mean, 1(2).

EXAMPLE (ii) If $\omega_i = w_i\,\omega$, $w_i \in \mathbb{R}_+^*, \omega : I \mapsto \mathbb{R}_+^*$, $1 \le i \le n$, then (1) becomes

$$\mathfrak{M}_n(\underline{a}; \underline{w}\,\omega) = \mathcal{M}^{-1}\left(\frac{\sum_{i=1}^n w_i\omega(a_i)\mathcal{M}(a_i)}{\sum_{i=1}^n w_i\omega(a_i)}\right). \tag{2}$$

EXAMPLE (iii) A special case of (2) is the Gini mean, $\mathfrak{G}_n^{p,q}(\underline{a}; \underline{w})$ of III 5.2.1, that is obtained by taking $\omega(x) = x^q$, and $\mathcal{M}(x) = x^{p-q}$, or $\log x$, according as $p > q$, or $p = q$.

These general means were first defined in [*Bajraktarević 1958,1963,1969; Daróczy 1964*] where 2 Theorem 5 is extended to cover these generalizations.
The next theorem due to Losonczi extends 5 Theorem 3; see [*Bullen 1973a; Losonczi 1971b*].

THEOREM 1 *Let $\mathfrak{K}_n(\underline{a}; \underline{\kappa}), \mathfrak{L}_n(\underline{a}; \underline{\lambda})$ and $\mathfrak{M}_n(\underline{a}; \underline{\nu})$ be three means as defined in (1), with common interval I and the generating functions $\mathcal{K}, \mathcal{L}, \mathcal{M}$ being differentiable and strictly monotonic, in particular $\mathcal{M}$ strictly increasing; and let, $f : I^2 \mapsto I$ be differentiable. Then*

$$f\big(\mathfrak{K}_n(\underline{a}; \underline{\kappa}), \mathfrak{L}_n(\underline{b}; \underline{\lambda})\big) \ge \mathfrak{M}_n\big(f(\underline{a}, \underline{b}); \underline{\nu}\big), \tag{3}$$

if and only if for all $u, v, s, t \in I$, and i, $1 \le i \le n$,

$$\begin{aligned}\left(\frac{\mathcal{M}\circ f(u,v) - \mathcal{M}\circ f(s,t)}{\mathcal{M}'\circ f(s,t)}\right)&\left(\frac{\nu_i\circ f(u,v)}{\nu_n\circ f(s,t)}\right) \\ \le \left(\frac{\mathcal{K}(u) - \mathcal{K}(s)}{\mathcal{K}'(s)}\right)&\left(\frac{\kappa_i(u)}{\kappa_n(s)}\right) f_1'(s,t) + \left(\frac{\mathcal{L}(u) - \mathcal{L}(t)}{\mathcal{L}'(t)}\right)\left(\frac{\lambda_i(v)}{\lambda_n(t)}\right) f_2'(s,t).\end{aligned} \tag{4}$$

If the sign in (3) is reversed or replaced by equality the same is true of (4).

□ (a) We first prove the necessity of (4). Given i, $1 \le i \le n$, put $a_i = u$, $b_i = v$ and $a_k = s$, $b_k = t$, $k \ne i$; then (3) becomes

$$\begin{aligned}\big(\nu_i\circ f(u,v)\big)&\Big(\mathcal{M}\circ f(u,v) - \mathcal{M}\circ f(\mathfrak{K}_n, \mathfrak{L}_n)\big)\Big) \\ &\le \Big(\mathcal{M}\circ f(\mathfrak{K}_n, \mathfrak{L}_n) - \mathcal{M}\circ f(s,t)\Big)\sum_{k=1,k\ne i}^{n} \nu_k\circ f(s,t),\end{aligned} \tag{5}$$

where $\mathfrak{K}_n, \mathfrak{L}_n$ denote respectively $\mathfrak{K}_n(\underline{a}; \underline{\kappa}), \mathfrak{L}_n(\underline{b}; \underline{\lambda})$ with the above choice of $\underline{a}, \underline{b}$.
Using the differentiability of $\mathcal{M}$ and f we have that

$$\begin{aligned}\mathcal{M}\circ f(\mathfrak{K}_n,&\mathfrak{L}_n) - \mathcal{M}\circ f(s,t) \\ &= \big(\mathcal{M}'\circ f(s,t) + \epsilon_1\big)\Big(\big(f_1'(s,t) + \epsilon_2\big)(\mathfrak{K}_n - s) + \big(f_2'(s,t) + \epsilon_3\big)(\mathfrak{L}_n - t)\Big),\end{aligned} \tag{6}$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ tend to zero as $(\mathfrak{K}_n, \mathfrak{L}_n)$ tends to (s,t).

If the definitions of λ_i and ν_i are extended by putting $\lambda_i = \lambda_n, \nu_i = \nu_n$, $i \geq n$, then it is not difficult to show that $\lim_{n\to\infty} \mathfrak{K}_n = s, \lim_{n\to\infty} \mathfrak{L}_n = t$; hence $\lim_{n\to\infty} \epsilon_1 = \lim_{n\to\infty} \epsilon_2 = \lim_{n\to\infty} \epsilon_3 = 0$. In fact:

$$\lim_{n\to\infty} (\mathfrak{K}_n - s) \sum_{k=1, k\neq i}^{n} \nu_k \circ f(s,t) = \left(\frac{\mathcal{K}(u) - \mathcal{K}(s)}{\mathcal{K}'(s)} \right) \frac{\kappa_i(u)\nu_n \circ f(s,t)}{\kappa_n(s)};$$

$$\lim_{n\to\infty} (\mathfrak{L}_n - t) \sum_{k=1, k\neq i}^{n} \nu_k \circ f(s,t) = \left(\frac{\mathcal{L}(u) - \mathcal{L}(t)}{\mathcal{L}'(t)} \right) \frac{\lambda_i(u)\nu_n \circ f(s,t)}{\lambda_n(t)}.$$

Using these values in (5) and (6) leads to (4).

(b) Now we consider the sufficiency of (4). Put $u = a_i$, $v = b_i$, $1 \leq i \leq n$, $s = \mathfrak{K}_n(\underline{a}; \underline{\kappa})$, $t = \mathfrak{L}_n(\underline{a}; \underline{\lambda})$ in (4) and add the n inequalities obtained. This leads to

$$\frac{\mathcal{M}\Big(\mathfrak{M}_n\big(f(\underline{a},\underline{b}); \underline{\nu}\big)\Big) - \mathcal{M} \circ f\big(\mathfrak{K}_n(\underline{a};\underline{\kappa}), \mathfrak{L}_n(\underline{a};\underline{\lambda})\big)}{\mathcal{M}' \circ f\big(\mathfrak{K}_n(\underline{a};\underline{\kappa}), \mathfrak{L}_n(\underline{a};\underline{\lambda})\big)} \leq 0,$$

which gives (3).

(c) Obviously the inequality sign in one of (3) or (4) can be reversed, or replaced by equality, if it is so replaced in the other. □

REMARK (i) Let $E_i = \big\{(u,v,s,t);$ *(4) holds with equality*$\big\}$ then, from the above proof, there is equality in (3) if and only if $\big(a_i, b_i, \mathfrak{K}_n(\underline{a};\underline{\kappa}), \mathfrak{L}_n(\underline{b};\lambda)\big) \in E_i$, $1 \leq i \leq n$.

REMARK (ii) Theorem 1 can be extended quite easily to the case of an $f : I^m \mapsto I$.

EXAMPLE (iv) If $\underline{\kappa}, \underline{\lambda}, \underline{\nu}$ are equal and constant, see Example (i) above, (4) reduces to

$$\frac{\mathcal{M} \circ f(u,v) - \mathcal{M} \circ f(s,t)}{\mathcal{M}' \circ f(s,t)} \leq \left(\frac{\mathcal{K}(u) - \mathcal{K}(s)}{\mathcal{K}'(s)} \right) f_1'(s,t) + \left(\frac{\mathcal{L}(u) - \mathcal{L}(t)}{\mathcal{L}'(t)} \right) f_2'(s,t).$$

If we now assume that $\mathcal{K}, \mathcal{L}, \mathcal{M}, f$ all have continuous second derivatives this last inequality can be seen to be equivalent to $H_{11}''h^2 + 2H_{12}''hk + H_{22}^2k^2 \leq 0$, for all h,k; where H is the function defined in 5 Theorem 3. This just says that this quadratic form is positive semi-definite, equivalently H is concave; see I 4.6 Theorem 40(g).

Theorem 1 can be used to obtain a condition under which these means are comparable.

COROLLARY 2 *Let $\mathfrak{K}_n(\underline{a};\underline{\kappa})$ and $\mathfrak{M}_n(\underline{a};\underline{\kappa})$ be two means as defined in (1), with common interval I and the generating functions $\mathcal{K},\mathcal{M}$ being differentiable and strictly monotonic, in particular $\mathcal{M}$ strictly increasing, then for all $\underline{a}\in I^n$*

$$\mathfrak{M}_n(\underline{a};\underline{\nu}) \le \mathfrak{K}_n(\underline{a};\underline{\kappa})$$

if for all $u,s\in I$,

$$\Big(\frac{\mathcal{M}(u)-\mathcal{M}(s)}{\mathcal{M}'(s)}\Big)\frac{\nu_i(u)}{\nu_n(s)} \le \Big(\frac{\mathcal{K}(u)-\mathcal{K}(s)}{\mathcal{K}'(s)}\Big)\frac{\kappa_i(u)}{\kappa_n(s)},\ 1\le i\le n.$$

In particular: (a) if $\underline{\nu}=\underline{\kappa}$ then the means $\mathfrak{M}_n(\underline{a};\underline{\nu}),\mathfrak{K}_n(\underline{a};\underline{\nu})$ are comparable if $\mathcal{K}$ is increasing, respectively decreasing, and $\mathcal{M}$ is convex, respectively concave, with respect to $\mathcal{K}$; (b) if $\mathcal{M}=\mathcal{K}$ and $\nu_1=\cdots=\nu_n=\nu$, $\kappa_1=\cdots=\kappa_n=\kappa$ then the means $\mathfrak{M}_n(\underline{a};\nu),\mathfrak{M}_n(\underline{a};\kappa)$ are comparable if κ/ν is monotonic.

□ The main statement is an immediate deduction from Theorem 1; just take $f(x,y)=x$, and (b) is immediate.
As for (a) this follows from the fact if $\mathcal{K}$ is increasing and $\mathcal{M}$ is convex with respect to $\mathcal{K}$ then $\big(\mathcal{M}(u)-\mathcal{M}(s)\big)/\mathcal{M}'(s)\le\big(\mathcal{K}(u)-\mathcal{K}(s)\big)/\mathcal{K}'(s)$; see I 4.1 Lemma 2 and I 4.5.3 Definition 33(c). □

REMARK (iii) The argument in Corollary 2 can easily be extended to show that for any $F: I\mapsto I$, $\mathfrak{M}_n\big(F(\underline{a});\underline{\nu}\big)\le F\big(\mathfrak{K}_n(\underline{a};\underline{\kappa})\big)$ if and only if

$$\Big(\frac{\mathcal{M}\circ F(u)-\mathcal{M}\circ F(s)}{\mathcal{M}'\circ F(s)}\Big)\frac{\nu_i(u)}{\nu_n(s)} \le \Big(\frac{\mathcal{K}(u)-\mathcal{K}(s)}{\mathcal{K}'(s)}\Big)\frac{\kappa_i(u)}{\kappa_n(s)}F'(s),\quad 1\le i\le n.$$

COROLLARY 3 *Let $\mathfrak{K}_n(\underline{a};\underline{\kappa})$ and $\mathfrak{M}_n(\underline{a};\underline{\kappa})$ be two means as defined in (1), with common interval I and differentiable strictly increasing generating functions, then $\mathfrak{M}_n(\underline{a};\underline{\nu})=\mathfrak{K}_n(\underline{a};\underline{\kappa})$ for all $\underline{a}\in I^n$ if and only if $\mathcal{K}=(\alpha\mathcal{M}+\beta)/(\gamma\mathcal{M}+\delta)$ and $\kappa_i=\epsilon\nu_i(\gamma\mathcal{M}+\delta)$, $1\le i\le n$, where the constants $\alpha,\beta,\gamma,\delta\,\epsilon$ satisfy the conditions: (i) $\epsilon(\gamma^2+\delta^2)(\alpha\gamma-\beta\delta)\ne 0$, and (ii) if $\gamma\ne 0$ then $-\delta/\gamma$ is not a value of $\mathcal{M}$ and α/γ is not a value of $\mathcal{K}$.*

□ Using Corollary 2 and the cases of equality in Theorem 1, with $f(x,y)=x$, the above equality holds if and only if

$$\Big(\frac{\mathcal{M}(u)-\mathcal{M}(s)}{\mathcal{M}'(s)}\Big)\frac{\nu_i(u)}{\nu_n(s)} = \Big(\frac{\mathcal{K}(u)-\mathcal{K}(s)}{\mathcal{K}'(s)}\Big)\frac{\kappa_i(u)}{\kappa_n(s)},\ 1\le i\le n. \tag{7}$$

Fixing $s=s_0\in I$ and putting $m(u)=\big(\mathcal{M}(u)-\mathcal{M}(s_0)\big)/\mathcal{M}'(s_0)$, and $k(u)=\big(\mathcal{K}(u)-\mathcal{K}(s_0)\big)/\mathcal{K}'(s_0)$ it is easily seen that if $u,s\in I\setminus\{s_0\}$ then

$$\Big(\frac{m(u)-m(s)}{m'(s)}\Big)\frac{m(t)}{m(u)} = \Big(\frac{k(u)-k(s)}{k'(s)}\Big)\frac{k(t)}{k(u)}.$$

Now fixing $s = s_1 \neq s_0$ we get that $\mathcal{K}(u) = \big(\alpha\mathcal{M}(u) + \beta\big)/(\gamma\mathcal{M}(u) + \delta)$, $u \neq s_0$; where the constants $\alpha, \beta, \gamma, \delta$ are determined from the values of $\mathcal{M}, \mathcal{M}', \mathcal{K}, \mathcal{K}'$ at s_0, s_1. The formula holds at s_0 as well, by continuity.
Further since $\mathcal{K}$ is not constant $\alpha^2 + \beta^2 > 0$ and $\alpha\delta - \beta\gamma \neq 0$. If $\gamma \neq 0$ then $K(u) - \alpha/\gamma = (\alpha\delta - \beta\gamma)/\big(\gamma^2 M(u) - \delta\gamma\big)$, which implies the last condition in the statement of the theorem.
Substituting for $\mathcal{K}$ in (7) gives

$$\Big(\frac{\kappa_i(u)}{\nu_i(u)}\Big)\frac{1}{\gamma\mathcal{M}(u) + \delta} = \Big(\frac{\kappa_n(u)}{\nu_n(u)}\Big)\frac{1}{\gamma\mathcal{M}(t) + \delta},\ 1 \leq i \leq n,\ u \neq t;$$

this shows that the left-hand side is a constant, both as a function of i and as a function of u. This completes this part of the proof.
The converse is easily verified. □

COROLLARY 4 *With the notations and assumptions of Theorem 1 we have, for all $\underline{a}, \underline{b}, \in I^n$ with $\underline{a} + \underline{b} \in I^n$, that $\mathfrak{K}_n(\underline{a}; \underline{\kappa}) + \mathfrak{L}_n(\underline{b}; \underline{\lambda}) \leq \mathfrak{M}_n(\underline{a} + \underline{b}; \underline{\nu})$ if and only if for all $u, v, s, t \in I$,*

$$\begin{aligned}\Big(\frac{\mathcal{M}(u+v) - \mathcal{M}(s+t)}{\mathcal{M}'(s+t)}\Big)&\Big(\frac{\nu_i(u+v)}{\nu_n(s+t)}\Big) \\ &\leq \Big(\frac{\mathcal{K}(u) - \mathcal{K}(s)}{\mathcal{K}'(s)}\Big)\Big(\frac{\kappa_i(u)}{\kappa_n(s)}\Big) + \Big(\frac{\mathcal{L}(u) - \mathcal{L}(t)}{\mathcal{L}'(t)}\Big)\Big(\frac{\lambda_i(v)}{\lambda_n(t)}\Big).\end{aligned} \tag{8}$$

□ This is immediate on taking $f(x, y) = x + y$ in Theorem 1. □

REMARK (iv) It might be noted that inequality (8) is unchanged if f $\underline{\kappa}, \underline{\lambda}, \underline{\nu}$ are replaced by $\tilde{\underline{\kappa}}, \tilde{\underline{\lambda}}, \tilde{\underline{\nu}}$, where for some suitable n-tuple $\underline{w}$, $\tilde{\underline{\kappa}} = (w_1\kappa_1, \ldots, w_n\kappa_n), \tilde{\underline{\lambda}} = (w_1\lambda_1, \ldots, w_n\lambda_n), \tilde{\underline{\nu}} = (w_1\nu_1, \ldots, w_n\nu_n)$.
A particular case of Corollary 4 gives, using Example (iii), the following property of Gini means.

COROLLARY 5 *If either $p \geq q$ and $\max\{1 + q - p\} \leq q \leq 1$, or $p < q$ and $\max\{q - p, 1\} \leq q \leq 1 + q - p$ then*

$$\mathfrak{G}_n^{p,q}(\underline{a} + \underline{b}; \underline{w}) \leq \mathfrak{G}_n^{p,q}(\underline{a}; \underline{w}) + \mathfrak{G}_n^{p,q}(\underline{b}; \underline{w}); \tag{9}$$

if either $p \geq q$ and $q - p \leq \min\{1 + q - p, 0\}$, or $p < q$ and $0 \leq q \leq \min\{q - p, 1\}$ then (~9) holds.

□ First let $\mathcal{K} = \mathcal{L} = \mathcal{M}$ and for some suitable n-tuple $\underline{w}$, and function ω, $\underline{\kappa} = \underline{\lambda} = \underline{\nu} = \underline{w}\omega$. see Example (ii). Then (8) becomes

$$G(u + v, s + t) \leq G(u, s) + G(v, t) \tag{10}$$

where $G(x,y) = \Big(\big(\mathcal{M}(x) - \mathcal{M}(y)\big)/\mathcal{M}'(y)\Big)\big(\omega(x)/\omega(y)\big)$.
In the case of Gini means we further have that $\mathcal{M}(x) = x^{p-q}$, or $\log x$, $\omega(x) = x^q$, and so $G(x,y) = yg(x/y)$ where

$$g(t) = \begin{cases} \dfrac{1}{p-q}(t^p - t^q), & \text{if } p > q, \\ t^q \log t, & \text{if } p = q. \end{cases}$$

It is easily checked that the convexity of g would imply (10), and that $g''(t) > 0$ if $t > 0$ and p, q satisfy either of the first set of conditions.
The second set of conditions imply that g is concave and which in turn implies ($\sim$10). This completes the proof. □

If we assume that $I = \mathbb{R}$ and that $\mathcal{M}'', \nu'$ exist then (10) can be solved in a manner that is essentially unique.

COROLLARY 6 *If $\mathcal{M}$ is a strictly increasing twice differentiable function on $\mathbb{R}$ and if ω is a positive differentiable function on $\mathbb{R}$ then*

$$\mathfrak{M}_n(\underline{a} + \underline{b}; \underline{w}\omega) \le \mathfrak{M}_n(\underline{a}; \underline{w}\omega) + \mathfrak{M}_n(\underline{b}; \underline{w}\omega) \tag{11}$$

for all real n-tuples $\underline{a}, \underline{b}$ if and only if ω is constant and $\mathcal{M}(t) = \alpha + \beta t$, for some $\alpha, \beta \in \mathbb{R}$, $\beta > 0$; that is to say if and only if $\mathfrak{M}_n(\underline{a}; \underline{w}\omega) = \mathfrak{A}_n(\underline{a}; \underline{w})$.

□ As in Corollary 5, (11) holds if and only if (10) holds for all $u, v, s, t \in \mathbb{R}$. If $u > 0$ and $s = 0$, (10) can be written as $\dfrac{G(u+v,t) - G(v,t)}{u} \le \dfrac{G(u,0) - G(0,0)}{u}$.
Letting u tend to zero we get that $G_1'(v,t) \le G_1'(0,0)$. A similar argument with $u < 0$ leads to the opposite inequality and so $G_1'(u,v) = G_1'(0,0) = \alpha$, say.
In the same way we can show that $G_2'(u,v) = G_2'(0,0) = \beta$, say.
Hence, $G(u,v) = \alpha u + \beta v$, with $\alpha + \beta = 0$, since $G(u,u) = 0$.
That is to say,

$$\Big(\frac{\mathcal{M}(x) - \mathcal{M}(y)}{\mathcal{M}'(y)}\Big)\frac{\omega(x)}{\omega(y)} = \alpha(x - y). \tag{12}$$

Putting $y = 0$ in (12) gives

$$\mathcal{M}(x) = \mathcal{M}(0) + \alpha\omega(0)\mathcal{M}'(0)\frac{x}{\omega(x)}. \tag{13}$$

On substituting this in (12) we get that

$$\frac{x\omega(y) - y\omega(x)}{\omega(y) - y\omega'(y)} = \alpha(x - y).$$

Again putting $y = 0$, we get that $\alpha = 1$; than putting $y = 1$, $\omega(x) = Ax + B$. Since $\omega > 0$ we have then that $A = 0, B > 0$.
Hence from (13) $\mathcal{M}(x) = \mathcal{M}(0) + \mathcal{M}'(0)x = \alpha + \beta x$, $\beta > 0$, and the proof is completed by using 1.2 Theorem 5. □

REMARK (v) Although the main result in this section is due to Losonczi many results involving particular means of this type had been obtained earlier by other authors; see [*Aczél & Daróczy; Bajraktarević 1951; Danskin; Daróczy 1971; Daróczy & Losonczi; Losonczi 1970,1971b; Páles 1987*] as well as references in III 4.1, 4.2.

7.2 FURTHER RESULTS

7.2.1 DEVIATION MEANS Even more general means have been studied by various authors; see [*Daróczy 1972; Daróczy & Páles 1982; Losonczi 1973,1981; Páles 1982,1984,1988a,b; Smoliak*]. We give some results concerning the *deviation means* introduced by Losonczi; the symmetric means of Daróczy can be obtained as special cases. Let I be a real interval and denote by $\mathcal{D}(I)$ the set of functions $D : I^2 \mapsto \mathbb{R}$, satisfying:
(i) for all $x \in I$, $D(x, \cdot) : I \mapsto \mathbb{R}$ is continuous and strictly increasing,
(ii) $D(x, x) = 0$, $x \in I$;
such functions are called *deviation functions*.
Let $\underline{D} = (D_1, \ldots, D_n)$ where $D_i \in \mathcal{D}(I)$, $1 \le i \le n$, and let $a_i \in I, 1 \le i \le n$, then $K : y \mapsto \sum_{i=1}^n D_i(a_i, y)$ is strictly increasing and $K(\min \underline{a}) \le 0 \le K(\max \underline{a})$. So the equation

$$\sum_{i=1}^{n} D_i(a_i, y) = 0,$$

has a unique solution y such that $\min \underline{a} \le y \le \max \underline{a}$, called the a *deviation mean*, or more precisely a *$\underline{D}$-mean of $\underline{a}$*, written $\mathfrak{D}_n(\underline{a}; \underline{D})$.

EXAMPLE (i) If $D_i = D$, $1 \le i \le n$, we obtain the *symmetric mean of Daróczy*; [*Daróczy 1972*].

EXAMPLE (ii) If $D_i(x, y) = \omega_i(x)\big(\mathcal{M}(y) - \mathcal{M}(x)\big)$, $1 \le i \le n$, where $\mathcal{M}$ is continuous and strictly increasing, we get the mean of Bajraktarević, $\mathfrak{M}_n(\underline{a}; \underline{\omega})$, defined in 7.1 (1).

Let $\underline{D} = (D_1, \ldots)$ be a sequence of functions in $\mathcal{D}(I)$, and $\underline{D}_m = (D_1, \ldots, D_m)$, $m \ge 1$; then $\underline{D}$ is called a *reproducing sequence* when for all $\underline{a} \in I^k$, $k \ge 1$, and $t \in I$, we have that: $\lim_{m \to \infty} m\big(\mathfrak{D}_{k+m}(\underline{a}^{(k+m)}; \underline{D}_{k+m}) - t\big) = D(t) \sum_{i=1}^{k} D_i(a_i, t)$,

where $\underline{a}^{(k+m)} = (a_1, \dots a_k, \overbrace{t, t, \dots, t}^{m \text{ elements}})$ and D is a non-positive function that depends on $\underline{D}$. The set of all reproducing sequences will be written $\mathcal{R}(I)$, and we will write $D_i^*(x, y) = D(y) D_i(x, y)$.

The following result, due to Losonczi, generalizes results in 7.1.

THEOREM 7 *Let I, I_1 and I_2 be intervals in $\mathbb{R}$, $f : I_1 \times I_2 \mapsto I$ a differentiable function, $\underline{C} \in \mathcal{R}(I)$, $\underline{D} \in \mathcal{R}(I_1)$, $\underline{E} \in \mathcal{R}(I_2)$; then for all $\underline{a} \in I_1^n, \underline{b} \in I_2^n$, $n \geq 1$,*

$$\mathfrak{D}_n\big(f(\underline{a}, \underline{b}); \underline{C}_n\big) \leq f\big(\mathfrak{D}_n(\underline{a}; \underline{D}_n), \mathfrak{D}_n(\underline{b}; \underline{E}_n)\big), \tag{14}$$

if and only if for all $u, s \in I_1$, and $v, t \in I_2$.

$$D_k^*\big(f(u, v), f(s, t)\big) \leq D_k^*(u, s) f_1'(s, t) + D_k^*(v, t) f_2'(s, t), \; k \geq 1. \tag{15}$$

If condition (15) is satisfied and if Δ_k, $k \geq 1$, is the set of (u, v, s, t) for which there is equality in (15) then there is equality in (14) if and only if for all k, $1 \leq k \leq n$, $\big(a_k, b_k, \mathfrak{D}_n(\underline{a}; \underline{D}_n), \mathfrak{D}_n(\underline{b}; \underline{E}_n)\big) \in \Delta_k$.

REMARK (i) The properties of "means on the move", see 4.5, have been investigated for deviation means; [*Páles 1991*].

Deviation means have been even further generalized to *quasi-deviation means* based on the concept of a *quasi-deviation function*, that is a function $E : I^2 \mapsto \mathbb{R}$ satisfying:

(i) for all $(x, t) \in I^2$, $\operatorname{sign} E(x, t) = \operatorname{sign}(x - t)$;
(ii) for all $x \in I$, $E(x, \cdot) : I \mapsto \mathbb{R}$ is continuous;
(iii) for all $x, y \in I$, $x < y$, $t \mapsto E(x, t)/E(y, t)$, $x < t < y$, is strictly increasing.

If now $\underline{a} \in I^n$ there is a unique solution of the equation

$$\sum_{i=1}^{n} E(a_i, t) = 0,$$

called the *quasi-deviation mean of $\underline{a}$ generated by E*. Such means have been the object of much study; see [*Páles 1982,1983b,1988a,b*].

7.2.2 ESSENTIAL INEQUALITIES In an interesting paper Páles, [*Páles 1990b*], introduced the following concepts. Given k, $k \geq 2$, continuous, strictly increasing functions $\mathcal{M}^{(i)} : I = [m, M] \mapsto \mathbb{R}$, $1 \leq i \leq k$, with associated equal weighted quasi-arithmetic means $\mathfrak{M}_n^{(i)}$, $1 \leq i \leq k$, $n \in \mathbb{N}^*$, let

$$\sigma_I\big(\mathfrak{M}_n^{(1)}, \dots, \mathfrak{M}_n^{(k)}\big) = \overline{\Big\{\underline{u};\; \underline{u} = \big(\mathfrak{M}_n^{(1)}(\underline{a}), \dots, \mathfrak{M}_n^{(k)}(\underline{a})\big), \underline{a} \in I^n, \, n \in \mathbb{N}^*\Big\}};$$

This set is called the *space of means associated with the functions* $\mathcal{M}^{(i)}$, $1 \le i \le k$. Its importance is that if for some continuous $F : I^k \mapsto \mathbb{R}$ an inequality that involves all these means, say $F\big(\mathfrak{M}_n^{(1)}(\underline{a}), \ldots, \mathfrak{M}_n^{(k)}(\underline{a})\big) \ge 0$, $\underline{a} \in I^n$, $n \in \mathbb{N}^*$, holds if and only if

$$F(\underline{u}) \ge 0,\ \underline{u} \in \sigma_I\big(\mathfrak{M}_n^{(1)}, \ldots, \mathfrak{M}_n^{(k)}\big);$$

that is the only such inequalities are those that give a description of the space $\sigma_I\big(\mathfrak{M}_n^{(1)}, \ldots, \mathfrak{M}_n^{(k)}\big)$. Inequalities of this type are called *essential inequalities* for these means.

The basic theorem proved by Páles is the following; as the proof uses the Hahn-Banach-Minkowski separation theorem it will be omitted.

THEOREM 8 *With the above notation and putting* $\mathcal{M} = \alpha_0 + \sum_{i=1}^k \alpha_i \mathcal{M}^{(i)}$ *then* $\underline{u} \in \sigma_I\big(\mathfrak{M}_n^{(1)}, \ldots, \mathfrak{M}_n^{(k)}\big)$ *if and only if for any choice of* $\alpha_i \in \mathbb{R}$, $0 \le i \le k$,

$$\mathcal{M} \ge 0 \Longrightarrow \alpha_0 + \sum_{i=1}^k \alpha_i \mathcal{M}^{(i)}(u_i) \ge 0.$$

In the case of the power means the following can be obtained by using Theorem 8.

THEOREM 9 *(a) If* $t < s < r$ *and* $t \le 0 \le r$ *then*

$$\sigma_{\mathbb{R}_+^*}\big(\mathfrak{M}_n^{[t]}, \mathfrak{M}_n^{[s]}, \mathfrak{M}_n^{[r]}\big) = \big\{(u, v, w);\ 0 < u \le v \le w)\big\}.$$

(b) If $0 < t < s < r$ *then*

$$\sigma_{\mathbb{R}_+^*}\big(\mathfrak{M}_n^{[t]}, \mathfrak{M}_n^{[s]}, \mathfrak{M}_n^{[r]}\big) = \big\{(u, v, w);\ 0 < u \le v,\ v^{s(r-t)/r(s-t)} u^{t(s-r)/r(s-t)} \le w)\big\}.$$

(c) If $r < s$ *and* $I = [m, M]$ *then*

$$\sigma_I\big(\mathfrak{M}_n^{[r]}, \mathfrak{M}_n^{[s]}\big) = \big\{(u, v);\ m < u \le v \le M,\ \textit{and}\, v \le \mathfrak{M}_2^{[s]}(m, M; \lambda_r(x))\big\},$$

where

$$\lambda_r(x) = \begin{cases} \dfrac{M^r - x^r}{M^r - m^r}, & \textit{if}\, r \ne 0, \\[2ex] \dfrac{\log M - \log x}{\log M - \log m}, & \textit{if}\, r = 0. \end{cases}$$

(a) The inclusion

$$\sigma_{\mathbb{R}_+^*}\big(\mathfrak{M}_n^{[t]}, \mathfrak{M}_n^{[s]}, \mathfrak{M}_n^{[r]}\big) \subseteq \big\{(u, v, w);\ 0 < u \le v \le w)\big\},$$

is immediate from (r;s). Now assume that $rst \neq 0$. If $0 < u \leq v \leq w$, then by Theorem 8 we must show that

$$\alpha x^t + \beta x^s + \gamma x^r + \delta \geq 0,\ x > 0 \implies \alpha u^t + \beta v^s + \gamma w^r + \delta \geq 0. \tag{16}$$

Divide the left-hand inequality in (16) by x^t and then let $x \to 0$ we see that $\alpha \geq 0$, dividing by x^r and letting $x \to \infty$ we get that $\gamma \geq 0$. So

$$\alpha u^t + \beta v^s + \gamma w^r + \delta \geq \alpha u^t + \beta u^s + \gamma u^r + \delta \geq 0,$$

by the left-hand inequality in (16).
The proofs of the other cases are similar.
(b) The proof is similar to that of (a) but uses Liapunov's inequality, III 2.1 (8), in addition to (r;s).
(c) (*i*): $\sigma_I\big(\mathfrak{M}_n^{[r]}, \mathfrak{M}_n^{[s]}\big) \subseteq \big\{(u,v);\ m < u \leq v \leq M,\ and\, v \leq \mathfrak{M}_2^{[s]}(m, M; \lambda_r(x))\big\}$.
If $(u,v) \in \sigma_I\big(\mathfrak{M}_n^{[r]}, \mathfrak{M}_n^{[s]}\big)$ then $m < u \leq v \leq M$ is immediate from (r;s). Further if $0 \leq \lambda \leq 1$ then $\mathfrak{M}_2^{[r]}(m, M; \lambda) \leq \mathfrak{M}_2^{[s]}(m, M; \lambda)$, which on substituting $\lambda = \lambda_r(x)$ gives

$$x \leq \mathfrak{M}_2^{[s]}(m, M; \lambda_r(x)),\ m \leq x \leq M. \tag{17}$$

Now assume that $rs \neq 0$ and that $u = \mathfrak{M}_n^{[r]}(\underline{a}),\ v = \mathfrak{M}_n^{[s]}(\underline{a})$, when from (17)

$$\operatorname{sign} s(a_i)^s \leq \operatorname{sign} s\Big(\frac{M^r - a_i^r}{M^r - m^r} m^s + \frac{a_i^r - m^r}{M^r - m^r} M^s\Big),\ 1 \leq i \leq n.$$

Adding these gives $v \leq \mathfrak{M}_2^{[s]}(m, M; \lambda_r(x))$.
The other cases are similar.
(*ii*): $\sigma_I\big(\mathfrak{M}_n^{[r]}, \mathfrak{M}_n^{[s]}\big) \supseteq \big\{(u,v);\ m < u \leq v \leq M,\ and\, v \leq \mathfrak{M}_2^{[s]}(m, M\lambda_r(x))\big\}$.
Again assume that $rs \neq 0$, and that (u,v) is in the set on the right. By Theorem 8 we have to prove that

$$\alpha x^r + \beta x^s + \gamma \geq 0,\ m \leq x \leq M \implies \alpha u^r + \beta v^s + \gamma \geq 0. \tag{18}$$

Take a convex combination of the inequalities obtained from the left-hand inequality in (18) with $x = m, x = M$, using as weights $\lambda_r(x), 1 - \lambda_r(x)$, to get that

$$\alpha u^r + \beta\big(\mathfrak{M}_2^{[s]}(m, M; \lambda_r(x))\big)^s + \gamma \geq 0.$$

In addition, from (18), $\alpha u^r + \beta u^s + \gamma \geq 0$. Take a convex combination of these last two inequalities choosing the weights, $\theta, 1 - \theta$, so that the resulting coefficient of β is v^s gives the right-hand inequality in (18).

The other cases are similar. □

REMARK (i) Part (c) can be used to obtain the converse inequality III 4.1 Theorem 3, and in particular Schweitzer's inequality, III 4.1 (12); [*Páles 1990b*].

7.2.3 CONJUGATE MEANS Let $\mathfrak{D}$ be a symmetric, continuous and internal mean of n-tuples with entries in the open interval I and let ϕ be a continuous strictly monotonic function on I then the *conjugate $\mathfrak{D}$-mean, by ϕ,* is defined by:

$$\mathfrak{D}^*_\phi(\underline{a}) = \phi^{-1}\left(\frac{\sum_{i=1}^n \phi(a_i) - \phi(\mathfrak{D}(\underline{a}))}{n-1}\right).$$

the definition in the case $n = 2$ is due to Daróczy, who solved the comparison problem for two such means; [*Daróczy 1999*]. Later Daróczy & Páles showed which of the conjugate arithmetic means are also quasi-arithmetic means and made the above extension to n-tuples; [*Daróczy & Páles 2001a,b*]. Their main results are given in the following theorem.

THEOREM 10 *(a) Let ϕ and ψ be continuous and strictly monotonic, ψ increasing, then*

$$\mathfrak{D}^*_\phi(\underline{a}) \le \mathfrak{D}^*_\psi(\underline{a})$$

for all $a \in I$ if and only if ψ is convex with respect to ϕ.
(b) If $n \ge 3$ a conjugate arithmetic mean is a quasi-arithmetic mean if and only if it is an arithmetic mean.

□ The proofs of these results, and a discussion of the case $n = 2$ of (b), can be found in the references. □

7.2.4 SENSITIVITY OF MEANS Given two equal weighted quasi-arithmetic means $\mathfrak{M}, \mathfrak{N}$ defined for positive n-tuples, where $\mathfrak{M}$ differentiable with positive partial derivatives, we say that $\mathfrak{M}$ *has small sensitivity with respect to* $\mathfrak{N}$ if

$$\mathfrak{N}(\partial\mathfrak{M}/\partial x_1, \ldots, \partial\mathfrak{M}/\partial x_n) \le \frac{1}{n}.$$

THEOREM 11 *The equal weighted quasi-arithmetic mean $\mathfrak{M}$ has small sensitivity with respect to the power mean $\mathfrak{M}_n^{[r]}, r \in \mathbb{R}^*$, if and only if*

$$\frac{\mathcal{M}'''}{\mathcal{M}'} < (2-r)\left(\frac{\mathcal{M}''}{\mathcal{M}'}\right)^2;$$

equivalently if and only if $|\mathcal{M}' \circ \mathcal{M}^{-1}|^r$ is convex if $r > 0$, concave if $r < 0$.

□ The proof can be found in [*Dux & Goda*]. □

- end of Chapter IV -

V SYMMETRIC POLYNOMIAL MEANS

The elementary and complete symmetric polynomials have a history that goes back to Newton at the beginning of the modern mathematical era. They are used to define means that generalize the geometric and arithmetic means in a completely different way to the generalizations of Chapters III and IV. These new means give extensions of the geometric mean-arithmetic mean inequality. In this chapter we study the properties of these means. In addition generalizations of these means due to Whiteley and Muirhead are discussed

1 Elementary Symmetric Polynomials and Their Means

[1]The elementary symmetric polynomials[2] arose naturally in the study of algebraic equations, see for instance, [*Milovanović, Mitrinović & Rassias pp.7–8,52–58; Uspensky,1948, Chap.IX*]. Consequently many of the results have been known for a long time — the basic inequality, 2(1) below, being due to Newton and Campbell; see [*Newton*], [*Campbell*]

The properties of these polynomials and the associated elementary symmetric polynomial means[3] have been studied in most of the basic references, [*AI pp.95–107; BB pp.33–35; CE pp.516–517; DI pp.76–77,243–244; EM9 pp.95–96,98; HLP pp.44–64*], and of course in various papers; in particular see [*Bellman 1941; Brunn; Fujisawa; Mardessich; Muirhead 1902/03*] where many of the fundamental results are often rediscovered.

DEFINITION 1 *Let $\underline{a}$ be an n-tuple, r and n are integers, $1 \le r \le n$, then the* rth elementary symmetric polynomial of $\underline{a}$ *is defined by*

$$e_n^{[r]}(\underline{a}) = \frac{1}{r!}\sum_r !\left(\prod_{j=1}^{r} a_{i_j}\right); \tag{1}$$

---

1 We revert now to Convention 3 of II 1; this convention was abandoned in the previous chapter.

2 Also called *elementary symmetric functions*. We will prefer the word polynomial as the usage symmetric function has a more general meaning in this book.

3 Again, these means are often called *symmetric means* but that usage has a more general meaning in this book.

and the rth elementary symmetric polynomial mean of $\underline{a}$ *is defined by*

$$\mathfrak{S}_n^{[r]}(\underline{a}) = \left(\frac{e_n^{[r]}(\underline{a})}{\binom{n}{r}}\right)^{1/r}. \tag{2}$$

EXAMPLE (i) It is easy to check that: $e_n^{[1]}(\underline{a}) = a_1 + \cdots + a_n$, $e_n^{[n]}(\underline{a}) = a_1 a_2 \ldots a_n$, and $e_n^{[2]}(\underline{a}) = a_1 a_2 + a_1 a_3 + \cdots + a_{n-1} a_n$.

EXAMPLE (ii) We have from Example (i):

$$\begin{aligned} \mathfrak{S}_n^{[1]}(\underline{a}) &= \mathfrak{A}_n(\underline{a}), \qquad \mathfrak{S}_n^{[n]}(\underline{a}) = \mathfrak{G}_n(\underline{a}), \\ \mathfrak{S}_n^{[2]}(\underline{a}) &= \sqrt{\frac{a_1 a_2 + a_1 a_3 + \cdots + a_{n-1} a_n}{\frac{1}{2} n(n-1)}}. \end{aligned} \tag{3}$$

EXAMPLE (iii) If the polynomial $c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$, $c_n \neq 0$, has zeros $-a_1, \ldots, -a_n$ then

$$\begin{aligned} c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 &= c_n (x + a_1)(x + a_2) \cdots (x + a_n) \\ &= c_n \big(x^n + e_n^{[1]}(\underline{a}) x^{n-1} + e_n^{[2]}(\underline{a}) x^{n-2} + \cdots + e_n^{[n]}(\underline{a})\big). \end{aligned}$$

Hence

$$e_n^{[r]}(\underline{a}) = \frac{c_{n-r}}{c_n}, \ 1 \le r \le n;$$

see I 1.1 Example (i).

EXAMPLE (iv) A simple formula for the tangent of the sum of n numbers can be given using the elementary symmetric polynomials; [*Pietra*].

$$\tan\left(\sum_{k=1}^{n} a_k\right) = \frac{\sum_{k=1}^{[n/2]+1} (-1)^{k+1} e_n^{[2^{k-1}]}(\tan \underline{a})}{1 + \sum_{k=1}^{[n/2]} (-1)^k e_n^{[2^k]}(\tan \underline{a})}.$$

A different definition of the rth elementary symmetric polynomial mean is often given. In the references [*AI p.95; HLP p.51*] for instance, the definition is

$$\mathfrak{s}_n^{[r]}(\underline{a}) = \left(\mathfrak{S}_n^{[r]}(\underline{a})\right)^r = \frac{e_n^{[r]}(\underline{a})}{\binom{n}{r}}. \tag{4}$$

EXAMPLE (v)

$$\mathfrak{s}_n^{[n-1]}(\underline{a}) = \frac{\mathfrak{G}_n^n(\underline{a})}{\mathfrak{H}_n(\underline{a})}; \quad \text{or} \quad \frac{\mathfrak{s}_n^{[n]}(\underline{a})}{\mathfrak{s}_n^{[n-1]}(\underline{a})} = \mathfrak{H}_n(\underline{a}). \tag{5}$$

The justification for (2) is that $\mathfrak{S}_n^{[r]}$ has more of the properties of a mean than does $\mathfrak{s}_n^{[r]}$; for instance (3) above and Lemma 2(a) below. However some of the algebraic properties are easier to state in terms of $\mathfrak{s}_n^{[r]}$, see for instance (10) below. In addition $\mathfrak{s}_n^{[r]}(\underline{a})$, like $\mathfrak{e}_n^{[r]}(\underline{a})$, but unlike $\mathfrak{S}_n^{[r]}(\underline{a})$, is defined for all real $\underline{a}$. For these reasons, and also because $\mathfrak{s}_n^{[r]}$ is traditional we will continue to use it as well as $\mathfrak{S}_n^{[r]}$.

When convenient the range of r in the above definitions can be extended as follows:

$$\begin{aligned} e_n^{[0]}(\underline{a}) &= \mathfrak{S}_n^{[0]}(\underline{a}) = \mathfrak{s}_n^{[0]}(\underline{a}) = 1; \\ e_n^{[r]}(\underline{a}) &= \mathfrak{S}_n^{[r]}(\underline{a}) = \mathfrak{s}_n^{[r]}(\underline{a}) = 0, \ r > n, \ r < 0. \end{aligned} \tag{6}$$

Other notations such as $e_{n}^{[r]}(\underline{a}), e_{n-1}^{[r]}(\underline{a})$ etc. will be used without further explanation, see II 1.1 Convention 2, and II 3.2.2.

The ratios

$$\mathfrak{N}_n^{[k]}(\underline{a}) = \frac{\mathfrak{s}_n^{[k]}(\underline{a})}{\mathfrak{s}_n^{[k-1]}(\underline{a})}, \ 1 \le k \le n,$$

have been called *Newton means*; [*Martens & Nowicki; Nowicki 2001*]. We see from (6), Examples (ii), (v) that $\mathfrak{N}_n^{[1]}(\underline{a}) = \mathfrak{A}_n(\underline{a})$, and $\mathfrak{N}_n^{[n]}(\underline{a}) = \mathfrak{H}_n(\underline{a})$.

As is seen from Example (ii) the rth elementary symmetric polynomial mean coincides with arithmetic and geometric means at the extreme values of r, $r = 1$ and $r = n$. So it would be reasonable to expect that $\mathfrak{S}_n^{[r]}$, $1 \le r \le n$, is a scale of comparable means between the geometric and arithmetic means. The justification of this is given below in 2 Theorem 3(b), S(r;s).

It is important to note that the elementary symmetric polynomials can be generated as follows, see Example (iii).

$$\prod_{k=1}^{n}(x + a_k) = \sum_{k=1}^{n} e_n^{[k]}(\underline{a})x^{n-k} = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{s}_n^{[r]}(\underline{a})x^{n-k}; \tag{7}$$

or, equivalently,

$$\prod_{k=1}^{n}(1 + a_k x) = \sum_{k=1}^{n} e_n^{[k]}(\underline{a})x^{k} = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{s}_n^{[r]}(\underline{a})x^{k}. \tag{8}$$

REMARK (i) It is possible to define weighted elementary symmetric polynomial means; if $\underline{w}$ is another n-tuple,

$$\mathfrak{S}_n^{[r]}(\underline{a};\underline{w}) = \left(\frac{e_n^{[r]}(\underline{a}\,\underline{w})}{e_n^{[r]}(\underline{w})}\right)^{1/r}. \tag{9}$$

is called the *rth elementary symmetric polynomial mean of $\underline{a}$ with weight $\underline{w}$*. However as the properties of this weighted mean are not very satisfactory we will not consider it in any detail; see [*Bullen 1965*].

LEMMA 2 *(a) The rth elementary symmetric polynomial mean, $\mathfrak{S}_n^{[r]}(\underline{a})$ has the properties (Sy), (Ho), (Re), (Mo) and it is strictly internal,*

$$\min \underline{a} \leq \mathfrak{S}_n^{[r]}(\underline{a}) \leq \max \underline{a},$$

with equality if and only if $\underline{a}$ is constant.
(b) If $1 \leq r \leq n-1$ then

$$e_n^{[r]}(\underline{a}) = e_{n-1}^{[r]}(\underline{a}) + a_n e_{n-1}^{[r-1]}(\underline{a}); \quad \mathfrak{s}_n^{[r]}(\underline{a}) = \frac{n-r}{n}\mathfrak{s}_{n-1}^{[r]}(\underline{a}) + \frac{r}{n}a_n\mathfrak{s}_{n-1}^{[r-1]}(\underline{a}). \tag{10}$$

The first part of this simple lemma extends results for means defined earlier; see II 1.1 Theorem 2 and III 1 Theorem 2. In addition it helps to justify the use of $\mathfrak{S}_n^{[r]}(\underline{a})$ as the mean rather than $\mathfrak{s}_n^{[r]}(\underline{a})$. The second part states some simple and useful relations that are readily checked.

2 The Fundamental Inequalities

The following simple result is so basic that we will give several proofs.

THEOREM 1 *If $\underline{a}$ is an n-tuple and if ris an integer, $1 \leq r \leq n-1$, then*

$$\left(\mathfrak{s}_n^{[r]}(\underline{a})\right)^2 \geq \mathfrak{s}_n^{[r-1]}(\underline{a})\mathfrak{s}_n^{[r+1]}(\underline{a}); \tag{1}$$

$$\left(e_n^{[r]}(\underline{a})\right)^2 > e_n^{[r-1]}(\underline{a})e_n^{[r+1]}(\underline{a}). \tag{2}$$

Inequality (1) is strict unless $\underline{a}$ is constant.

□ (*i*) We first prove (1) by induction on n.
If $n=2$ then (1) reduces to the $n=2$ case of (GA), see 1 (3) and (6), so suppose that $n>2$. and that (1), together with the case of equality has been proved for all r and m where $1 \leq r \leq m-1$, $2 \leq m \leq n-1$.
Also assume that $\underline{a}'_n$ is not constant. If $1 \leq r \leq n-1$, then by 1(10):

$$\mathfrak{s}_n^{[r-1]}\mathfrak{s}_n^{[r+1]} - \left(\mathfrak{s}_n^{[r]}\right)^2 = \frac{1}{n^2}(A + Ba_n + Ca_n^2), \qquad \text{where}$$

$$A = ((n-r)^2-1)\mathfrak{s}_{n-1}^{[r-1]}\mathfrak{s}_{n-1}^{[r+1]} - \left((n-r)\mathfrak{s}_{n-1}^{[r]}\right)^2,$$
$$B = (n-r+1)(r+1)\mathfrak{s}_{n-1}^{[r-1]}\mathfrak{s}_{n-1}^{[r]} + (n-r-1)(r-1)\mathfrak{s}_{n-1}^{[r-2]}\mathfrak{s}_{n-1}^{[r+1]} - 2r(n-r)\mathfrak{s}_{n-1}^{[r-1]}\mathfrak{s}_{n-1}^{[r]},$$
$$C = (r^2-1)\mathfrak{s}_{n-1}^{[r-2]}\mathfrak{s}_{n-1}^{[r]} - \left(r\mathfrak{s}_{n-1}^{[r-1]}\right)^2.$$

By the induction hypothesis

$$\left(\mathfrak{s}_{n-1}^{[r]}\right)^2 > \mathfrak{s}_{n-1}^{[r-1]}\mathfrak{s}_{n-1}^{[r+1]}; \quad \left(\mathfrak{s}_{n-1}^{[r-1]}\right)^2 > \mathfrak{s}_{n-1}^{[r]}\mathfrak{s}_{n-1}^{[r-2]}; \quad \mathfrak{s}_{n-1}^{[r-1]}\mathfrak{s}_{n-1}^{[r]} > \mathfrak{s}_{n-1}^{[r-2]}\mathfrak{s}_{n-1}^{[r+1]}. \tag{3}$$

This implies that $A \leq -\left(\mathfrak{s}_{n-1}^{[r]}\right)^2$, $B \leq 2\mathfrak{s}_{n-1}^{[r-1]}\mathfrak{s}_{n-1}^{[r]}$, $C < -\left(\mathfrak{s}_{n-1}^{[r-1]}\right)^2$; and so

$$\mathfrak{s}_n^{[r-1]}\mathfrak{s}_n^{[r+1]} - \left(\mathfrak{s}_n^{[r]}\right)^2 < -\frac{1}{n^2}\left(\mathfrak{s}_{n-1}^{[r]} - a_n\mathfrak{s}_{n-1}^{[r-1]}\right)^2. \tag{4}$$

This proves (1) in this case.

Now suppose that $\underline{a}'_n$ is constant, $a_1 = \cdots = a_{n-1} = a \neq a_n$, when (4) becomes an equality since the inequalities (3) all become equalities in this case. But the right-hand side of (4) can be written

$$-\frac{1}{n^2}\big(\mathfrak{s}_{n-1}^{[r-1]}\big)^2\bigg(\frac{\mathfrak{s}_{n-1}^{[r]}}{\mathfrak{s}_{n-1}^{[r-1]}} - a_n\bigg)^2 = -\frac{1}{n^2}\big(\mathfrak{s}_{n-1}^{[r-1]}\big)^2(a-a_n)^2 < 0,$$

which completes the proof.

(*ii*) (2) is an easy consequence of (1) which can be rewritten using 1(4) as

$$\Big(e_n^{[r]}(\underline{a})\Big)^2 \geq \frac{(r+1)(n-r+1)}{r(n-r)} e_n^{[r-1]}(\underline{a}) e_n^{[r+1]}(\underline{a}). \tag{5}$$

Noting that the numerical factor on the right-hand side of (5) is greater than 1 establishes (2).

(*iii*) A direct proof (2) can also be given.

A typical term in the expansion of $e_n^{[r-1]}(\underline{a})e_n^{[r+1]} - \Big(e_n^{[r]}\Big)^2$ is $\displaystyle (\prod_{k=1}^{r-s} a_k^2)(\prod_{k=r-s+1}^{r+s} a_k)$; and this term has coefficient $\displaystyle\binom{2s}{s-1} - \binom{2s}{s}$ which is negative.

(*iv*) Of course taking note of 1 Example (iii) and 1(7) both (1) and (2) can be deduced easily from I 1.1 Corollary 8; see [*Campbell; Green; Nowicki 2001*]. This proof shows that these inequalities hold for real n-tuples $\underline{a}$.

(*v*) The following identity is proved in [*Dougall; Muirhead 1902/03*].

$$\Big(\mathfrak{s}_n^{[r]}\Big)^2 - \mathfrak{s}_n^{[r-1]}\mathfrak{s}_n^{[r+1]} = \frac{1}{r(r+1)\binom{n}{r}\binom{n}{r+1}} \sum_{k=1}^{n-1}\binom{2k}{k}\frac{(r,k)}{k+1},$$

where $(r,k) = \sum\Big(\prod_{j=k}^{r-k-1} a_j^2\Big)\Big(\prod_{j=r-k}^{r+k-1} a_j\Big)(a_{r+k} - a_{r+k+1})^2$, the summation being over all such products obtainable from $a_1, a_2, \ldots, a_n$. This identity gives another proof of (1).

(*vi*) Another identity due to Jolliffe, [*Jolliffe*], also gives an immediate proof of (1).

$$\Big(\frac{n!}{(r-1)!(n-r-1)!}\Big)^2\Big((\mathfrak{s}_n^{[r]})^2 - \mathfrak{s}_n^{[r-1]}\mathfrak{s}_n^{[r+1]}\Big) =$$

$$(n-1)\sum\big((a_i - a_j)C_{n-2,r-1}\big)^2$$

$$+\frac{2!(n-3)}{(r-1)(n-r-1)}\sum\big((a_i - a_j)(a_k - a_\ell)C_{n-4,r-2}\big)^2$$

$$+\frac{3!(n-5)}{(r-1)(r-2)(n-r-2)}\sum\big((a_i - a_j)(a_k - a_\ell)(a_p - a_q)C_{n-6,r-3}\big)^2$$

$$+\cdots,$$

where $C_{n-2,r-1}$ is the sum of the products of $(r-1)$ factors from the $(n-2)$ factors differing from $(a_i - a_j)$, and the summation is over all possible terms of that type; $C_{n-4,r-2}, C_{n-6,r-3}, \ldots$ are defined similarly. □

REMARK (i) In [*Jecklin 1949c*] there is another inequality similar to (5).

$$e_n^{[r]}(\underline{a}^2) \geq \left(\frac{2}{\binom{n}{r} - 1}\right) \sum_{k=1}^{r} (-1)^{k+1} e_n^{[r-k]}(\underline{a}) e_n^{[r+k]}(\underline{a}).$$

REMARK (ii) Theorem 1, in the form of I 1.1 Corollary 8, was originally stated by Newton without proof; [*Newton pp.347–349*]. The first proof was given in [*Maclaurin*]; later proofs can be found in [*Angelescu; Bonnensen; Darboux 1902; Durand; Fujisawa; Hamy; Ness 1964; Perel'dik; Schlömilch 1858a; Sylvester*]. Dunkel has given a very full treatment of the whole topic, [*Dunkel 1908/9, 1909/10*]. Reference should also be made to [*Kellogg*]. The inductive proof was discovered independently by Dixon, Jolliffe and M. H. A. Newman, and is given in [*HLP pp.53–54*]. Cubic extensions of these inequalities have been given; see [*Rosset*].

COROLLARY 2 *(a) If* $1 \leq r < s \leq n$, *then*

$$e_n^{[r-1]}(\underline{a}) e_n^{[s]}(\underline{a}) < e_n^{[r]}(\underline{a}) e_n^{[s-1]}(\underline{a}), \quad \textit{and} \quad \mathfrak{s}_n^{[r-1]}(\underline{a}) \mathfrak{s}_n^{[s]}(\underline{a}) \leq \mathfrak{s}_n^{[r]}(\underline{a}) \mathfrak{s}_n^{[s-1]}(\underline{a}),$$

where the second inequality is strict unless $\underline{a}$ *is constant.*
(b) If $1 \leq r \leq n-1$, *and* $e_n^{[r-1]}(\underline{a}) > e_n^{[r]}(\underline{a})$, *respectively* $\mathfrak{s}_n^{[r-1]}(\underline{a}) > \mathfrak{s}_n^{[r]}(\underline{a})$, *then*

$$e_n^{[r]}(\underline{a}) > e_n^{[r+1]}(\underline{a}), \quad \textit{respectively} \quad \mathfrak{s}_n^{[r]}(\underline{a}) > \mathfrak{s}_n^{[r+1]}(\underline{a}).$$

(c) If $1 \leq r+s \leq n$ *then*

$$e_n^{[r+s]}(\underline{a}) < e_n^{[r]}(\underline{a}) e_n^{[s]}(\underline{a}) \quad \textit{and} \quad \mathfrak{s}_n^{[r+s]}(\underline{a}) \leq \mathfrak{s}_n^{[r]}(\underline{a}) \mathfrak{s}_n^{[s]}(\underline{a}).$$

□ (a) and (b) are immediate corollaries of (1) and (2) written as

$$\frac{\mathfrak{s}_n^{[r]}(\underline{a})}{\mathfrak{s}_n^{[r+1]}(\underline{a})} \geq \frac{\mathfrak{s}_n^{[r-1]}(\underline{a})}{\mathfrak{s}_n^{[r]}(\underline{a})}, \quad \text{and} \quad \frac{e_n^{[r]}(\underline{a})}{e_n^{[r+1]}(\underline{a})} > \frac{e_n^{[r-1]}(\underline{a})}{e_n^{[r]}(\underline{a})},$$

respectively. Then (c) follows from (a) by repeated application: for instance

$$e_n^{[r]}(\underline{a}) e_n^{[s]}(\underline{a}) > e_n^{[r-1]}(\underline{a}) e_n^{[s+1]}(\underline{a}) > \cdots > e_n^{[0]}(\underline{a}) e_n^{[s+r]}(\underline{a}) = e_n^{[r+s]}(\underline{a}).$$

□

REMARK (iii) Put $a_k^{(1)} = \mathfrak{s}_n^{[k]}(\underline{a})/\mathfrak{s}_n^{[k-1]}(\underline{a})$, $1 \le k \le n$. Then by Corollary 2(a) $a_1^{(1)} \ge a_2^{(1)} \ge \cdots \ge a_n^{(1)}$, and if $\underline{a}$ is not constant these inequalities are strict. Further $\prod_{k=1}^n a_k^{(1)} = \prod_{k=1}^n a_k$. Now iterate this procedure using $a_k^{(1)}$, $1 \le k \le n$ to get for each k a sequence $a_k^{(m)}, m = 1, 2, \ldots$ and k, $\lim_{m\to\infty} a_k^{(m)} = \mathfrak{G}_n(\underline{a})$; see [*Stieltjes*].

REMARK (iv) If $e_n^{[r]}(\underline{a}) \le e_n^{[r]}(\underline{b})$ then one can ask for what functions f is it true that $e_n^{[r]}(f(\underline{a})) \le e_n^{[r]}(f(\underline{b}))$? This question has been answered and in particular this is the case if $f(x) = x^p$, $0 < p \le 1$; see [*Efroymson, Swartz & Wendroff*].

We now establish the *elementary symmetric polynomial mean inequality*, S(r;s), the basic result of this section, giving several proofs.

THEOREM 3 *If* $1 \le r < s \le n$ *then*
(a)
$$\mathfrak{N}_n^{[s]}(\underline{a}) \le \mathfrak{N}_n^{[r]}(\underline{a});$$
(b)
$$\mathfrak{S}_n^{[s]}(\underline{a}) \le \mathfrak{S}_n^{[r]}(\underline{a}), \qquad S(\mathrm{r};\mathrm{s})$$
with equality, in either inequality, if and only if $\underline{a}$ *is constant.*

□ (a) This is just a rewriting of (1) using the definition of Newton means given in 1.
(b) (*i*) If $1 \le m \le r$ then, from (1), $\left(\mathfrak{s}_n^{[m-1]}\mathfrak{s}_n^{[m+1]}\right)^m \le \left(\mathfrak{s}_n^{[m]}\right)^{2m}$; multiplying all of these over m gives
$$\left(\mathfrak{s}_n^{[r+1]}\right)^r \le \left(\mathfrak{s}_n^{[r]}\right)^{r+1} \quad \text{or} \quad \mathfrak{S}_n^{[r+1]} \le \mathfrak{S}_n^{[r]}, \tag{6}$$
which clearly implies S(r;s). The case of equality is immediate.
(*ii*) It is of interest to see that S(r;s) can be proved from the weaker inequality (2). The method of proof is similar to Crawford's proof of (GA), II 2.4.1 proof (*vi*). Suppose that $0 < a_1 \le \cdots \le a_n$, $a_1 \ne a_n$, and define a new n-tuple $\underline{b}$ by: $b_1 = \mathfrak{S}_n^{[r]}(\underline{a}), b_k = a_k$, $2 \le n \le n-1$ and b_n chosen so that $\mathfrak{S}_n^{[r]}(\underline{b}) = \mathfrak{S}_n^{[r]}(\underline{a})$, or equivalently $e_n^{[r]}(\underline{a}) = e_n^{[r]}(\underline{b})$. If then we can prove that for any $s > r$, $\mathfrak{S}_n^{[s]}(\underline{b}) > \mathfrak{S}_n^{[s]}(\underline{a})$ the result follows as in the proof of (GA) mentioned above. Let $\underline{c} = (a_2, \ldots, a_{n-1}) = (b_2, \ldots, b_{n-1})$ then by the definition of $\underline{b}$,
$$\begin{aligned} e_n^{[r]}(\underline{a}) &= a_1 a_n e_{n-2}^{[r-2]}(\underline{c}) + (a_1 + a_n) e_{n-2}^{[r-1]}(\underline{c}) + e_{n-2}^{[r]}(\underline{c}),\\ e_n^{[r]}(\underline{a}) = e_n^{[r]}(\underline{b}) &= b_1 b_n e_{n-2}^{[r-2]}(\underline{c}) + (b_1 + b_n) e_{n-2}^{[r-1]}(\underline{c}) + e_{n-2}^{[r]}(\underline{c}).\end{aligned}$$
Hence,
$$(b_1 b_n - a_1 a_n) e_{n-2}^{[r-2]}(\underline{c}) = -(b_1 + b_n - a_1 - a_n) e_{n-2}^{[r-1]}(\underline{c}), \tag{7}$$

or

$$\left(b_1 e_{n-2}^{[r-2]}(\underline{c}) + e_{n-2}^{[r-1]}(\underline{c})\right) b_n = a_1 a_n e_{n-2}^{[r-2]}(\underline{c}) + (a_1 + a_n - b_1) e_{n-2}^{[r-1]}(\underline{c}).$$

Since by internality, 1 Lemma 2, $a_n > b_1$ this last identity shows that $b_n > 0$. Now

$$\begin{aligned} e_n^{[s]}(\underline{b}) - e_n^{[s]}(\underline{a}) =& (b_1 b_n - a_1 a_n)(e_{n-2}^{[s-2]}(\underline{c}) + (b_1 + b_n - a_1 - a_n) e_{n-2}^{[s-1]}(\underline{c}) \\ =& (b_1 + b_n - a_1 - a_n) e_{n-2}^{[s-2]}(\underline{c}) \left(\frac{e_{n-2}^{[s-1]}(\underline{c})}{e_{n-2}^{[s-2]}(\underline{c})} - \frac{e_{n-2}^{[r-1]}(\underline{c})}{e_{n-2}^{[r-2]}(\underline{c})} \right), \text{ by (7).} \end{aligned}$$

The second factor is negative by Corollary 2(a). As to the first factor we have from (7) that: $\operatorname{sign}(b_1 + b_n - a_1 - a_n)$

$$\begin{aligned} &= \operatorname{sign}(a_1 a_n - b_1 b_n) = \operatorname{sign}(b_1(b_1 + b_n - a_1 - a_n) + a_1 a_n - b_1 b_n) \\ &= \operatorname{sign}((b_1 - a_1)(b_1 - a_n)) = -1. \end{aligned}$$

This proves that $e_n^{[s]}(\underline{b}) > e_n^{[s]}(\underline{a})$, which is equivalent to $\mathfrak{S}_n^{[s]}(\underline{b}) > \mathfrak{S}_n^{[s]}(\underline{a})$, and the proof is complete.

(*iii*) A direct proof of inequality (6) has been given in [*Perel'dik*]. For simplicity let us use the notation: $e_k = e_n^{[k]}(\underline{a})$, $(e_k)^i = e_k^i = e_{n-1}^{[k]}(\underline{a}_i')$, $e_k^{i,j} = (e_k^i)^j$, etc.

Let $\epsilon > 0$, and consider those $\underline{a}$ for which $e_r = \epsilon$, and find for which of these e_{r+1} has a maximum value. We show that this is a unique maximum occurring when $\underline{a}$ is constant, a say. Then if this is the case

$$\epsilon = \binom{n}{r} a^r, \quad \text{and} \quad e_{r+1} \le [e_{r+1}]_{\max} = \binom{n}{r+1} a^{r+1} = \frac{\binom{n}{r+1}}{\binom{n}{r}^{1+1/r}} \epsilon^{1+1/r}.$$

By 1 (2) this is (6) and because of the uniqueness of the maximum we also get the case of equality.

Let $u(\underline{a}) = e_{r+1} - \lambda(e_r - \epsilon)$, when $\partial u/\partial a_k = e_r^k - \lambda e_{r-1}^k$, $1 \le k \le n$. For a maximum we must have $\partial u/\partial a_k = 0$, $1 \le k \le n$, so

$$\lambda = \frac{e_r^k}{e_{r-1}^k}, \ 1 \le k \le n. \tag{8}$$

This implies that $\lambda = \dfrac{\sum_{k=1}^n e_r^k}{\sum_{i=1}^n e_{r-1}^k}$. The last identity shows that λ is a symmetric function of $\underline{a}$, and so in particular is invariant under interchange of any pair a_i and a_j, $1 \le i, j \le n$. Take $k = 1$ in (8), and use 1(10) to get

$$\lambda = \frac{e_r^{1,k} + a_k e_{r-1}^{1,k}}{e_{r-1}^{1,k} + a_k e_{r-2}^{1,k}} = \frac{A + a_k B}{B + a_k C}, \text{ say, } 2 \le k \le n,$$

where A, B and C involve neither a_1 nor a_k. Interchanging a_1 and a_k, and using the above noted symmetry of λ, we get that

$$\frac{A + a_k B}{B + a_k C} = \frac{A + a_1 B}{B + a_1 C}, \quad \text{or,} \quad (a_1 - a_k)(AC - B^2) = 0. \tag{9}$$

Hence, reordering $\underline{a}$ if necessary, there are two possibilities: either $\underline{a}$ is constant, or for some i, $1 \le i \le n-1$, $a_1 \ne a_j$, $2 \le j \le i+1$ and $a_1 = a_j, i+2 \le j \le n$. Suppose the second case holds; then from (9) we have that if $2 \le k \le i+1$ then $AC - B^2 = 0$, or $\dfrac{e_r^{1,k}}{e_{r-1}^{1,k}} = \dfrac{e_{r-1}^{1,k}}{e_{r-2}^{1,k}}$, $2 \le k \le i+1$. Hence

$$\lambda = \frac{e_r^{1,k}}{e_{r-1}^{1,k}}, \; 2 \le k \le i+1. \tag{10}$$

Now repeat the above argument taking $k = 2$ in (10), and use 1(10) to get

$$\lambda = \frac{e_r^{1,2,k} + a_k e_{r-1}^{1,2,k}}{e_{r-1}^{1,2,k} + a_k e_{r-2}^{1,2,k}}, \; 3 \le k \le i+1;$$

and so, since we are considering the second case $\dfrac{e_r^{1,2,k}}{e_{r-1}^{1,2,k}} = \dfrac{e_{r-1}^{1,2,k}}{e_{r-2}^{1,2,k}}$, $3 \le k \le i+1$.
If $r \le n-i-1$ this process can be repeated until finally $\lambda = \dfrac{e_r^{1,2,\ldots i+1}}{e_{r-1}^{1,2,\ldots,i+1}} = \dfrac{n-i-r}{r} a_1$, which by the symmetry of λ is a contradiction since $a_1 \ne a_2, \ldots, a_{i+1}$.
If $r = n - m > n - i - 1$ this process leads to

$$\lambda = \frac{e_r^{1,2,\ldots m}}{e_{r-1}^{1,2,\ldots,m}} = \frac{a_{m+1} a_{m+2} \ldots a_n}{e_{r-1}^{1,2,\ldots,m+1} + a_{m+1} e_{r-2}^{1,2,\ldots,m+1}}.$$

This is again a contradiction since $a_1 \ne a_{m+1}$.
Hence $\partial u / \partial a_k = 0$, $1 \le k \le n$, implies that $\underline{a}$ is constant, a say. It remains to prove that it is a maximum of e_{r+1} subject to $e_r = \epsilon$. Simple calculations show that $du = \sum_{i=1}^n (\epsilon + (\lambda - a_i) e_{r-1}^i) da_i$, and that

$$\begin{aligned} d^2 u &= \sum_{i=1}^n e_{r-1}^1 d^2 a_i + \sum_{i=1}^n \Big((\lambda - a_i) \sum_{j=1}^n e_{r-2}^{i,j} da_j \Big) da_i \\ &= - \frac{\binom{n-2}{r-2}}{r(r-1)} a^{r-1} \Big((n-r) \sum_{i=1}^n d^2 a_i + n(r-1) \sum_{i=1}^n da_i)^2 \Big) < 0. \end{aligned}$$

This completes the proof. □

REMARK (v) The inequality S(r;s) is another extension of (GA) and proofs of S(r;s) give further proofs of (GA). S(r;s) was first stated by Maclaurin; it was also proved, probably independently, by Schlömilch and Dunkel. Proof (*ii*) above is in [*HLP p.53*]. Muirhead states that Schlömilch gave precedence to Fort who had given a proof in the eighteenth century; see [*Brenner 1978; Dunkel 1909/10; Kreis 1948; Maclaurin; Muirhead 1900/01; Schlömilch 1858a*].

REMARK (vi) It was pointed out by Bauer and Alzer that using 1(3) and 1(5) inequality S(1;n-1) is just Sierpiński's inequality, II 3.8 (58); [*Alzer1989b; Bauer 1986a*].

REMARK (vii) The second proof of (b) applies with no change to the weighted mean defined in 1(9) provided the n-tuples $\underline{a}$ and $\underline{w}$ are similarly ordered; see [*Bullen 1965*].

REMARK (viii) Inequality in Theorem 3(a) is a generalization of (HA).

In the case $n = 3$ S(r;s) can be given a simple geometric interpretation as follows.

COROLLARY 4 *Let a_1, a_2, a_3 be the sides of a parallelepiped, and A, B, C the sides of a cube of the same perimeter, area and volume, respectively, then: $A \geq B \geq C$, with equality if and only if the parallelepiped is a cube.*

□ This follows from S(r;s) because $A = \mathfrak{S}_3^{[1]}(\underline{a}), B = \mathfrak{S}_3^{[2]}(\underline{a}), C = \mathfrak{S}_3^{[3]}(\underline{a})$. □

REMARK (ix) This last result can be extended to n-dimensions; see [*Jecklin 1948a*].

REMARK (x) If $\underline{x}, \underline{y}, \underline{A}$ are the n-tuples defined in II 5.6 Theorem 16, then it has been proved that $\mathfrak{G}_n(\underline{y}) \leq \mathfrak{S}_n^{[2]}(\underline{A}) \leq \mathfrak{A}_n(\underline{x})$; see the references given in that earlier section.

As a corollary of S(r;s) we get a proof of an invariance property of the elementary symmetric polynomial means ; see II 5.2 (c) for the same results for arithmetic means.

COROLLARY 5 *The sequences $\underline{a}$ and $\mathfrak{S}_1^{[r]}(\underline{a}), \mathfrak{S}_2^{[r]}(\underline{a}), \ldots$ increase, strictly, together.*

□ By 1(10) we have that $\mathfrak{s}_{n+1}^{[r]} = \dfrac{n+1-r}{n+1}\mathfrak{s}_n^{[r]} + \dfrac{r}{n+1}a_{n+1}\mathfrak{s}_n^{[r-1]}$, or equivalently

$$\mathfrak{s}_{n+1}^{[r]} - \mathfrak{s}_n^{[r]} = \frac{r}{n+1}\mathfrak{s}_n^{[r-1]}\left(a_{n+1} - \frac{\mathfrak{s}_n^{[r]}}{\mathfrak{s}_n^{[r-1]}}\right).$$

From S(r;r+1), (6), we easily then see that $\dfrac{\mathfrak{s}_n^{[r]}}{\mathfrak{s}_n^{[s-1]}} \le \mathfrak{S}_n^{[r]} \le \mathfrak{A}_n$. Hence

$$\mathfrak{s}_{n+1}^{[r]} - \mathfrak{s}_n^{[r]} \ge \frac{r}{n+1}\mathfrak{s}_n^{[r-1]}(a_n - \mathfrak{A}_n) \ge 0,$$

by the internality of the arithmetic mean.
Further if $\underline{a}$ is strictly increasing the above inequalities are strict. □

The inequality S(r;s) implies that if $1 \le r \le n$ then $\mathfrak{G}_n(\underline{a}) \le \mathfrak{S}_n^{[r]}(\underline{a}) \le \mathfrak{A}(\underline{a})$, and it is natural to ask if the outer means can be replaced by other by other power means. Shi has shown that if $1 < r < n$ then for some r_1, r_2, $0 \le r_1 < r_2 < 1$,

$$\mathfrak{M}_n^{[r_1]}(\underline{a}) \le \mathfrak{S}_n^{[r]}(\underline{a}) \le \mathfrak{M}_n^{[r_2]}(\underline{a});$$

further if $n = 3$ then $r_1 = 0$ is best possible, and $r_2 = 2(1 - \log 2/\log 3)$ is best possible; [*Shi*].
In addition we can ask for an r, $r = r_k(\underline{a})$, $0 \le r \le 1$, such that $\mathfrak{M}_n^{[r]}(\underline{a}) = \mathfrak{S}_n^{[k]}(\underline{a})$? Using Theorem 3 we easily see that $1 = r_1(\underline{a}) \ge \cdots \ge r_n(\underline{a}) = 0$. A lower bound is readily obtained since if $\underline{a} = (1, 0, \ldots, 0)$ then $r_k(\underline{a}) = 0$, $k \ge 2$. On the other hand it has been shown that if $k \ge 3$ then

$$r_k(\underline{a}) \le k\frac{\log\big(n/(n-1)\big)}{\log\big(n/(n-k)\big)}.$$

Further this upper bound is sharp, being attained when $\underline{a} = (1, 1, \ldots, 1, 0)$; [*Kuczma 1992*].

Mitrinović has obtained the following interesting generalizations of inequality (2) and Corollary 2(b); see [*Mitrinović 1967b,c,1968a*]. [We write $\Delta^k e_n^{[s-\nu]}$ for $\Delta^k c_\nu$ where $c_\nu = e_n^{[s-\nu]}$.]

THEOREM 6 *Let $\underline{a}$ be an n-tuple and r an integer, $1 \le r < n$, then*

$$(\Delta^\nu e_n^{[s-\nu]})^2 \ge (\Delta^\nu e_n^{[r-\nu-1]})(\Delta^\nu e_n^{[r-\nu+1]}), \quad 0 \le \nu \le r-1. \tag{11}$$

If in addition

$$(-1)^p \Delta^p e_n^{[r-\nu+1]} > 0, \qquad 0 \le p \le \nu \le r, \tag{12}$$

then

$$(-1)^\nu \Delta^\nu e_n^{[r-\nu]} > 0. \tag{13}$$

□ For (11) modify proof (*iv*) of Theorem 1 by applying I 1.1 Corollary 8 to the polynomial $(x-1)^\nu \prod_{i=1}^n (x + a_i)$, noting 1(7).

Inequality (13) is proved by induction on ν, noting that if $\nu = 1$ the result is just Corollary 2(b). So assume the result has been proved $1 \le \nu \le k-1$. From (11)

$$(\Delta^{k-1}e_n^{[r-k+1]})^2 \ge (\Delta^{k-1}e_n^{[r-k]})(\Delta^{k-1}e_n^{[r-k+2]}),$$

which by the induction process , (13) with $\nu = k-1$, is equivalent to

$$\frac{\Delta^{k-1}e_n^{[r-k+1]}}{\Delta^{k-1}e_n^{[r-k]}} \ge \frac{\Delta^{k-1}e_n^{[r-k+2]}}{\Delta^{k-1}e_n^{[r-k+1]}} \tag{14}$$

Hypothesis (12), with $p = \nu = k$ is equivalent to

$$(-1)^{k-1}\Delta^{k-1}e_n^{[r-k+2]} > (-1)^{k-1}\Delta^{k-1}e_n^{[r-k+1]}; \tag{15}$$

and the right-hand side of (15) is positive, by (14) with $p = k-1, \nu = k$. Hence from (15) the right-hand side of (14) is greater than 1; this of course makes the left-hand side of (14) greater than 1. Using this and (15), and reversing the argument implies (13) with $\nu = k$, and completes the induction. □

REMARK (xi) If $\nu = 0$ (11) reduces to (2), and as we noted above if $\nu = 1$ (13) is just Corollary 2(b).

REMARK (xii) Clearly similar results hold with $s_n^{[r]}(\underline{a})$ replacing $e_n^{[r]}(\underline{a})$

REMARK (xiii) By considering polynomials of the form $\prod_{j=1}^{\nu}(x-\alpha_j)\prod_{i=1}^{n}(x+a_i)$ then I 1 Corollary 8 can be used to obtain even more general results; [*Mitrinović 1967a*]. For instance applying 1 1 Corollary 8 to $(x-\alpha)\prod_{i=1}^{n}(x+a_i)$ we get that $(e_n^{[r-1]} - \alpha e_n^{[r-2]})(e_n^{[r+1]} - \alpha e_n^{[r]}) \le (e_n^{[r]} - \alpha e_n^{[r-1]})^2$. That is, for every α

$$\alpha^2(e_n^{[r-2]}e_n^{[r]} - (e_n^{[r-1]})^2) + \alpha(e_n^{[r-1]}e_n^{[r]} - e_n^{[r-2]}e_n^{[r+1]}) + (e_n^{[r-1]}e_n^{[r+1]} - (e_n^{[r]})^2 \le 0.$$

By (2) the coefficient of α^2 is negative and so

$$(e_n^{[r-1]}e_n^{[r]} - e_n^{[r-2]}e_n^{[r+1]})^2 \le 4\left(e_n^{[r-2]}e_n^{[r]} - (e_n^{[r-]})^2\right)\left(e_n^{[r-1]}e_n^{[r+1]} - (e_n^{[r]})^2\right).$$

COROLLARY 7 *If (a)* $(-1)^\nu\Delta^\nu e_{n-1}^{[r-\nu-2]}(\underline{a}_1') > 0$, *(b)* $(-1)^\nu\Delta^\nu e_{n-1}^{[r-\nu-1]}(\underline{a}_1') > 0$, *and (c)* $(-1)^\nu\Delta^\nu e_{n-1}^{[r-\nu]}(\underline{a}_1') > 0$ *then*

$$\begin{aligned}\Delta^\nu e_n^{[r-\nu-1]}(\underline{a})\Delta^\nu e_n^{[r-\nu+1]}(\underline{a}) - (\Delta^\nu e_n^{[r-\nu]}(\underline{a}))^2 \\ \le \Delta^\nu e_{n-1}^{[r-\nu-1]}(\underline{a}_1')\Delta^\nu e_{n-1}^{[r-\nu+1]}(\underline{a}_1') - (\Delta^\nu e_{n-1}^{[r-\nu]}(\underline{a}_1'))^2.\end{aligned} \tag{16}$$

□ Put $a_1 = x$ and denote the left-hand side of (16) by $f(x)$, when (16) becomes $f(x) \le f(0)$. Simple calculations show that

$$f''(x) = 2\Big(\Delta^\nu e_{n-1}^{[r-\nu-2]}(\underline{a}_1')\Delta^\nu e_{n-1}^{[r-\nu]}(\underline{a}_1') - \big(\Delta^\nu e_{n-1}^{[r-\nu-1]}(\underline{a}_1')\big)^2\Big) \le 0, \text{ by (11)}.$$

Hence $f'(x) \le f'(0)$, so that the result will follow if we show that $f'(0) \le 0$. Again, simple calculations show that

$$f'(0) = \Delta^\nu e_{n-1}^{[r-\nu-2]}(\underline{a}_1')\Delta^\nu e_{n-1}^{[r-\nu+1]}(\underline{a}_1') - \Delta^\nu e_{n-1}^{[r-\nu-1]}(\underline{a}_1')\Delta^\nu e_{n-1}^{[r-\nu]}(\underline{a}_1').$$

The hypotheses, together with (11) imply that

$$\begin{aligned}(-1)^\nu\Delta^\nu e_{n-1}^{[r-\nu-2]}(\underline{a}_1')&\Delta^\nu e_{n-1}^{[r-\nu+1]}(\underline{a}_1')\Delta^\nu e_{n-1}^{[r-\nu-1]}(\underline{a}_1')\\ &\le\big(\Delta^\nu e_{n-1}^{[r-\nu]}(\underline{a}_1')\big)^2(-1)^\nu\Delta^\nu e_{n-1}^{[r-\nu-2]}(\underline{a}_1')\\ &\le(-1)^\nu\Delta^\nu e_{n-1}^{[r-\nu]}(\underline{a}_1')\big(\Delta^\nu e_{n-1}^{[r-\nu-1]}(\underline{a}_1')\big)^2,\end{aligned}$$

which gives $\Delta^\nu e_{n-1}^{[r-\nu-2]}(\underline{a}_1')\Delta^\nu e_{n-1}^{[r-\nu+1]}(\underline{a}_1') \le \Delta^\nu e_{n-1}^{[r-\nu]}(\underline{a}_1')\Delta^\nu e_{n-1}^{[r-\nu-1]}(\underline{a}_1')$. Substituting this in $f'(0)$ proves that $f'(0) \le 0$ as was to be shown. □

REMARK (xiv) Inequality (16) can be strengthened if $\underline{a}_1'$ is replaced by $\underline{a}_i', 1 \le i \le n$, and then taking the minimum over all i.

COROLLARY 8 *If $\Delta^p e_n^{[r-p]} > 0,\ 0 \le p \le m$, then $\Delta^m e_n^{[r-m+1]} > 0$.*

□ The case $m = 0$ is trivial and the case $m = 1$ is just Corollary 2(b).
Assume that $\Delta^p e_n^{[r-p]} > 0,\ 0 \le p \le m-1$ implies that $\Delta^{m-1} e_n^{[r-m+1]} > 0$ and we prove that if also $\Delta^m e_n^{[r-m]} > 0$ then $\Delta^m e_n^{[r-m+1]} > 0$.
From (11) $(\Delta^{m-1} e_n^{[r-m+1]})^2 \ge \Delta^{m-1} e_n^{[r-m]}\Delta^{m-1} e_n^{[r-m+2]}$, which by the induction hypothesis is equivalent to

$$\frac{\Delta^{m-1} e_n^{[r-m+1]}}{\Delta^{m-1} e_n^{[r-m+2]}} \ge \frac{\Delta^{m-1} e_n^{[r-m]}}{\Delta^{m-1} e_n^{[r-m+1]}}. \tag{17}$$

The inequality $\Delta^m e_n^{[r-m]} > 0$ is equivalent to $\Delta^{m-1} e_n^{[r-m]} > \Delta^{m-1} e_n^{[r-m+1]}$, or by the induction hypothesis to

$$\frac{\Delta^{m-1} e_n^{[r-m]}}{\Delta^{m-1} e_n^{[r-m+1]}} > 1. \tag{18}$$

So, from (17) and (18), $\dfrac{\Delta^{m-1} e_n^{[r-m+1]}}{\Delta^{m-1} e_n^{[r-m+2]}} > 1$, or $\Delta^{m-1} e_n^{[r-m+1]} > \Delta^{m-1} e_n^{[r-m+2]}$, which is just $\Delta^m e_n^{[r-m+1]} > 0$, as had to be proved. This completes the induction.
□

3 Extensions of S(r;s) of Rado-Popoviciu Type

Since the basic inequality S(r;s) is a generalization of (GA) is natural to ask whether generalizations of Rado or Popoviciu type are possible. A fairly complete analogue of (P), II 3.1, due to Bullen, [*Bullen 1965*], is given below in Theorem 2, but extensions of (R) are much more incomplete. Unlike the similar extension of (r;s), III 3.2.3, our present results cannot be deduced from a general result, as in IV 3.2, but must be proved separately. However the techniques used are the two basic ones used to prove II 3.1 Theorem 1— the use of elementary calculus, and the use of the basic inequality, S(r;s) in this situation, on suitably chosen sequences.

As in II 3.2.2 if $\underline{a} = (a_1, \ldots, a_{n+m})$ let us write $\tilde{\underline{a}} = (a_{n+1}, \ldots, a_{n+m})$, and the put $\tilde{e}_m^{[r]} = e_m^{[r]}(\tilde{\underline{a}})$, etc.

The following simple lemma extends the identities 1(10).

LEMMA 1 *(a)*

$$e_{n+m}^{[s]} = \begin{cases} \sum_{t=0}^{s} e_n^{[s-t]}\, \tilde{e}_m^{[t]}, & \text{if } s \le \min\{n,m\},\\ \sum_{t=0}^{n+m-s} e_n^{[n-t]}\, \tilde{e}_m^{[s-n+t]}, & \text{if } s > \max\{n,m\},\\ \sum_{t=0}^{m} e_n^{[s-t]}\, \tilde{e}_m^{[t]}, & \text{if } m < s \le n. \end{cases}$$

(b) If $1 \le s \le n+m$, $u = \max\{s-n, 0\}$, $r = \min\{s, m\}$, *and if*

$$\lambda(s,t) = \frac{\binom{n}{s-n}\binom{s}{t+n}}{\binom{n+m}{s}},\ 0 \le t \le s,$$

then

$$\mathfrak{s}_{n+m}^{[s]} = \sum_{t=u}^{r} \lambda(s,t)\mathfrak{s}_n^{[s-t]}\, \tilde{\mathfrak{s}}_{n+m}^{[t]}, \tag{1}$$

□ (a) follows immediately from 1(1), (7) or (8) , and using 1(4) it is easily seen to imply (b). □

REMARK (i) In particular if $a_{n+1} = \cdots = a_{n+m} = \beta$ then (1) reduces to

$$\mathfrak{s}_{n+m}^{[s]} = \sum_{t=u}^{r} \lambda(s,t)\mathfrak{s}_n^{[s-t]}\beta^t, \tag{2}$$

and if in addition $a_1 = \cdots = a_n = \alpha$,

$$\mathfrak{s}_{n+m}^{[s]} = \sum_{t=u}^{r} \lambda(s,t)\alpha^{s-t}\beta^t, \tag{3}$$

REMARK (ii) Identities 1(10) follow as particular cases of this lemma; take $m = 1$ and replace n by $(n-1)$.

We now prove the main result of this section.

THEOREM 2 *Let $\underline{a}$ be an $(n+m)$-tuple, r and k integers with $1 \le r < k \le n+m$, and put $u = \max\{r-n, 0\}$, $v = \min\{r, m\}$, $w = \max\{k-n, 0\}$, $x = \min\{k, m\}$.*
(a) If $v \le w$ and $r - u \le k - x$ then

$$\left(\frac{\mathfrak{S}_{n+m}^{[r]}(\underline{a})}{\mathfrak{S}_{n+m}^{[k]}(\underline{a})}\right)^k \ge \left(\frac{\mathfrak{S}_{n}^{[r-u]}(\underline{a})}{\mathfrak{S}_{n}^{[k-x]}(\underline{a})}\right)^{k-x} \left(\frac{\tilde{\mathfrak{S}}_{m}^{[v]}(\underline{a})}{\tilde{\mathfrak{S}}_{m}^{[w]}(\underline{a})}\right)^{w}. \tag{4}$$

(b) If $v \ge w$ then

$$\left(\frac{\mathfrak{S}_{n+m}^{[r]}(\underline{a})}{\mathfrak{S}_{n+m}^{[k]}(\underline{a})}\right)^k \le \left(\frac{\tilde{\mathfrak{S}}_{m}^{[v]}(\underline{a})}{\tilde{\mathfrak{S}}_{m}^{[w]}(\underline{a})}\right)^{w}. \tag{5}$$

□ (a) Rewrite (4) as $L \le R$ where

$$L = \left(\frac{\mathfrak{s}_{n+m}^{[k]}}{\mathfrak{s}_{n}^{[k-x]}\tilde{\mathfrak{s}}_{m}^{[w]}}\right)^r; \quad R = \frac{(\mathfrak{s}_{n+m}^{[r]})^k}{(\mathfrak{s}_{n}^{[r-u]})^{r(k-x)/(r-u)}(\tilde{\mathfrak{s}}_{m}^{[v]})^{wr/v}}.$$

By (1) and S(r;s)

$$\begin{aligned}(\mathfrak{s}_{n+m}^{[r]})^k &= \left(\sum_{t=u}^{v} \lambda(r,t)\mathfrak{s}_{n}^{[r-t]}\tilde{\mathfrak{s}}_{m}^{[t]}\right)^k \\ &\ge \left(\sum_{t=u}^{v} \lambda(r,t)(\mathfrak{s}_{n}^{[r-u]})^{(r-t)/(r-u)}(\tilde{\mathfrak{s}}_{m}^{[v]})^{t/v}\right)^k.\end{aligned} \tag{6}$$

This inequality is strict unless $a_1 = \cdots = a_{n+m}$. In certain cases this step is vacuous; in particular if $r = 1, k = n+m$, when all the means in (4) are either arithmetic or geometric means.

It follows from (3) that the last expression is the (kr)-th power of the rth elementary symmetric polynomial mean of $\underline{b}$ where

$$b_i = \begin{cases} \left(\mathfrak{s}_{n}^{[r-u]}\right)^{1/(r-u)} & \text{if } 1 \le i \le n, \\ \left(\tilde{\mathfrak{s}}_{m}^{[v]}\right)^{1/v} & \text{if } n+1 \le i \le n+m. \end{cases}$$

Hence by S(r;s) and (3) again

$$\begin{aligned}(\mathfrak{s}_{n+m}^{[r]})^k &\ge \left(\sum_{t=w}^{x} \lambda(k,t)(\mathfrak{s}_{n}^{[r-u]})^{(k-t)/(r-u)}(\tilde{\mathfrak{s}}_{m}^{[v]})^{t/v}\right)^r \\ &= (\mathfrak{s}_{n}^{[r-u]})^{r(k-x)/(r-u)}(\tilde{\mathfrak{s}}_{m}^{[v]})^{wr/v}\left(\sum_{t=w}^{x} \lambda(k,t)(\mathfrak{s}_{n}^{[r-u]})^{(x-t)/(r-u)}(\tilde{\mathfrak{s}}_{m}^{[v]})^{(t-w)/v}\right)^r.\end{aligned}$$

This inequality is strict unless $\mathfrak{S}_n^{[r-u]} = \tilde{\mathfrak{S}}_m^{[v]}$. If the previous application of S(r;s) had not given a strict inequality, then neither can the present application. However if, as noted above, the previous use had been vacuous, then strict inequality could occur here. From this last expression we have that $R \geq S$, where $S = \left(\sum_{t=w}^{x} \lambda(k,t)(\mathfrak{s}_n^{[r-u]})^{(x-t)/(r-u)}(\tilde{\mathfrak{s}}_m^{[v]})^{(t-w)/v}\right)^r$.
In a similar way using (1) and S(r;s),

$$(\mathfrak{s}_{n+m}^{[k]})^r = \Big(\sum_{t=w}^{x} \lambda(k,t)\mathfrak{s}_n^{[k-t]}\tilde{\mathfrak{s}}_m^{[t]}\Big)^r \leq \Big(\sum_{t=w}^{x} \lambda(k,t)(\mathfrak{s}_n^{[k-x]})^{(k-t)/(k-x)}(\tilde{\mathfrak{s}}_m^{[w]})^{t/w}\Big)^r, \tag{7}$$

the inequality being strict unless $a_1 = \cdots = a_{n+m}$. So

$$(\mathfrak{s}_{n+m}^{[k]})^r \leq (\mathfrak{s}_n^{[k-x]})^r(\tilde{\mathfrak{s}}_m^{[w]})^r\Big(\sum_{t=w}^{x} \lambda(k,t)(\mathfrak{s}_n^{[k-x]})^{(x-t)/(r-x)}(\tilde{\mathfrak{s}}_m^{[w]})^{(t-w)/w}\Big)^r,$$

which gives $L \leq T$ where, $T = \left(\sum_{t=w}^{x} \lambda(k,t)(\mathfrak{s}_n^{[k-x]})^{(x-t)/(k-x)}(\tilde{\mathfrak{s}}_m^{[w]})^{(t-w)/w}\right)^r$.
But by S(r;s) and the hypotheses in (a) $T \leq S$, and this inequality is strict unless $v = w$ and $r - u = k - x$, or $a_1 = \cdots = a_n + m$. This completes the proof of (a).
(b) The proof is similar except that when S(r;s) is applied in (6) and (7) it is applied to the second term only, that is to $\tilde{\mathfrak{s}}_m^{[t]}$. □

REMARK (iii) Although the cases of equality were not stated in the above theorem, the proof is detailed enough for them to be obtained in any particular case.

REMARK (iv) If $r = 1, k = n + m$ then (4) reduces to the equal weight case of II 3.2.2 Theorem 8(b); while if $k = s + 1, n = 1, m = q$, $r \leq s$ (5) reduces to

$$\left(\frac{\mathfrak{S}_{q+1}^{[r]}(\underline{a})}{\mathfrak{S}_{q+1}^{[s+1]}(\underline{a})}\right)^{s+1} \geq \left(\frac{\mathfrak{S}_q^{[r]}(\underline{a})}{\mathfrak{S}_q^{[s+1]}(\underline{a})}\right)^s, \tag{8}$$

a direct generalization of the equal weight case of (P).
Results similar to (8) can be obtained for the elementary symmetric polynomials; see [*Mitrinović & Vasić 1968e*].

THEOREM 3 *If $\underline{a}$ is an $(n+1)$-tuple, and if r, s are integers, $1 \leq r < s \leq n$, and if $0 < p \leq q$ then,*

$$\left(\frac{e_n^{[r]}(\underline{a})}{e_{n+1}^{[r]}(\underline{a})}\right)^p \geq \left(\frac{e_n^{[s]}(\underline{a})}{e_{n+1}^{[s]}(\underline{a})}\right)^q. \tag{9}$$

□ Suppose that $0 < p < q$, and putting $x = a_{n+1}$, and $f(x) = \dfrac{(e_{n+1}^{[r]})^p}{(e_{n+1}^{[s]})^q}$ then, by 1 (10), $f(x) = \dfrac{(e_n^{[r]} + xe_n^{[r-1]})^p}{(e_n^{[s]} + xe_n^{[s-1]})^q}$ and $f'(x) = \dfrac{(e_{n+1}^{[r]})^{p-1}}{(e_{n+1}^{[s]})^{q+1}} A$ where

$$
\begin{aligned}
A =&\big((p-q)xe_n^{[r-1]}e_n^{[s-1]} + pe_n^{[s]}e_n^{[r-1]} - qe_n^{[r]}e_n^{[s-1]}\big)\\
=&(q-p)e_n^{[r-1]}e_n^{[s-1]}\Big(\frac{pe_n^{[s]}e_n^{[r-1]} - qe_n^{[r]}e_n^{[s-1]}}{(q-p)e_n^{[r-1]}e_n^{[s-1]}} - x\Big)\\
\le&(q-p)e_n^{[r-1]}e_n^{[s-1]}\left(\frac{q}{q-p}\frac{e_n^{[s]}e_n^{[r]}}{e_n^{[s-1]}e_n^{[r-1]}}\Big(\frac{e_n^{[r-1]}}{e_n^{[r]}} - \frac{e_n^{[s-1]}}{e_n^{[s]}}\Big) - x\right)
\end{aligned}
$$

So, using 2 Corollary 2(b), $f' \le 0$ if $x \ge 0$. hence $f(a_{n+1}) \le f(0)$, which is just (9).

The case of $p = q$ has a similar but easier proof. □

THEOREM 4 *(a) If $\underline{a}$ is an $(n+1)$-tuple, and if s an integer, $1 \le s \le n$, and if $0 < p \le q$, then,*

$$
\left(\frac{e_{n+1}^{[n+1]}(\underline{a})}{e_n^{[n]}(\underline{a})}\right)^p \le \frac{p^p}{q^q}(q-p)^{q-p}(e_n^{[n]}(\underline{a}))^{p-q}\frac{e_{n+1}^{[s]}(\underline{a})}{e_n^{[s-1]}(\underline{a})}. \tag{10}
$$

(b) If $\underline{a}$ is an $(n+1)$-tuple, and if s, k, m integers, $\lambda \ne 0$, with:

$$
1 \le s < k \le n;\ 1 \le m < s \le n \text{ if } \lambda > 0; \text{ or } 1 \le k < s < m \le n \text{ if } \lambda < 0. \tag{11}
$$

then

$$
\frac{e_{n+1}^{[k]}(\underline{a}) - \lambda e_{n+1}^{[m]}(\underline{a})}{e_{n+1}^{[s]}(\underline{a})} < \frac{e_n^{[k]}(\underline{a}) - \lambda e_n^{[m]}(\underline{a})}{e_n^{[s]}(\underline{a})}. \tag{12}
$$

If instead of (11) we have that $1 \le m < s < k \le n$ if $\lambda < 0$; or $1 \le s < k \le n$, $1 \le s < m \le n$ if $\lambda > 0$, then inequality (~12) holds.

(c) If $\underline{a}$ is an $(n+1)$-tuple, and if s an integer, $1 \le s \le n+1$, and if $\mu > 0$ then

$$
\frac{(e_{n+1}^{[1]}(\underline{a}) + \mu)^s}{s^s e_{n+1}^{[s]}(\underline{a})} \ge \frac{\left(e_n^{[1]}(\underline{a}) + \mu - \dfrac{e_n^{[s]}(\underline{a})}{e_n^{[s-1]}(\underline{a})}\right)^{s-1}}{(s-1)^{s-1}e_n^{[s-1]}(\underline{a})}, \tag{13}
$$

with equality if and only if $a_{n+1} = \dfrac{1}{s-1}\left(\mu + e_n^{[1]} - s\dfrac{e_n^{[s]}}{e_n^{[s-1]}}\right)$.

REMARK (v) If $p = s = 1$, $q = n$ then (10) reduces to the equal weight case of (P); while if $s = n+1$ in (13) this inequality becomes the equal weight case of the result in II 3.2.1 Example (ii).

While Theorem 2 can be regarded as extending (P) to elementary symmetric polynomial means no such complete extension of (R) is known. The following result is a partial extension; [*Mitrinović & Vasić 1968b*].

THEOREM 5 *If $\underline{a}$ is an $(n+1)$-tuple, and if r an integer, $1 \le r \le n+1$, and if $\lambda > 0$ then*

$$(n+1)\Big(\mathfrak{A}_{n+1}(\underline{a}) - \lambda \mathfrak{S}_{n+1}^{[r]}(\underline{a})\Big) \ge \tag{14}$$
$$n\left(\mathfrak{A}_n(\underline{a}) - \lambda^{r/(r-1)}\frac{(n+1)(r-1)}{rn}\mathfrak{S}_n^{[r-1]}(\underline{a}) - \frac{n+1-r}{rn}\frac{(\mathfrak{S}_n^{[r]}(\underline{a}))^r}{(\mathfrak{S}_n^{[r-1]}(\underline{a}))^{r-1}}\right),$$

with equality if and only if $a_{n+1} = (n+1)\big(\mathfrak{A}_{n+1} - \lambda\mathfrak{S}_{n+1}^{[r]}\big)$.

□ Putting $x = a_{n+1}$ let $f(x)$ denote the left-hand side of (14). On simplification, using 1(10), $f(x) = n\mathfrak{A}_n + x - (n-1)\lambda\left(\frac{n+1-r}{n+1}\mathfrak{s}_n^{[r]} + \frac{r}{n+1}\mathfrak{s}_n^{[r-1]}x\right)^{1/r}$.
So $f(x)$ is defined if $x \ge -\Big(\frac{n+1-r}{r}\Big)\frac{\mathfrak{s}_n^{[r]}}{\mathfrak{s}_n^{[r-1]}}$, and is easily seen to have a unique minimum at $x_0 = \frac{n+1}{r}\lambda^{r/(r-1)}\mathfrak{S}_n^{[r-1]} - \frac{n+1-r}{r}\frac{(\mathfrak{S}_n^{[r]})^r}{(\mathfrak{S}_n^{[r-1]})^{r-1}}$. On substituting, $f(x_0)$ is the right-hand side of (14) and this completes the proof. □

REMARK (vi) With $r = n+1$ (14) reduces to the equal weight case of II 3.2.1(10) with $\mu = 1$.

REMARK (vii) It would be of interest to generalize (14) by replacing $\mathfrak{A}_{n+1}$ by $\mathfrak{S}_{n+1}^{[s]}$ in some way.

4 The Inequalities of Marcus & Lopes

In the case of power means and certain more general quasi-arithmetic means there are inequalities connecting the values of these means at the n-tuples $\underline{a}$, $\underline{b}$ and $\underline{a}+\underline{b}$; see III 3.1.3 (8) and IV 5. We now consider such an inequality for elementary symmetric polynomial means. The basic result is inequality (5) below due to Marcus & Lopes; it is the corollary of a more general inequality (1), due to McLeod, and independently Bullen & Marcus; see [*Bullen & Marcus; Godunova 1967; Marcus & Lopes; McLeod; Yuan & You*].

THEOREM 1 *If $\underline{a}$ and $\underline{b}$ are n-tuples, r and s integers, $1 \le r \le s \le n$, then*

$$\left(\frac{e_n^{[s]}(\underline{a}+\underline{b})}{e_n^{[s-r]}(\underline{a}+\underline{b})}\right)^{1/r} \ge \left(\frac{e_n^{[s]}(\underline{a})}{e_n^{[s-r]}(\underline{a})}\right)^{1/r} + \left(\frac{e_n^{[s]}(\underline{b})}{e_n^{[s-r]}(\underline{b})}\right)^{1/r}, \tag{1}$$

with equality if and only if either $\underline{a} \sim \underline{b}$, or $r = s = 1$.

□ First assume that r=1, when we can clearly also assume that $2 \le s \le n$, and that $\underline{a}$ and $\underline{b}$ are not proportional. We give two proofs of this case.

(*i*) Write $g_s(\underline{a}) = \dfrac{e_n^{[s]}(\underline{a})}{e_n^{[s-1]}(\underline{a})}$, $\phi_s(\underline{a},\underline{b}) = g_s(\underline{a}+\underline{b}) - g_s(\underline{a}) - g_s(\underline{b})$, and as usual $A_n = \sum_{i=1}^n a_i$, $B_n = \sum_{i=1}^n b_i$.
If $s = 2$ then $\phi_2(\underline{a},\underline{b}) = \dfrac{\sum_{i=1}^n (a_iB_n - b_iA_n)^2}{2A_nB_n(A_n+B_n)} > 0$, which completes the proof in this case.
Now assume that $s > 2$. The following identities are easily demonstrated.

$$(\alpha)\ se_n^{[s]}(\underline{a}) = \sum_{i=1}^n a_i e_{n-1}^{[s-1]}(\underline{a}_i'); \quad (\beta)\ e_n^{[s]}(\underline{a}) = a_i e_{n-1}^{[s-1]}(\underline{a}_i') + e_{n-1}^{[s]}(\underline{a}_i');$$

$$(\gamma)\ (n-s)e_n^{[s]}(\underline{a}) = \sum_{i=1}^n e_{n-1}^{[s]}(\underline{a}_i'); \quad (\delta)\ se_n^{[s]}(\underline{a}) = A_n e_n^{[s]}(\underline{a}) - \sum_{i=1}^n a_i^2 e_{n-1}^{[s-2]}(\underline{a}_i').$$

From these we can deduce the following:

$$g_s(\underline{a}) = A_n - \sum_{i=1}^n \frac{a_i^2}{a_i + g_{s-1}(\underline{a}_i')}, \tag{2}$$

$$\phi_s(\underline{a},\underline{b}) = \frac{1}{s}\sum_{i=1}^n \left(\frac{a_i^2}{a_i+g_{s-1}(\underline{a}_i')} + \frac{b_i^2}{b_i+g_{s-1}(\underline{b}_i')} - \frac{(a_i+b_i)^2}{a_i+b_i+g_{s-1}(\underline{a}_i'+\underline{b}_i')} \right). \tag{3}$$

We are now in a position to complete the proof of this case, by induction on s. By the induction hypothesis $g_{s-1}(\underline{a}_i'+\underline{b}_i') \ge g_{s-1}(\underline{a}_i') + g_{s-1}(\underline{b}_i')$, with equality if and only if $\underline{a}_i'$ is proportional to $\underline{b}_i'$. Then using (3), and the induction hypothesis,

$$\begin{aligned} &\phi_s(\underline{a},\underline{b}) \\ &\ge \frac{1}{s}\sum_{i=1}^n \left(\frac{(a_ig_{s-1}(\underline{b}_i') - b_ig_{s-1}(\underline{a}_i'))^2}{(a_i+g_{s-1}(\underline{a}_i'))(b_i+g_{s-1}(\underline{b}_i'))(a_i+b_i+g_{s-1}(\underline{a}_i')+g_{s-1}(\underline{b}_i'))} \right) \ge 0. \end{aligned} \tag{4}$$

Further if for at least one i, $\underline{a}_i'$ is not proportional to $\underline{b}_i'$ the first inequality in this expression is strict and so $\phi_s(\underline{a},\underline{b}) > 0$.
If on the other hand for all i there is a $\lambda_i > 0$ such that $\underline{a}_i' = \lambda_i \underline{b}_i'$ then

$$(a_ig_{s-1}(\underline{b}_i') - b_ig_{s-1}(\underline{a}_i'))^2 = (a_i - \lambda_i b_i)^2 g_{s-1}^2(\underline{b}_i'),$$

which is positive since $\underline{a}$ is not proportional to $\underline{b}$. So again $\phi_s(\underline{a},\underline{b}) > 0$. and the proof of this case, $r = 1, 1 \le s \le n$, is completed.
(*ii*) The inequality is immediate in this case from I 4.6 (~19) if we can show that $g_s(\underline{a})$ is concave. This we show by induction on s and note that it is immediate if $s = 1$. Now suppose $g_{s-1}(\underline{a})$ is concave and consider (2). The function $x^2/(x+y)$, $x, y \in \mathbb{R}_+^*$, is convex by I 4.6 Theorem 40 (g), increasing in x and decreasing in y. So, by (2), I 4.6 Theorem 40 (b) and (d), $g_s(\underline{a})$ is concave; [*Soloviov*].

Now suppose that $r > 1$. Then by (1) in the case $r = 1$,

$$\left(\frac{e_n^{[s]}(\underline{a}+\underline{b})}{e_n^{[s-r]}(\underline{a}+\underline{b})}\right)^{1/r} = \left(\prod_{j=1}^{r}\frac{e_n^{[s-j+1]}(\underline{a}+\underline{b})}{e_n^{[s-j]}(\underline{a}+\underline{b})}\right)^{1/r}$$
$$\geq \left(\prod_{j=1}^{r}\left(\frac{e_n^{[s-j+1]}(\underline{a})}{e_n^{[s-j]}(\underline{a})}+\frac{e_n^{[s-j+1]}(\underline{b})}{e_n^{[s-j]}(\underline{b})}\right)\right)^{1/r}.$$

Then using (H) in the form III 2.1(4),

$$\left(\frac{e_n^{[s]}(\underline{a}+\underline{b})}{e_n^{[s-r]}(\underline{a}+\underline{b})}\right)^{1/r} \geq \left(\prod_{j=1}^{r} e\frac{e_n^{[s-j+1]}(\underline{a})}{e_n^{[s-j]}(\underline{a})}\right)^{1/r} + \left(\prod_{j=1}^{r}\frac{e_n^{[s-j+1]}(\underline{b})}{e_n^{[s-j]}(\underline{b})})\right)^{1/r}$$
$$= \left(\frac{e_n^{[s]}(\underline{a})}{e_n^{[s-r]}(\underline{a})}\right)^{1/r} + \left(\frac{e_n^{[s]}(\underline{b})}{e_n^{[s-r]}(\underline{b})}\right)^{1/r}.$$

The cases of equality are immediate from those for $r = 1$ and (H). □

REMARK (i) Inequality (1) can be interpreted as saying that the function $g_{s,r}(\underline{a})$: $\underline{a} \mapsto \dfrac{e_n^{[s]}(\underline{a})}{e_n^{[s-r]}(\underline{a})}$ is concave; the case $r = 1$ was proved directly in proof (ii) above.

REMARK (ii) In particular (1) shows that $g_{s,r}(\underline{a})$ is an increasing function of a_i for each i, $1 \leq i \leq n$. This however is easily proved directly. Using the identity (β) above, and 2 Corollary 2(a),

$$\frac{\partial g_{s,r}(\underline{a})}{\partial a_i} = \frac{e_n^{[s-r]}(\underline{a})e_{n-1}^{[s-1]}(\underline{a}'_i) - e_n^{[s]}(\underline{a})e_{n-1}^{[s-r-1]}(\underline{a}'_i)}{(e_n^{[s-r]}(\underline{a}))^2}$$
$$= \frac{e_{n-1}^{[s-r]}(\underline{a}'_i)e_{n-1}^{[s-1]}(\underline{a}'_i) - e_{n-1}^{[s]}(\underline{a}'_i)e_{n-1}^{[s-r-1]}(\underline{a}'_i)}{(e_n^{[s-r]}(\underline{a}))^2} \geq 0.$$

COROLLARY 2 *If $\underline{a}$ and $\underline{b}$ are n-tuples, r and integer, $1 \leq r \leq n$, then*

$$\mathfrak{S}_n^{[r]}(\underline{a}+\underline{b}) \geq \mathfrak{S}_n^{[r]}(\underline{a}) + \mathfrak{S}_n^{[r]}(\underline{b}), \tag{5}$$

with equality if and only if $r = 1$, or $\underline{a} \sim \underline{b}$.

□ (i) This is just the case $s = r$ of Theorem 1.

(ii) Alternatively we can use the case $r = 1$ and the fact that the function $g_s(\underline{a})$ is concave, see proof (ii) in Theorem 1. Clearly $\left(\mathfrak{s}_n^{[r]}(\underline{a})\right)^{1/r} = \left(\prod_{i=1}^{r} g_i(\underline{a})\right)^{1/r}$. So

by I 4.6 Theorem 40(e) $\left(\mathfrak{s}_n^{[r]}(\underline{a})\right)^{1/r}$ is concave and the result is just I 4.6 ($\sim$19); [*Soloviov*]. □

REMARK (iii) Proof (*i*) in Theorem 1, and Corollary 2, is that in [*Bullen & Marcus; Marcus & Lopes*]; the proof in [*McLeod*] is entirely different. Proof (*ii*) is due to Soloviov.

5 Complete Symmetric Polynomial Means; Whiteley Means

5.1 THE COMPLETE SYMMETRIC POLYNOMIAL MEANS The definition of $e_n^{[r]}(\underline{a})$ can be rewritten as:

$$e_n^{[r]}(\underline{a}) = \sum\left(\prod a_i^{i_j}\right), \tag{1}$$

where the sum is taken over all $\binom{n}{r}$ n-tuples $(i_1, \ldots, i_n)$ with $i_j = 0$ or 1, $1 \le j \le n$, and $\sum_{j=1}^{n} i_j = r$.
A natural generalization is the *rth complete symmetric polynomial of* $\underline{a}$[4], $c_n^{[r]}(\underline{a})$, where the sum in (1) is taken over all $\binom{n+r-1}{r}$ non-negative integer n-tuples $(i_1, \ldots, i_n)$ with $\sum_{j=1}^{n} i_j = r$. In addition we define $c_n^{[0]}(\underline{a}) = 1$; [*AI pp.105–106; DI p.55; MPF p.165*].

EXAMPLE (i) It is easily checked that:

$$c_n^{[1]}(\underline{a}) = e_n^{[1]}(\underline{a}); \qquad c_n^{[2]}(\underline{a}) = a_1^2 + \cdots + a_n^2 + a_1a_2 + \cdots + a_{n-1}a_n;$$
$$c_3^{[3]}(\underline{a}) = a_1^3 + a_2^3 + a_3^3 + a_1^2a_2 + a_1^2a_3 + a_2^2a_3 + a_2^2a_1 + a_3^2a_1 + a_3^2a_2 + a_1a_2a_3$$

Associated with the rth complete symmetric polynomial is the *rth complete symmetric polynomial mean*[5] :

$$\mathfrak{Q}_n^{[r]}(\underline{a}) = (\mathfrak{q}_n^{[r]}(\underline{a}))^{1/r} = \left(\frac{c_n^{[r]}(\underline{a})}{\binom{n+r-1}{r}}\right)^{1/r}. \tag{3}$$

The complete symmetric polynomials can also be defined in a manner analogous to 1(8):

$$\prod_{i=1}^{n}(1 - a_i x)^{-1} = \sum_{k=0}^{\infty} c_n^{[k]}(\underline{a})x^k = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\mathfrak{q}_n^{[k]}(\underline{a})x^k. \tag{2}$$

LEMMA 1 *If $a_1, \ldots, a_n$ are n distinct real numbers then*

$$c_n^{[r]}(\underline{a}) = [\underline{a}; x^{n+r-1}]_{n-1}. \tag{4}$$

---

4 Also called *complete symmetric function*; see Footnote 2.

5 Also called the *complete symmetric mean*; see Footnote 3.

□ If the rational function on the left-hand side of (2) is written in terms of its partial fractions
$$\prod_{i=1}^{n}(1-a_ix)^{-1}=\sum_{i=1}^{n}\frac{A_i}{1-a_ix},$$
then evaluation of the coefficients A_i leads to (4); see I 4.7 Example (ii) and [*Milne-Thomson pp.7–8*]. □

COROLLARY 2 *If k is even and if for $y\in\mathbb{R}$, $\sum_{i=0}^{2m}\binom{k+n+i-1}{n-1}b_iy^i\ge 0$ where $b_{2m}>0$, then $\sum_{i=0}^{2m}c_n^{[k+i]}(\underline{a})b_iy^i>0$ provided $\underline{a}$ is not constant. If k is odd the same result holds provided $\underline{a}$ is non-negative and not constant*

□ From Lemma 1 if
$$F(y)=\sum_{i=0}^{2m}c_n^{[k+i]}(\underline{a})b_iy^i \quad\text{and}\quad f(x)=x^{n+k-1}\sum_{i=0}^{2m}b_i(xy)^i$$
then $F(y)=[\underline{a};f]_{n-1}$, and so to say that F is positive is equivalent to saying that f is strictly $(n-1)$-convex, see I 4.7.
Now $f^{(n-1)}(x)=(n-1)!x^k\sum_{i=0}^{2m}\binom{k+n+i-1}{n-1}b_i(xy)^i$ which is non-negative under the above hypotheses. Since f is a polynomial of degree at least n we have from I 4.7 Lemma 45(d) that f is strictly $(n-1)$-convex, which completes the proof. □

REMARK (i) This is a result of Popoviciu and is the basis of the first proof of the next result that is analogous to S(r;s); [*Popoviciu 1934a, 1972*].

THEOREM 3 *If $\underline{a}$ is an n-tuple, r and s integers, with $1\le r<s$ then*
$$\mathfrak{Q}_n^{[r]}(\underline{a})\le\mathfrak{Q}_n^{[s]}(\underline{a}), \tag{5}$$
with equality if and only if $\underline{a}$ is constant.

□ Using the method of proof (i) of 2 Theorem 3(b) it is sufficient to prove that
$$(\mathfrak{q}_n^{[r]})^2\le\mathfrak{q}_n^{[r-1]}\mathfrak{q}_n^{[r+1]}, \tag{6}$$
with equality if and only if $\underline{a}$ is constant; this is an analogue of 2(1).
We give two proofs of (6).
(i) Put $m=1$ and $b_i=\binom{2}{i}/\binom{k+n+i-1}{n-1}$ in Corollary 2 then $\sum_{i=0}^{2}\binom{k+n+i-1}{n-1}b_iy^i=(1+y)^2\ge 0$. So by that result
$$\sum_{i=0}^{2}c_n^{[k+i]}(\underline{a})b_iy^i=\mathfrak{q}_n^{[k]}+2y\mathfrak{q}_n^{[k+1]}+y^2\mathfrak{q}_n^{[k+2]}\ge 0,$$
with equality if and only if $\underline{a}$ is constant. Putting $k=r-1$ implies (6).

(*ii*) Let $A = \{(x_1, \ldots, x_{n-1}); x_i > 0, 1 \le i \le n-1, x_1 + \cdots + x_{n-1} < 1\}$ and $x_n = 1 - x_1 - \cdots - x_{n-1}$. Then

$$\mathfrak{q}_n^{[r]}(\underline{a}) = (n-1)! \int_A (a_1x_1 + \cdots + a_nx_n)^r \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1},$$

and (6) follows by an application of (C)-$\int$, see VI 1.2.1 Theorem 3(b). □

REMARK (ii) This second proof is due to Schur; see [*HLP p.164*].

REMARK (iii) A completely different proof has been given by Neuman, who has given the following interesting generalization of (6):

$$\mathfrak{q}_n^{[u+v]} \le \left(\mathfrak{q}_n^{[up]}\right)^{1/p} \left(\mathfrak{q}_n^{[vp']}\right)^{1/p'}, \tag{7}$$

where $p > 1$, p' is the conjugate index, and $u+v, up, vp' \in \mathbb{N}^*$; putting $p = 2$, $u = (r-1)/2$, $v = (r+1)/2$ in (7) gives (6); [*Neuman 1985,1986; Neuman & Pečarić 1986*].

The above integral for $q_n^{[r]}(\underline{a})$ can be used to obtain the following extension of (6); see [*HLP p.164*].

THEOREM 4 *If $\underline{a}$ is a real non-constant n-tuple then $\sum_{r,s=1}^n \mathfrak{q}_n^{[r+s]}(\underline{a})x_rx_s$ is a strictly positive quadratic form; and if $\underline{a}$ is a positive non-constant n-tuple then $\sum_{r,s=1}^n \mathfrak{q}_n^{[r+s+1]}(\underline{a})x_rx_s$ is strictly positive.*

It is not known if the following inequality analogous to 4(1) is valid for the complete symmetric polynomials:

$$\left(\frac{c_n^{[s]}(\underline{a}+\underline{b})}{c_n^{[s-r]}(\underline{a}+\underline{b})}\right)^{1/r} \le \left(\frac{c_n^{[s]}(\underline{a})}{c_n^{[s-r]}(\underline{a})}\right)^{1/r} + \left(\frac{c_n^{[s]}(\underline{b})}{c_n^{[s-r]}(\underline{b})}\right)^{1/r},$$

although the particular cases $s = r, s = r+1$ have been proved by McLeod and Baston respectively; in particular this gives an analogue for 4(5), see below 5.2 Remark (iv): [*Baston 1976/7; McLeod*].

An inequality between the elementary symmetric polynomial means and the complete symmetric polynomial means is given below 6 Corollary 6.

For further results concerning the complete symmetric polynomials see the above references and [*Baston 1976,1978; DeTemple & Robertson; Hunter D*].

5.2 THE WHITELEY MEANS Identities (2) and 1(8) suggest the following generalizations of the elementary and complete symmetric polynomials.

Let $\underline{a}$ be a real n-tuple, $s \in \mathbb{R}, s \neq 0,\ k \in \mathbb{N}^*$, and define the *sth function of degree* k, $t_n^{[k,s]}(\underline{a})$, by

$$1+\sum_{k=1}^{\infty} t_n^{[k,s]}(\underline{a})x^k = \begin{cases} \prod_{i=1}^n (1+a_i x)^s & \text{if } s>0,\\ \prod_{i=1}^n (1-a_i x)^s & \text{if } s<0. \end{cases} \tag{8}$$

The *Whiteley means* are defined as[6]

$$\mathfrak{W}_n^{[k,s]}(\underline{a}) = \begin{cases} \left(\dfrac{t_n^{[k,s]}(\underline{a})}{\binom{ns}{k}}\right)^{1/k} & \text{if } s>0,\\ \left(\dfrac{t_n^{[k,s]}(\underline{a})}{(-1)^k\binom{ns}{k}}\right)^{1/k} & \text{if } s<0. \end{cases} \tag{9}$$

An alternative definition of $t_n^{[k,s]}(\underline{a})$ is: $t_n^{[k,s]}(\underline{a}) = \sum\left(\prod_{j=1}^n \lambda_{i_j} a_j^{i_j}\right)$, where

$$\lambda_i = \begin{cases} \binom{s}{i} & \text{if } s>0,\\ (-1)^i\binom{s}{i} & \text{if } s<0, \end{cases} \tag{10}$$

and the summation is over all non-negative integral n-tuples $(i_1,\ldots i_n)$ such that $\sum_{j=1}^n i_j = k$.

Remark (i) By analogy with $\mathfrak{s}_n^{[r]}$ and $\mathfrak{q}_n^{[r]}$ we will write $\mathfrak{w}_n^{[k,s]}$ for $(\mathfrak{W}_n^{[k,s]})^k$.

Example (i) If $s=1$, and $\lambda_0=1, \lambda_i=0$ otherwise, then from (10) we have $t_n^{[k,1]} = e_n^{[k]}$; this is also immediate from (8).

Example (ii) If $s=-1$ then $\lambda_i=1$ for all i, and so $t_n^{[k,-1]} = c_n^{[k]}$; this is also immediate from (8).

Example (iii) Simple calculations show that $\mathfrak{W}_n^{[1,s]}(\underline{a}) = \mathfrak{A}_n(\underline{a})$, and

$$\mathfrak{w}_n^{[2,s]} = \frac{1}{n(ns-1)}\Big((s-1)\sum_{i=1}^n a_i^2 + 2s\sum_{\substack{i,j=1\\ i<j}}^n a_i a_j\Big)$$

Remark (ii) If $s<0$ then the coefficients in $t_n^{[k,s]}$ are all positive; this is also the case if $s>0$ and either $0\le k<s+1$, s not an integer, or s is an integer and $0\le k\le ns$. This explains certain restrictions in the theorems given below.

[6] Note that if $s<0$ then $(-1)^k\binom{ns}{k}=\binom{-ns+k-1}{k}$.

We note that an expression for $t_n^{[k,s]}$ in terms of an integral can be obtained; [*BB p.36*]. If $s<0$ and $|t|$ is small enough then

$$(1-at)^s = \frac{1}{(-s-1)!}\int_0^\infty e^{-x(1-at)}x^{-s-1}\,\mathrm{d}x.$$

So

$$\prod_{i=1}^n (1-a_ix)^s = \frac{1}{\big((-s-1)!\big)^n}\int_0^\infty\!\!\cdots\!\int_0^\infty \exp\big(-\sum_{i=1}^n a_i(1-a_it)\big)\prod_{i=1}^n x_i^{-s-1}\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

Hence

$$t_n^{[k,s]}(\underline{a}) = \frac{1}{k!\big((-s-1)!\big)^n}\int_0^\infty\!\!\cdots\!\int_0^\infty \big(\sum_{i=1}^n a_ix_i\big)\exp\Big(-\sum_{i-1}^n x_1\Big)\prod_{i=1}^n x_i^{-s-1}\mathrm{d}x_1\cdots\mathrm{d}x_n. \quad (11)$$

In this section we extend various properties of the elementary and completely symmetric polynomial means to the Whiteley means and in 5.3, by considering even more general means, other properties will be extended.

First we note that the Whiteley means have the property of strict internality.

LEMMA 5 *If $\underline{a}$ is an n-tuple, k an integer, $1\le k\le n, s\in\mathbb{R}, s\ne 0$, then*

$$\min\underline{a} \le \mathfrak{W}_n^{[k,s]}(\underline{a}) \le \max\underline{a},$$

with equality if and only if $\underline{a}$ is constant.

□ This is immediate from (8) and (9). □

Before generalizing other properties of the elementary and completely symmetric means we establish some lemmas of Whiteley; see [*Whiteley 1962b*].

LEMMA 6 *If $1\le i\le n$, then*

$$\frac{\partial t_n^{[k,s]}(\underline{a})}{\partial a_i} + a_i\frac{\partial t_n^{[k-1,s]}(\underline{a})}{\partial a_i} = st_n^{[k-1,s]}(\underline{a}), \qquad s>0,$$
$$\frac{\partial t_n^{[k,s]}(\underline{a})}{\partial a_i} - a_i\frac{\partial t_n^{[k-1,s]}(\underline{a})}{\partial a_i} = (-s)t_n^{[k-1,s]}(\underline{a}), \quad s<0.$$

□ If $s<0$ we have from (8):

$$(1-a_ix)\sum_{k=0}^\infty \frac{\partial t_n^{[k,s]}(\underline{a})}{\partial a_i}x^k = -sx\prod_{k=1}^n(1-a_kx)^s = -sx\sum_{k=0}^\infty t_n^{[k,s]}(\underline{a})x^k.$$

This gives the result in this case.

The proof for $s>0$ is similar. □

COROLLARY 7
$$\sum_{i=1}^{n} \frac{\partial w_n^{[k,s]}(\underline{a})}{\partial a_i} = k\mathfrak{w}_n^{[k-1,s]}(\underline{a}).$$

□ Sum the results of Lemma 6 over i and the lemma follows easily from Euler's theorem on homogeneous functions, see I 4.6 Remark(ii).

□

LEMMA 8 *If $s < 0$ then*
$$\mathfrak{W}_n^{[1,s]}(\underline{a}) \le \mathfrak{W}_n^{[2,s]}(\underline{a}).$$
If $s > 1$ this inequality is reversed and in both cases there is equality if and only if $\underline{a}$ is constant.

□ From Example (iii)
$$\begin{aligned} n^2\Big((\mathfrak{W}_n^{[1,s]}(\underline{a}))^2 - (\mathfrak{W}_n^{[2,s]}(\underline{a}))^2 \Big) &= \frac{1}{ns-1}\big((n-1)\sum_{i=1}^{n} a_i^2 - 2\sum_{i<j} a_i a_j\big) \\ &= \frac{1}{ns-1}\sum_{i<j}(a_i - a_j)^2, \end{aligned}$$
from which the result is immediate. □

The next result, the main result of this section, generalizes both S(r;s) and 5.1 Theorem 3.

THEOREM 9 *If $\underline{a}$ is a non-negative n-tuple, and $s > 0$, and if k, m are integers with $1 \le k \le m < s+1$ when s is not an integer, and $1 \le k < m \le ns$ if s is an integer and then*
$$\mathfrak{W}_n^{[m,s]}(\underline{a}) \le \mathfrak{W}_n^{[k,s]}(\underline{a}). \tag{12}$$
If $s < 0$, (~12) holds. In both cases there is equality if $\underline{a}$ is constant.

It follows from the proof of 5.1 Theorem 3, and of proof (i) of 2 Theorem 3(b) that it is sufficient to prove the following generalization of 2 (1) and 5.1(6); [*Whiteley 1962a*].

THEOREM 10 *If $\underline{a}$ is a non-negative n-tuple, and $s > 0$, and if k an integer with $1 \le k < s$ when s is not an integer, and $1 \le k < ns$ if s is an integer and then*
$$\big(\mathfrak{w}_n^{[k,s]}(\underline{a})\big)^2 \ge \mathfrak{w}_n^{[k-1,s]}(\underline{a})\mathfrak{w}_n^{[k+1,s]}(\underline{a}). \tag{13}$$
If $s < 0$, (~13) holds. In both cases there is equality if $\underline{a}$ is constant.

□ If $n = 1$ the result is immediate for all k; if $k = 1$ the result reduces to Lemma 8. The proof is completed by a double induction on k and n.

Assume the result is known for all k and all n', $n' < n$, and all n and all k', $k' < k$ and put $A = \{\underline{a};\ \underline{a}$ is a non-negative n-tuple and $\mathfrak{w}_n^{[k-1,s]}(\underline{a})\mathfrak{w}_n^{[k+1,s]}(\underline{a}) = 1\}$. Under the given hypotheses A is compact and so, on A, $\left(\mathfrak{w}_n^{[k,s]}(\underline{a})\right)^2$ must attain its minimum if $s > 0$, its maximum, if $s < 0$. In either case call this value M.
By Euler's theorem on homogeneous functions, see I 4.6 Remark(ii),

$$\sum_{i=1}^{n} a_i \frac{\partial\left(\mathfrak{w}_n^{[k-1,s]}(\underline{a})\mathfrak{w}_n^{[k+1,s]}(\underline{a})\right)}{\partial a_i} = 2k\mathfrak{w}_n^{[k-1,s]}(\underline{a})\mathfrak{w}_n^{[k+1,s]}(\underline{a}) = 2k,$$

so that in A not all the partial derivatives of $\mathfrak{w}_n^{[k-1,s]}(\underline{a})\mathfrak{w}_n^{[k+1,s]}(\underline{a})$ vanish together.
Let us first assume, that M is attained at a positive n-tuple, $\underline{a}^0$ in A say.
Apply Lagrange multiplier conditions, see II 2.4.3 Footnote 10, at this point:

$$\frac{\partial\left(\mathfrak{w}_n^{[k,s]}(\underline{a}^0)\right)^2}{\partial a_i^0} - \lambda \frac{\partial\left(\mathfrak{w}_n^{[k-1,s]}(\underline{a}^0)\mathfrak{w}_n^{[k+1,s]}(\underline{a}^0)\right)}{\partial a_i^0} = 0,\ 1 \le i \le n,$$

or

$$2\mathfrak{w}_n^{[k-1,s]}\frac{\partial \mathfrak{w}_n^{[k,s]}}{\partial a_i^0} = \lambda\left(\mathfrak{w}_n^{[k+1,s]}\frac{\partial \mathfrak{w}_n^{[k-1,s]}}{\partial a_i^0} + \mathfrak{w}_n^{[k-1,s]}\frac{\partial \mathfrak{w}_n^{[k+1,s]}}{\partial a_i^0}\right),\ 1 \le i \le n. \tag{14}$$

Taking each of the identities in (14), multiplying by a_i, adding and using Euler's theorem on homogeneous functions gives $\lambda = M$.
We now obtain an upper bound for λ. Add the identities in (14) and use Corollary 7 to obtain

$$2k\mathfrak{w}_n^{[k,s]}\mathfrak{w}_n^{[k-1,s]} = \lambda\left((k-1)\mathfrak{w}_n^{[k+1,s]}w_n^{[k-2,s]} + (k-1)\mathfrak{w}_n^{[k-1,s]}\mathfrak{w}_n^{[k,s]}\right),$$

which on simplifying gives

$$2k - \lambda(k+1) = \lambda(k-1)\frac{\mathfrak{w}_n^{[k+1,s]}\mathfrak{w}_n^{[k-2,s]}}{\mathfrak{w}_n^{[k-1,s]}\mathfrak{w}_n^{[k,s]}}. \tag{15}$$

Now as $\lambda = M = (\mathfrak{w}_n^{[k,s]})^2/\mathfrak{w}_n^{[k-1,s]}\mathfrak{w}_n^{[k+1,s]}$, putting $\mu = (\mathfrak{w}_n^{[k-1,s]})^2/(\mathfrak{w}_n^{[k-2,s]}\mathfrak{w}_n^{[k,s]})$ (15) becomes

$$2k - \lambda(k+1) = \frac{k-1}{\mu}. \tag{16}$$

The inductive hypothesis is $\mu \le 1$, if $s > 0$, and $\mu \ge 1$ if $s < 0$; substituting in (16) leads to $\lambda \le 1$, if $s > 0$, and $\lambda \ge 1$ if $s < 0$ which completes the induction. In addition if $\lambda = 1$ then $\mu = 1$ and the induction applies to the case of equality as well.

Now consider the case when M is attained at point with only p non-zero coordinates, $p < n$. This reduces to the previous case with n replaced by p and the proof is complete. □

REMARK (iii) From Examples (i) and (ii) it is immediate that (13) is a generalization of 5.1 (6) and of 2(1). However here we need $\underline{a}$ to be non-negative, see the comment in proof *(iv)* of 2 Theorem 1.

EXAMPLE (iv) Consider for instance $s = -1, n = 2, k = 2, a_1 = -a_2 = 1$. Then

$$\mathfrak{w}_2^{[2,-1]}(a, -a) = a^2 \quad \text{but} \quad \mathfrak{w}_2^{[1,-1]}(a, -a) = \mathfrak{w}_2^{[3,-1]}(a, -a) = 0.$$

COROLLARY 11 *If $s > 0$ and if k an integer with $1 \le k < s$ when s is not an integer, and $1 \le k < ns$ if s is an integer and if $\underline{a}$ is a non-negative n-tuple then*

$$\big(t_n^{[k,s]}(\underline{a})\big)^2 > \big(t_n^{[k-1,s]}(\underline{a})\big)\big(t_n^{[k+1,s]}(\underline{a})\big);$$

the reverse inequality holds if $s < 0$.

□ This an immediate consequence of Theorem 10, and the observation that

$$\frac{\binom{ns}{k-1}^2}{\binom{ns}{k+1}\binom{ns}{k}} \begin{cases} > 1 & \text{if } s > 0, \\ < 1 & \text{if } s < 0. \end{cases}$$

□

COROLLARY 12 *If $s > 0$ and if k, m integers with $1 \le k < m < s$ when s is not an integer, and $1 \le k < m < ns$ if s is an integer and if $\underline{a}$ is a non-negative n-tuple then*

$$\big(t_n^{[k,s]}(\underline{a})\big)^{1/k} > \big(t_n^{[m,s]}(\underline{a})\big)^{1/m};$$

the reverse inequality holds if $s < 0$.

□ This follows from Corollary 11 just as 2(6) follows from 2(1). □

Whiteley has also extended 4 Corollary 2; [*Whiteley 1962a*].

THEOREM 13 *If $s > 0$ and if k is an integer, $k < s + 1$ if s is not an integer, then for non-negative n-tuples $\underline{a}, \underline{b}$,*

$$\mathfrak{W}_n^{[k,s]}(\underline{a} + \underline{b}) \ge \mathfrak{W}_n^{[k,s]}(\underline{a}) + \mathfrak{W}_n^{[k,s]}(\underline{b}). \tag{17}$$

If $s < 0$ (~18) holds, and the inequalities are strict unless $k = 1$ or $\underline{a} \sim \underline{b}$.

□ (*i*) Inequality (17) is equivalent to an analogous one in terms of $t_n^{[k,s]}(\underline{a})$ which follows from (11), using a form of (M)-$\int$, see VI 1.2.1 Theorem 3(c).
(*ii*) A direct proof following a method similar to that used to prove Theorem 10 is given in [*Whiteley 1962a*]. However since the three cases, s negative, s positive and not an integer, and s a positive integer, need separate treatments the proof is rather long. A neater proof of a more general result is given in the next section.□

REMARK (iv) If $s = 1$ then (17) reduces to 4(5), while if $s = -1$ we get Mcleod's analogous result for the complete symmetric polynomial means: [*McLeod*].

$$\mathfrak{Q}_n^{[r]}(\underline{a}+\underline{b}) \le \mathfrak{Q}_n^{[r]}(\underline{a}) + \mathfrak{Q}_n^{[r]}(\underline{b}).$$

REMARK (v) If $s = -\delta$, where δ is a small positive number then $t_n^{[k,-\delta]}(\underline{a}) = \sum_{i=1}^n a_i^k + O(\delta^2)$. Applying (17) in this case and letting $\delta \to 0$ we see that (17) implies (M).

REMARK (vi) Inequality (17) is equivalent to saying that the surface $\mathfrak{W}_n^{[k,s]}(\underline{a}) = 1$ in the "positive quadrant " of $\mathbb{R}^n$ is convex if $s > 0$, and concave if $s < 0$.

REMARK (vii) Other results can be obtained by noting that (13) says that $\mathfrak{w}_n^{[k,s]}$ is a log-convex function of k .

5.3 SOME FORMS OF WHITELEY Let $\alpha_{i,j}$, $1 \le i \le n, i = 1, 2, \ldots$, be n sequences of positive numbers and define inductively the n sequences of positive numbers $\beta_{i,j}$, $1 \le i \le n, i = 1, 2, \ldots$, by $\alpha_{i,r} = \dfrac{1}{r!}\displaystyle\prod_{j=1}^{r} \beta_{i,j}$.

If $\underline{a}$ is a non-negative n-tuple, $\theta > 0$ define the *functions* $g_n^{[k]}(\underline{a})$ *of degree* k by

$$\sum_{k=0}^{\infty} g_n^{[k]}(\underline{a})x^k = \theta \prod_{i=1}^{n}\left(1 + \sum_{r=1}^{\infty} \alpha_{i,r}(a_i x)^r\right) = \theta \prod_{i=1}^{n}\left(1 + \sum_{r=1}^{\infty} \left(\frac{\prod_{j=1}^{r}\beta_{i,j}}{r!}\right)(a_i x)^r\right); \tag{18}$$

[*Bullen 1975; Whiteley 1962b*].

EXAMPLE (i) The $t_n^{[k,s]}$ of the previous section are particular cases of $g_n^{[k]}$; in (18) take $\beta_{i,j} = s - j + 1$ if $s > 0$ and $\beta_{i,j} = -s + j - 1$ if $s < 0$.

EXAMPLE (ii) We can generalize the previous example by letting $\underline{\sigma}$ be a positive or negative n-tuple, and taking $\beta_{i,j} = \sigma_i - j + 1$ if $\underline{\sigma} > 0$ and $\beta_{i,j} = -\sigma_i + j - 1$ if $\underline{\sigma} < 0$. In this case $g_n^{[k]}$ would be written $t_n^{[k;\underline{\sigma}]}$. Clearly if $\underline{\sigma}$ is constant, s say,

then $t_n^{[k,\underline{\sigma}]} = t_n^{[k,s]}$. These particular functions of degree k can of course be defined directly by writing:

$$1 + \sum_{k=1}^{\infty} t_n^{[k;\underline{\sigma}]}(\underline{a})x^k = \begin{cases} \prod_{i=1}^n (1 + a_i x)^{\sigma_i}, & \text{if } \underline{\sigma} > 0, \\ \prod_{i=1}^n (1 - a_i x)^{\sigma_i}, & \text{if } \underline{\sigma} < 0. \end{cases}$$

The functions $t_n^{[k;\underline{\sigma}]}$ and their associated means were introduced by Gini, and studied for statistical purposes by various authors, who in addition used these functions to define various biplanar means; see 7.3. However because the main interest was in finding suitable statistical means these authors seemed to have introduced more means than they proved theorems about them; [*Gini 1926; Gini & Zappa; Pietra; Pizzetti 1939; Zappa 1939,1940*]. Later these functions were studied by Menon, who obtained most of their known properties; in particular he extended 5.2 Theorems 9 and 10 to this more general situation; [*Menon 1969c,1970*].
Certain restrictions have to be placed on k to ensure that the coefficients in $t_n^{[k;\underline{\sigma}]}$ are positive; this is similar to the case of $t_n^{[k,s]}$, see 5.2 Remark (ii). If $\underline{\sigma} < 0$ then no restrictions are needed, all k are permitted. When $\underline{\sigma} > 0$, and $\min \underline{\sigma}$ is not an integer then we require that $1 \leq k \leq 1 + \min \underline{\sigma}$. If $\min \underline{\sigma}$ is an integer then the restrictions are more complicated to state, and for this reason this case has usually been excluded from consideration; [*Menon 1970,1971*]. However the proofs only require the coefficients mentioned above to be positive and so extend to this case when it is known that the coefficients are in fact positive. Thus if $\underline{\sigma}$ is constant, s say, then it is sufficient to have $1 \leq k \leq n$, as we have seen.

EXAMPLE (iii) Menon has made further extensions of the previous examples; [*Menon 1971*]. If $0 \leq q < 1$ then the *q-binomial coefficients* are defined by[7]:

$$\begin{bmatrix} s \\ k \end{bmatrix} = \begin{cases} \displaystyle\prod_{i=1}^{k} \frac{1 - q^{s-i+1}}{1 - q^i}, & \text{if } k > 0, \\ 1, & \text{if } k = 0, \\ 0, & \text{if } k < 0. \end{cases}$$

If then $\underline{\sigma}$ is a positive n-tuple define $e_n^{[k;\underline{\sigma}]}(q;\underline{a})$, and $c_n^{[k;\underline{\sigma}]}(q;\underline{a})$ as $g_n^{[k]}(\underline{a})$ with α_{ir} given by $\begin{bmatrix} \sigma_i \\ r \end{bmatrix}$ and $\begin{bmatrix} \sigma_i + r - 1 \\ r \end{bmatrix}$ respectively. In the above reference Menon has extended Theorem 10 to this situation.

[7] Note that $\lim_{q \to 1} \left[\begin{smallmatrix} s \\ k \end{smallmatrix}\right] = \binom{s}{k}$.

EXAMPLE (iv) Menon also used the q-binomial coefficients to generalize $\mathfrak{s}_n^{[r]}(\underline{a})$ as follows:

$$\mathfrak{s}_{n,q}^{[r]}(\underline{a}) = \frac{e_n^{[r]}(\underline{a})}{\begin{bmatrix} n \\ r \end{bmatrix}}.$$

Clearly: $\mathfrak{s}_{n,1}^{[r]}(\underline{a}) = \mathfrak{s}_n(\underline{a})$; $\mathfrak{s}_{n,0}^{[r]}(\underline{a}) = e_n^{[r]}(\underline{a})$.
As we see in the proof of Corollary 15 below $\begin{bmatrix} s \\ k \end{bmatrix}$ is log-concave as a function of k. Using this Menon proves

$$\mathfrak{s}_{n,q}^{[r-1]}(\underline{a})\mathfrak{s}_{n,q}^{[r+1]}(\underline{a}) \le \left(\mathfrak{s}_{n,q}^{[r]}(\underline{a})\right)^2,$$

an inequality containing both 2(1) and 2(2) as special cases; [*Ilori; Menon 1972*].

The original proof of 5.2 Theorem 13, [*Whitely 1958*], used 5.2 Corollary 11, or 5.2 Theorem 9, in a fundamental manner. Such results are not available in the present more general situation. However as Whiteley pointed out slightly weaker results will suffice; [*Whiteley 1962b*].
Let us consider the case $s > 0$; then inequalities (13) and (12) imply the following weaker and simpler inequalities, with the restrictions on k and m given in Theorems 9 and 10;

$$\left(t_n^{[k,s]}\right)^2 \ge \left(\frac{k+1}{k}\right) t_n^{[k-1,s]}\, t_n^{[k+1,s]}; \qquad \left(k!t_n^{[k,s]}\right)^{1/k} \ge \left(m!t_n^{[m,s]}\right)^{1/m}. \tag{19}$$

If $s < 0$ then the inequalities ($\sim$19) hold.
It is these results that will be extended, and which will then be used to obtain Theorem 16 below, a theorem that contains 5.2 Theorem 13 as a special case.
If $s > 0$ then inequalities (19) imply the following still simpler and easier inequalities:

$$\left(t_n^{[k,s]}\right)^2 \ge t_n^{[k-1,s]}\, t_n^{[k+1,s]}; \qquad \left(t_n^{[k,s]}\right)^{1/k} \ge \left(t_n^{[m,s]}\right)^{1/m}. \tag{20}$$

However if $s < 0$ we cannot deduce ($\sim$20); in fact as we will see below, (24), the same inequalities, (20), are valid if $s < -1$.

THEOREM 14 *(a) If $\underline{a}$ is a non-negative n-tuple and if in (18) for each i, $1 \le i \le n$, the sequences α_{ir}, $r = 1, 2, \dots$ are log-concave, or equivalently $\beta_{i,j-1} \ge \dfrac{j-1}{j}\beta_{i,j}$, $1 \le i \le n$, $j = 1, 2, \dots$, then*

$$\left(g_n^{[k]}\right)^2 \ge g_n^{[k-1]}\, g_n^{[k+1]},\ k \ge 1; \quad \left(g_n^{[k]}\right)^{1/k} \ge \left(g_n^{[m]}\right)^{1/m},\ 1 \le k < m. \tag{21}$$

(b) If instead for each i, $1 \le i \le n$, the sequences α_{ir}, $r = 1, 2, \ldots$ are strongly log-concave, or equivalently $\beta_{i,j-1} \ge \beta_{i,j}$, $1 \le i \le n$, $j = 1, 2, \ldots$, then

$$\left(g_n^{[k]}\right)^2 \ge \left(\frac{k+1}{k}\right) g_n^{[k-1]}\, g_n^{[k+1]},\ k \ge 1; \quad \left(g_n^{[k]}\right)^{1/k} \ge \left(g_n^{[m]}\right)^{1/m},\ 1 \le k < m. \tag{22}$$

If the hypothesis is changed to weakly log-convex then inequalities ($\sim$22) hold.

□ (a) In the case of positive sequences the left-hand inequality in (21) follows by a simple induction from I 3.2 Theorem 8(a). If some of the a_i are zero then consideration of (18) shows that the inequality follows from the inequality for smaller values of n.
The right-hand inequality in (21) follows from the left-hand inequality using the arguments of 2 Theorem 3.
(b) This has the same proof as (a) except that now we use I 3.2 Theorem 6. □

Using the Remarks (iv) and (v) that follow I 3.2 Theorems 6 and 8 we can make the following observations about the cases of equality in the above result.

(i) The left-hand inequality in (21) is strict unless $\underline{a}$ is zero, or if $a_i = 0, i \ne j$, and $\alpha_{j,k}^2 = \alpha_{j,k-1}\alpha_{j,k+1}$.

(ii) The right-hand inequality in (21) is strict unless $\underline{a}$ is zero, or if $a_i = 0, i \ne j$, and $\alpha_{j,s}^2 = \alpha_{j,s-1}\alpha_{j,s+1}$, $1 \le s \le k$.

(iii) The left-hand inequality in (22) is strict unless $\underline{a}$ is zero, or if $\alpha_{i,j}^2 = \alpha_{i,j-1}\alpha_{i,j+1}$, $1 \le i \le n$, $1 \le j \le k+1$.

(iv) The right-hand inequality in (22) is strict unless $\underline{a}$ is zero, or if $a_i = 0, i \ne j$ and $\alpha_{j,k}^2 = \alpha_{j,k-1}\alpha_{j,k+1}$.

Corollary 15 *If $\underline{a}$ is a positive n-tuple, $\underline{\sigma}$ a real n-tuple, $0 < q < 1$, k a positive integer then:*

$$\left(t_n^{[k;\underline{\sigma}]}(\underline{a})\right)^2 > \frac{k+1}{k}\left(t_n^{[k-1;\underline{\sigma}]}(\underline{a})\right)\left(t_n^{[k+1;\underline{\sigma}]}(\underline{a})\right),\ \underline{\sigma} > 0; \tag{23}$$

if $\underline{\sigma} < 0$ then ($\sim$ 23) holds;

$$\left(t_n^{[k;\underline{\sigma}]}(\underline{a})\right)^2 > \left(t_n^{[k-1;\underline{\sigma}]}(\underline{a})\right)\left(t_n^{[k+1;\underline{\sigma}]}(\underline{a})\right), \qquad \underline{\sigma} \le -1; \tag{24}$$

$$\left(e_n^{[k;\underline{\sigma}]}(q;\underline{a})\right)^2 > \left(e_n^{[k-1;\underline{\sigma}]}(q;\underline{a})\right)\left(e_n^{[k+1;\underline{\sigma}]}(q;\underline{a})\right),\ \underline{\sigma} > 0; \tag{25}$$

$$\left(c_n^{[k;\underline{\sigma}]}(q;\underline{a})\right)^2 > \left(c_n^{[k-1;\underline{\sigma}]}(q;\underline{a})\right)\left(c_n^{[k+1;\underline{\sigma}]}(q;\underline{a})\right),\ \underline{\sigma} \ge 1. \tag{26}$$

□ From Example (i) it is easily seen that $\beta_{i,j}$ satisfies the condition in Theorem 14(b) if $\underline{\sigma} > 0$, while if $\underline{\sigma} \le -1$ it satisfies the condition in Theorem 14(a). This

by Theorem 14 implies inequalities (23) and (24); that they are strict follows from the above discussion of the cases of equality.

Similarly using the definitions given above inequalities (25) and (26) follow from Theorem 14(a) if we show that the sequences $\begin{bmatrix} s \\ r \end{bmatrix}$ and $\begin{bmatrix} s+r-1 \\ r \end{bmatrix}$ are log-concave when $s > 0$.

We will only consider the first case, tho other can be handled in a similar manner. Since

$$\frac{\begin{bmatrix} s \\ r \end{bmatrix}^2}{\begin{bmatrix} s \\ r-1 \end{bmatrix}\begin{bmatrix} s \\ r+1 \end{bmatrix}} = \frac{(1-q^{s-r+1})/(1-q^r)}{(1-q^{s-r})/(1-q^{r+1})},$$

it suffices to show that $f(x) = (1-q^{s-x+1})/(1-q^x)$ is decreasing, $1 \le x \le s+1$. Now $f'(x) = \dfrac{\log q}{(1-q^x)^2}\big(q^{s-x+1}(1-q^x) + q^x(1-q^{s-x+1})\big)$, which is negative since $0 < q < 1$. □

REMARK (i) Inequality (24) justifies the remarks made above about (20). Of course inequalities stronger than (23) are known when $\underline{\sigma}$ is constant. All the results in Corollary 15 are due to Menon, who also obtained similar results for a type of function not being considered here that also generalizes $t_n^{[k;\underline{\sigma}]}$; [*Menon 1969b,1970,1971*]. In particular (25) implies the inequality of Menon quoted in Example (iv).

We are now in a position to prove a generalization of 5.2 Theorem 13.

THEOREM 16 *If $\underline{a}$ and $\underline{b}$ are non-negative n-tuples, and if the sequences $\alpha_{i,r}$, $r = 1, 2, \ldots,$ $1 \le i \le n$, are all strictly strongly log-concave, or equivalently if*

$$\beta_{i,j-1} > \beta_{i,j},\ 1 \le i \le n,\ j = 1, 2, \ldots, \tag{27}$$

then

$$\left(g_n^{[k]}(\underline{a}+\underline{b})\right)^{1/k} \ge \left(g_n^{[k]}(\underline{a})\right)^{1/k} + \left(g_n^{[k]}(\underline{b})\right)^{1/k},\ k \ge 1. \tag{28}$$

If the hypothesis is changed to weakly log-convex then (~27) holds. In both cases there is equality if and only if $k = 1$ or $\underline{a} \sim \underline{b}$

Before starting the proof we make some preliminary remarks.

REMARK (ii) The above discussion shows 5.2 Theorem 13 is a particular case of this result. As a result, the following proof completes the discussion of that theorem. The method of proof is similar to that used in 5.2 Theorem 10, as stated in the discussion of that theorem. However Theorem 14 allows us to make a subtler

use of the Lagrange multiplier conditions. Further the use here of the more general class of functions of degree k, $k \in \mathbb{N}$, allows us to differentiate and stay within this class. As is easily seen, if $g_n^{[k]}$ is a function of degree k, $k \geq 1$, then $\partial g_n^{[k]}/\partial a_m$ is a function of degree $k-1$; further if the coefficients of $g_n^{[k]}$ satisfy the above hypotheses so do those of the partial derivatives $\partial g_n^{[k]}/\partial a_m$, $1 \leq m \leq n$.

REMARK (iii) It is easy to show that if $\underline{a} = (\underline{a}', \underline{a}'')$, $\underline{a}' = (a_1, \ldots, a_m)$, $\underline{a}'' = (a_{m+1}, \ldots, a_n)$ then

$$g_n^{[k]}(\underline{a}) = \sum_{r=0}^{k} g_m^{[r]}(\underline{a}') g_{n-m}^{[k-r]}(\underline{a}'').$$

□ The proof is by induction on k. If $k = 1$ there is nothing to prove as (28) is trivial in this case. However to consider the cases of equality the induction must start from $k = 2$. In that case squaring (28) twice leads to the equivalent inequality $\sum_{\substack{i,j=1\\ i>j}}^{n} \beta_{i,1}\beta_{j,1}(\beta_{i,2}\beta_{j,2} - \beta_{i,1}\beta_{j,1})(a_i b_j - a_j b_i)^2 \leq 0$, which is an immediate consequence of (27).

Now put $A = \{(\underline{a}, \underline{b});\ \underline{a} \geq 0, \underline{b} \geq 0,\ \text{and}\ g_n^{[k]}(\underline{a} + \underline{b}) = 1\}$. This set is a compact subset in $\mathbb{R}^{2n}$. So the function $\left(g_n^{[k]}(\underline{a})\right)^{1/k} + \left(g_n^{[k]}(\underline{b})\right)^{1/k}$ will attain its maximum on A if (27) holds, and its minimum if ($\sim$27) holds. In either case call this value M. It suffices, from considerations of homogeneity to prove in the first case that $M \leq 1$, and in the second case that $M \geq 1$. Since both cases proceed similarly let us consider the first case only.

Suppose that M is attained at a point $(\underline{a}, \underline{b})$ of A satisfying $a_i > 0$, $i = i_1, \ldots, i_{n_1}$, and $a_i = 0$ otherwise; $b_j > 0$, $j = j_1, \ldots, i_{n_2}$, and $b_j = 0$ otherwise. We can assume that $1 \leq n_1, n_2 \leq n$, since the cases when either, or both, of n_1, n_2 is zero are trivial.

The proof breaks into two cases:

(i): for some q, $a_q b_q > 0$;

(ii): for all i, $a_i b_i = 0$; then in this case $(i_1, \ldots, i_{n_1}), (j_1, \ldots, i_{n_2})$ are distinct subsets of $(1, \ldots, n)$; put $\underline{a}' = (a_{i_1}, \ldots a_{i_{n_1}})$, $\underline{b}' = (b_{j_1}, \ldots b_{j_{n_2}})$.

`Proof of case(`i`).`

Apply the Lagrange multiplier conditions, see II 2.4.3 Footnote 11, to the non-zero variables a_i, b_j and proceed as in 5.2 Theorem 10 to obtain

$$\frac{\partial}{\partial a_i}\left(g_n^{[k]}(\underline{a})\right)^{1/k} - \lambda \frac{\partial}{\partial a_i}\left(g_n^{[k]}(\underline{a} + \underline{b})\right)^{1/k} = 0,\ i = i_1, \ldots, i_{n_1},$$
$$\frac{\partial}{\partial b_j}\left(g_n^{[k]}(\underline{b})\right)^{1/k} - \lambda \frac{\partial}{\partial b_j}\left(g_n^{[k]}(\underline{a} + \underline{b})\right)^{1/k} = 0,\ j = j_1, \ldots, j_{n_2}.$$

In other words,

$$\big(g_n^{[k]}(\underline{a})\big)^{(1-k)/k}\,\frac{\partial g_n^{[k]}(\underline{a})}{\partial a_i}=\lambda\big(g_n^{[k]}(\underline{a}+\underline{b})\big)^{(1-k)/k}\,\frac{\partial g_n^{[k]}(\underline{a}+\underline{b})}{\partial a_i},\tag{29}$$

$$\big(g_n^{[k]}(\underline{b})\big)^{(1-k)/k}\,\frac{\partial g_n^{[k]}(\underline{b})}{\partial b_j}=\lambda\big(g_n^{[k]}(\underline{a}+\underline{b})\big)^{(1-k)/k}\,\frac{\partial g_n^{[k]}(\underline{a}+\underline{b})}{\partial b_j}.\tag{30}$$

Multiply (29) by a_i, and (30) by b_j, and add, $i=i_1,\ldots,i_{n_1}$ $j=j_1,\ldots,j_{n_2}$, and use Euler's theorem on homogeneous functions, see I 4.6 Remark(ii), to get

$$\big(g_n^{[k]}(\underline{a})\big)^{1/k}+\big(g_n^{[k]}(\underline{b})\big)^{1/k}=\lambda\big(g_n^{[k]}(\underline{a}+\underline{b})\big)^{1/k}.\tag{31}$$

Now take $i=j=q$ in (29) and (30), raise both to the power $1/(k-1)$, and use (31) to get

$$\Big(\frac{\partial g_n^{[k]}(\underline{a})}{\partial a_q}\Big)^{1/(k-1)}+\Big(\frac{\partial g_n^{[k]}(\underline{b})}{\partial b_q}\Big)^{1/(k-1)}=\lambda^{k/(k-1)}\Big(\frac{\partial g_n^{[k]}(\underline{a}+\underline{b})}{\partial a_q}\Big)^{1/(k-1)}.\tag{32}$$

By Remark (ii) $\dfrac{\partial g_n^{[k]}(\underline{a})}{\partial a_q}$ is a function of degree $(k-1)$, satisfying (27). So the inductive hypothesis tells us that $\lambda\le 1$; but by (31) $M=\lambda$, so $M\le 1$ as had to be proved.

`Proof of case(`*ii*`).`

Using the notation introduced above (28) reduces to

$$\big(g_n^{[k]}(\underline{a}'+\underline{b}')\big)^{1/k}\ge\big(g_n^{[k]}(\underline{a}')\big)^{1/k}+\big(g_n^{[k]}(\underline{b}')\big)^{1/k}.$$

or, using Remark (iii),

$$\Big(\sum_{r=0}^{k}g_{n_1}^{[k]}(\underline{a}')g_{n_2}^{[k]}(\underline{b}')\Big)^{1/k}\ge\big(g_{n_1}^{[k]}(\underline{a}')\big)^{1/k}+\big(g_{n_2}^{[k]}(\underline{b}')\big)^{1/k}.$$

Raising this to the kth power leads to the following inequality that has to be proved.

$$\sum_{r=0}^{k}g_{n_1}^{[r]}(\underline{a}')g_{n_2}^{[k-r]}(\underline{b}')\ge\sum_{r=0}^{k}\binom{k}{r}\big(g_{n_1}^{[k]}(\underline{a}')\big)^{r/k}\big(g_{n_2}^{[k]}(\underline{b}')\big)^{(k-r)/k}.\tag{33}$$

If $1\le r\le k-1$ we have by the right-hand inequality (22) that $g_{n_1}^{[r]}(\underline{a}')>\dfrac{\big(k!g_{n_1}^{[k]}(\underline{a}')\big)^{r/k}}{r!}$, and $g_{n_2}^{[k-r]}(\underline{b}')>\dfrac{\big(k!g_{n_2}^{[k]}(\underline{a}')\big)^{(k-r)/k}}{(k-r)!}$. Inequality (33) is then an immediate consequence of these inequalities. □

REMARK (iv) By applying Theorem 16 to the functions $t_n^{[k;\underline{\sigma}]}$ Theorem 12 can be extended to such functions, as suggested in [*Whiteley 1958, p.50*].

5.4 ELEMENTARY SYMMETRIC POLYNOMIAL MEANS AS MIXED MEANS Elementary symmetric polynomial means are particular cases of the mixed means introduced in III 5.3: $\mathfrak{S}_n^{[r]}(\underline{a}) = \mathfrak{M}_n(r,0;r;\underline{a})$. So we could compare, using a matrix analogous to that in III 5.3, particular cases of III 5.3(12) with S(r;s) as follows.

$$\begin{pmatrix} \mathfrak{M}_n(0,1;1;\underline{a}) & \mathfrak{M}_n(0,1;2;\underline{a}) & \dots & \mathfrak{M}_n(0,1;n-1;\underline{a}) & \mathfrak{M}_n(0,1;n;\underline{a}) \\ \mathfrak{M}_n(1,0;n;\underline{a}) & \mathfrak{M}_n(1,0;n-1;\underline{a}) & \dots & \mathfrak{M}_n(1,0;2;\underline{a}) & \mathfrak{M}_n(1,0;1;\underline{a}) \\ \mathfrak{S}_n^{[n]}(\underline{a}) & \mathfrak{S}_n^{[n-1]}(\underline{a}) & \dots & \mathfrak{S}_n^{[2]}(\underline{a}) & \mathfrak{S}_n^{[1]}(\underline{a})) \end{pmatrix}.$$

Then by S(r;s) and III 5.3(12) the rows increase strictly to the right, and the entries in the second row are strictly less than those in the first row, and strictly greater than those in the last row, except for the first column all of whose entries are $\mathfrak{G}_n(\underline{a})$, and for the last column all of whose entries are $\mathfrak{A}_n(\underline{a})$.
An immediate problem suggests itself— are there any further relations between the elementary symmetric means in the last line and the mixed arithmetic and geometric means in the first? Carlson, Meany & Nelson conjectured that if $r+m > n$ then

$$\mathfrak{M}_n(r,0;r;\underline{a}) \le \mathfrak{M}_n(0,1;m;\underline{a}); \tag{34}$$

[*Carlson, Meany & Nelson 1971b*].
If this were proved to be correct the above matrix could be rewritten

$$\begin{pmatrix} \mathfrak{M}_n(1,0;n;\underline{a}) & \mathfrak{M}_n(1,0;n-1;\underline{a}) & \dots & \mathfrak{M}_n(1,0;2;\underline{a}) & \mathfrak{M}_n(1,0;1;\underline{a}) \\ \mathfrak{S}_n^{[n]}(\underline{a}) & \mathfrak{S}_n^{[n-1]}(\underline{a}) & \dots & \mathfrak{S}_n^{[2]}(\underline{a}) & \mathfrak{S}_n^{[1]}(\underline{a})) \\ \mathfrak{M}_n(0,1;1;\underline{a}) & \mathfrak{M}_n(0,1;2;\underline{a}) & \dots & \mathfrak{M}_n(0,1;n-1;\underline{a}) & \mathfrak{M}_n(0,1;n;\underline{a}) \end{pmatrix}.$$

Now the rows again increase strictly to the right, but the columns strictly increase, except of course the first and last.
The particular case $n = 3$ of (34), $\mathfrak{M}_3(2,0;2;a,b,c) \le \mathfrak{M}_3(0,1;2;a,b,c)$, was proved in [*Carlson 1970a*]. It is the following lemma that should be compared with III 5.3(17).

LEMMA 17 *If a, b and c are positive numbers then*

$$\sqrt{\frac{ab+bc+ca}{3}} \le \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}. \tag{35}$$

□

$$\begin{aligned}(a+b)(b+c)(c+a) &= (a+b+c)(ab+bc+ca) - abc\\ &= (a+b+c)(ab+bc+ca) - \sqrt[1/3]{(ab)(bc)(ca)}\ \sqrt[1/3]{abc}\\ &\geq \frac{8}{9}(a+b+c)(ab+bc+ca), \qquad \text{by (GA)},\\ &\geq 8\ \sqrt[3/2]{\frac{ab+bc+ca}{3}}, \qquad \text{by S(r; s)};\end{aligned}$$

and this is just (35). □

More generally Kuczma has shown that:

$\mathfrak{M}_n(2,0;2;\underline{a}) \leq \mathfrak{M}_n(0,1;n-1;\underline{a})$, and $\mathfrak{M}_n(n-1,0;2;\underline{a}) \leq \mathfrak{M}_n(0,1;n-1;\underline{a})$.

Both of these inequalities include (35); [*Kuczma 1994*].

6 Muirhead Means

A very different generalization of the elementary symmetric polynomial means is given as follows. Let $\underline{\alpha}$ be a non-negative n-tuple, and write $|\underline{\alpha}| = \alpha_1 + \cdots + \alpha_n$; then if $\underline{a}$ is an n-tuple we write

$$\mathfrak{m}_n(\underline{a};\underline{\alpha}) = \frac{1}{n!}\sum_{n}! \prod_{j=1}^{n} a_{i_j}^{\alpha_j}, \tag{1}$$

and define the *Muirhead $\underline{\alpha}$-mean* or just *Muirhead mean of $\underline{a}$* as,

$$\mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) = \mathfrak{A}_{n,(\alpha_1,\ldots,\alpha_n)}(\underline{a}) = \big(\mathfrak{m}_n(\underline{a};\underline{\alpha})\big)^{1/|\underline{\alpha}|}; \tag{2}$$

of course if $|\underline{\alpha}| = 1$ then $\mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) = \mathfrak{m}_n(\underline{a};\underline{\alpha})$.

Remark (i) Clearly these means have the properties of (Co), (Ho), (Mo), (Re), (Sy), and are strictly internal.

Convention Since the order of the elements of in $\underline{\alpha}$ is immaterial we will always take them to be decreasing.

Example (i) The following are easily checked. $\mathfrak{A}_{2,\alpha,\beta}(a,b) = \left(\dfrac{a^\alpha b^\beta + a^\beta b^\alpha}{2}\right)^{1/\alpha+\beta}$;

$$\mathfrak{A}_n(\underline{a}) = \mathfrak{A}_{n,(1,0,\ldots,0)}(\underline{a}); \quad \mathfrak{G}_n(\underline{a}) = \mathfrak{A}_{n,(1/n,1/n,\ldots,1/n)}(\underline{a}) = \mathfrak{A}_{n,(1,\ldots,1)}(\underline{a});$$

$$\mathfrak{M}_n^{[r]}(\underline{a}) = \mathfrak{A}_{n,(r,0,\ldots,0)}(\underline{a}),\ r \neq 0; \quad \mathfrak{S}_n^{[r]}(\underline{a}) = \mathfrak{A}_{n,\underline{\alpha}}(\underline{a}),\ \underline{\alpha} = (\overbrace{1,\ldots,1}^{r \text{ terms}},0,\ldots,0).$$

Example (ii) Using 5.1 Example(i) we can readily see that

$$\mathfrak{Q}_3^{[3]}(\underline{a}) = \mathfrak{M}_3^{[3]}\big(\mathfrak{A}_{3,(1,1,1)}(\underline{a}), \mathfrak{A}_{3,(2,1,0)}(\underline{a}), \mathfrak{A}_{3,(3,0,0)}(\underline{a}); 1,6,3)\big)$$

The main purpose of this section is to obtain conditions under which two different Muirhead means are comparable. The answer, *Muirhead's theorem*, is in terms of the order relation defined in I 3.3.

Theorem 1 [Muirhead's Theorem] *Let $\underline{\alpha}^{(1)}$, and $\underline{\alpha}^{(2)}$ be non-identical non-negative n-tuples, $\underline{a}$ an n-tuple; then $\mathfrak{m}_n(\underline{a};\underline{\alpha}^{(1)})$ and $\mathfrak{m}_n(\underline{a};\underline{\alpha}^{(2)})$ are comparable if and only if one of $\underline{\alpha}^{(1)}$ or $\underline{\alpha}^{(2)}$ is an average of the other. More precisely*

$$\mathfrak{m}_n(\underline{a};\underline{\alpha}^{(1)}) \le \mathfrak{m}_n(\underline{a};\underline{\alpha}^{(2)}) \iff \underline{\alpha}^{(1)} \prec \underline{\alpha}^{(2)}, \tag{3}$$

or, $\underline{\alpha}^{(1)}$ is an average of $\underline{\alpha}^{(2)}$. Further the inequality in (3) is then strict unless $\underline{a}$ is constant.

□ (a) $\Longrightarrow$ Suppose that the inequality in (3) holds for all positive n-tuples $\underline{a}$. Choose $\underline{a}$ constant and equal to a, when the inequality becomes $a^{|\underline{\alpha}^{(1)}|} \le a^{|\underline{\alpha}^{(2)}|}$. By considering large a and small a this implies that $|\underline{\alpha}^{(1)}| = |\underline{\alpha}^{(2)}|$, that is the first part of I 3.3(16).
Now take $a_1 = \cdots = a_k = a, a_{k+1} = \cdots = a_n = 1$, $1 \le k < n$. Then the largest power of a in $\mathfrak{m}_n(\underline{a};\underline{\alpha}^{(1)})$ is $\sum_{i=1}^k \alpha_i^{(1)}$, whereas in $\mathfrak{m}_n(\underline{a};\underline{\alpha}^{(2)})$ it is $\sum_{i=1}^k \alpha_i^{(2)}$. If we now assume that a is large then the inequality implies the second part of I 3.3(16). Thus $\underline{\alpha}^{(1)} \prec \underline{\alpha}^{(2)}$.
(b) $\Longleftarrow$. We give two proofs of this implication.
(i) It is sufficient to note that $\phi(\underline{\alpha}) = \mathfrak{m}_n(\underline{a};\underline{\alpha})$ is both symmetric and convex and apply 1 4.8 Theorem 49.
(ii) By I 3.3 Lemma 13, Muirhead's lemma, it is sufficient to consider the case when $\underline{\alpha}^{(1)} = \underline{\alpha}^{(2)}T$ and $T = \lambda I + (1-\lambda)Q$, where Q is a permutation matrix that differs from I in just two rows. Without loss of generality we can assume that

$$\alpha_1^{(1)} = \lambda\alpha_1^{(2)} + (1-\lambda)\alpha_2^{(2)},\ \alpha_2^{(1)} = (1-\lambda)\alpha_1^{(2)} + \lambda\alpha_2^{(2)},\ \alpha_k^{(1)} = \alpha_k^{(2)}, 3 \le k \le n.$$

Then

$$\begin{aligned}
&2(n!)\big(\mathfrak{m}_n(\underline{a};\underline{\alpha}^{(2)}) - \mathfrak{m}_n(\underline{a};\underline{\alpha}^{(1)})\big)\\
&= \sum{}! \prod_{j=3}^n a_{i_j}^{\alpha_j^{(2)}} \big(a_{i_1}^{\alpha_1^{(2)}} a_{i_2}^{\alpha_2^{(2)}} + a_{i_1}^{\alpha_2^{(2)}} a_{i_2}^{\alpha_1^{(2)}} - a_{i_1}^{\alpha_1^{(1)}} a_{i_2}^{\alpha_2^{(1)}} - a_{i_1}^{\alpha_2^{(1)}} a_{i_2}^{\alpha_1^{(1)}}\big)\\
&= \sum{}! \prod_{j=3}^n a_{i_j}^{\alpha_j^{(2)}} (a_{i_1}a_{i_2})^{\alpha_2^{(2)}} \big(a_{i_1}^{\alpha_1^{(2)}-\alpha_2^{(2)}} + a_{i_2}^{\alpha_1^{(2)}-\alpha_2^{(2)}}\\
&\qquad - a_{i_1}^{\lambda(\alpha_1^{(2)}-\alpha_2^{(2)})} a_{i_2}^{(1-\lambda)(\alpha_1^{(2)}-\alpha_2^{(2)})} - a_{i_1}^{(1-\lambda)(\alpha_1^{(2)}-\alpha_2^{(2)})} a_{i_2}^{\lambda(\alpha_1^{(2)}-\alpha_2^{(2)})}\big)\\
&= \sum{}! \prod_{j=3}^n a_{i_j}^{\alpha_j^{(2)}} (a_{i_1}a_{i_2})^{\alpha_2^{(2)}} \big(a_{i_1}^{\lambda(\alpha_1^{(2)}-\alpha_2^{(2)})} - a_{i_2}^{\lambda(\alpha_1^{(2)}-\alpha_2^{(2)})}\big) \times\\
&\qquad\qquad \big(a_{i_1}^{(1-\lambda)(\alpha_1^{(2)}-\alpha_2^{(2)})} - a_{i_2}^{(1-\lambda)(\alpha_1^{(2)}-\alpha_2^{(2)})}\big) \ge 0.
\end{aligned}$$

The cases of equality are immediate. □

REMARK (ii) The second proof is essentially that in [*Muirhead 1902/03*]; see also [*AI p.167; DI pp.183–184; HLP pp.44–51; MO pp.87,110–111; PPT pp.361–364*].

REMARK (iii) Since $(1/n,\ 1/n,\dots,1/n) \prec (1,0,\dots,0)$ we see from Example (i) that the above theorem gives another proof of (GA). It is worth giving the details. Note first that $(1/n,\dots,1/n) = (1,0,\dots,0)\frac{1}{n}J$, where J is the $n\times n$ matrix all of whose entries are equal to 1. Also, see I 4.3 Lemma 13, $\frac{1}{n}J = \prod_{k=1}^{n-1} T_k$,

$$T_k = \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix} \text{ where } K = \frac{1}{n-k+1}\begin{pmatrix} 1 & n-k \\ n-k & 1 \end{pmatrix},\ 1\le k\le n-1.$$

Here I_j is the $j\times j$ unit matrix or is absent if $j=0$.

More directly let $\underline{\alpha}^{(k)}$ be the n-tuple $(n-k, \overbrace{1,\dots,1}^{k\text{ terms}}, 0,\dots,0,\ 0)$, $0\le k<n$. Then

$$\begin{aligned}\mathfrak{A}_n(\underline{a}^n) - \mathfrak{G}_n(\underline{a}^n) &= \sum_{k=0}^{n-2}\left(\mathfrak{m}_n(\underline{a};\underline{\alpha}^{(k)}) - \mathfrak{m}_n(\underline{a};\underline{\alpha}^{(k+1)})\right) \\ &= \frac{1}{n!}\sum_{k=0}^{n-2}\left(\sum!(a_{i_1}^{n-k-1} - a_{i_2}^{n-k-1})(a_{i_1}-a_{i_2})a_{i_3}\dots a_{i_{k+2}}\right) \\ &\ge 0;\end{aligned}$$

where if $k=0$ there are no factors in $\sum!$ beyond the brackets; see [*HLP p.50*].

COROLLARY 2 *(a) If $\underline{a}$ is an n-tuple and if $\underline{\alpha}^{(1)}$ and $\underline{\alpha}^{(2)}$ are distinct non-negative n-tuples with $\underline{\alpha}^{(1)} \prec \underline{\alpha}^{(2)}$ then*

$$\mathfrak{A}_{n,\underline{\alpha}^{(1)}}(\underline{a}) \le \mathfrak{A}_{n,\underline{\alpha}^{(2)}}(\underline{a}), \tag{4}$$

with equality if and only if $\underline{a}$ is constant.
(b) [SCHUR] *If $\underline{a}$ is a positive n-tuple and if $\underline{\alpha}$ a non-negative n-tuple with $|\alpha|=1$ then*

$$\mathfrak{G}_n(\underline{a}) \le \mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) \le \mathfrak{A}_n(\underline{a}), \tag{5}$$

with equality if and only $\underline{a}$ is constant, or, on the left if $\mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) = \mathfrak{G}_n(\underline{a})$, on the right if $\mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) = \mathfrak{A}_n(\underline{a})$.

□ (a) This is an immediate consequence of Theorem 1 and the definition of a Muirhead mean.
(b) We give three proofs.
(*i*) By I 3.3 Example(ii) if $|\underline{\alpha}|=1$ then $(1/n,1/n,\dots,1/n) \prec \underline{\alpha} \prec (1,0,\dots,0)$, and the result follows from (a) and Example (i); [*HLP p.50*].

(*ii*) A direct proof of (5) can be found in [*Bartoš & Znám*]. By (GA) we have, for each permutation $(i_1, \ldots, i_n)$ that

$$\prod_{j=1}^{n} a_{i_j}^{\alpha_j} \leq \sum_{j=1}^{n} \alpha_j a_j. \tag{6}$$

Adding all these inequalities over all permutations give the right-hand inequality in (5).

Again from (GA) and the definitions (1) and (2),

$$\mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) \geq \Big(\prod_{n}! \prod_{j=1}^{n} a_{i_j}^{\alpha_j}\Big)^{1/n!}\Big) = \mathfrak{G}_n(\underline{a}),$$

giving the left-hand inequality in (5).

The cases of equality follow from those in (GA).

(*iii*) Another proof, due to Segre, can be found in VI 4.5 Example (vi); [*Segre*].□

REMARK (iv) Part (a) remains valid for all real n-tuples $\underline{\alpha}^{(1)}$ and $\underline{\alpha}^{(2)}$, such that $\underline{\alpha}^{(1)} \prec \underline{\alpha}^{(2)}$ and $|\underline{\alpha}^{(1)}| > 0$.

The following interesting extension of Schur's result, (5), has been given; [*Gel'man*].

THEOREM 3 *Let F_m be a symmetric polynomial of n variables, homogeneous of degree m, and with non-negative coefficients. Then*

(a) $f(\underline{a}) = F_m(\underline{a}^{1/m})$ is concave, symmetric and homogeneous,(of degree 1);

(b) $g(\underline{a}) = \log \circ F_m(e^{\frac{1}{m}\underline{a}})$ is convex.

(c)

$$\mathfrak{G}_n(\underline{a}) \leq \left(\frac{F_m(\underline{a})}{F_m(\underline{e})}\right)^{1/m} \leq \mathfrak{M}_n^{[m]}(\underline{a}). \tag{7}$$

□ (a) The concavity of f is given by III 2.1 Remark (iii), and the rest is immediate.

(b) By (C),

$$f(\sqrt{\underline{ab}}) \leq \sqrt{f(\underline{a})f(\underline{b})}.$$

Now replace $\underline{a}$ and $\underline{b}$ by $e^{\underline{a}}$ and $e^{\underline{b}}$, respectively and take logarithms on both sides of the last inequality.

(c) Consider first the right-hand side of (7), and put $\underline{b} = \underline{a}^m$.

$$\begin{aligned} F_m(\underline{a}) = f(\underline{b}) &= \frac{1}{n}\sum_{i=1}^{n} f(b_i, \ldots, b_n, b_1 \ldots, b_{i-1}), \text{ by the symmetry of } f, \\ &\leq f\big(\mathfrak{A}_n(\underline{b}), \ldots, \mathfrak{A}_n(\underline{b})\big), \text{ by the concavity of } f, \\ &= \mathfrak{A}_n(\underline{b}) f(\underline{e}), \text{ by the homogeneity of } f, = \big(\mathfrak{M}_n^{[m]}(\underline{a})\big)^m F_m(\underline{e}). \end{aligned}$$

Now consider the left-hand side of (7) , and define $\underline{b}$ by $\underline{a} = \exp(\underline{b}/m)$. Then, as above, but using (b):

$$\begin{aligned} F_m(\underline{a}) =& g(\underline{b}) = \frac{1}{n}\sum_{i=1}^{n} f(b_i, \ldots, b_n, b_1, \ldots, b_{i-1}) \\ \geq& g\big(\mathfrak{A}_n(\underline{b}), \ldots, \mathfrak{A}_n(\underline{b})\big) = \log\big(F_m(e^{\frac{1}{m}\mathfrak{A}_n(\underline{b})}, \ldots, e^{\frac{1}{m}\mathfrak{A}_n(\underline{b})})\big); \end{aligned}$$

or, by the homogeneity of F_m, $F_m(\underline{a}) \geq F_m(\underline{e})e^{\mathfrak{A}_n(\underline{b})} = F_m(\underline{e})\big(\mathfrak{G}_n(\underline{a})\big)^m$, which completes the proof. □

COROLLARY 4 *If* $0 < p \leq 1$ *then*

$$\mathfrak{m}_n(\underline{a}; \underline{\alpha}^{(1)}) \leq \big(\mathfrak{m}_n(\underline{a}; \underline{\alpha}^{(2)})\big)^p \iff \underline{\alpha}^{(1)} \prec p\underline{\alpha}^{(2)}. \tag{8}$$

□ $\Longrightarrow$ This follows using the same argument as in (b) (i) of Theorem 1.

$\Longleftarrow$ By Theorem 1, and (r;s) with $r = p, s = 1$,

$$\mathfrak{m}_n(\underline{a}; \underline{\alpha}^{(1)}) \leq \mathfrak{m}_n(\underline{a}; p\underline{\alpha}^{(2)}) \leq \big(\mathfrak{m}_n(\underline{a}; \underline{\alpha}^{(2)})\big)^p.$$

□

REMARK (v) If we take $\underline{\alpha}^{(1)} = (r, 0, \ldots, 0)$, $\underline{\alpha}^{(2)} = (s, 0, \ldots, 0)$, $p = r/s$, then the inequality (8) is just (r;s).

REMARK (vi) The proof of the necessity does not use the restriction on p, but Remark (v) shows that if $p > 1$ then $\underline{\alpha}^{(1)} \prec p\underline{\alpha}^{(2)}$ is not sufficient for the inequality to hold.

COROLLARY 5 *Let* $\underline{a}, \underline{w}$ *be positive* n*-tuples then*

$$\prod_{i=1}^{n}\Big(\sum_{k=1}^{n} w_k a_k^{\alpha_i^{(1)}}\Big) \leq \prod_{i=1}^{n}\Big(\sum_{k=1}^{n} w_k a_k^{\alpha_i^{(2)}}\Big) \iff \underline{\alpha}^{(1)} \prec \underline{\alpha}^{(2)}.$$

□ $\Longrightarrow$ This follows along the lines of Theorem 1.

$\Longleftarrow$ By 1 3.3 Theorem 14 $\underline{\alpha}^{(1)} \prec \underline{\alpha}^{(2)}$ implies that $\underline{\alpha}^{(1)} = \underline{\alpha}^{(2)}S$, where $S = (s_{ij})_{1\leq i,j\leq n}$ is a doubly stochastic matrix. Hence, by III 2.1 (2), (H),

$$\sum_{k=1}^{n} w_k a_k^{\alpha_i^{(1)}} = \sum_{k=1}^{n} w_k a_k^{(\sum_{j=1}^{n} s_{ij}\alpha_i^{(2)})} \leq \prod_{j=1}^{n}\Big(\sum_{k=1}^{n} w_k a_k^{\alpha_j^{(2)}}\Big)^{s_{ij}},$$

and the result follows by multiplication. □

COROLLARY 6 *If $\underline{a}$ is a positive n-tuple, r and integer, $1 \le r \le n$, then*

$$\mathfrak{S}_n^{[r]}(\underline{a}) \le \mathfrak{Q}_n^{[r]}(\underline{a}), \tag{9}$$

with equality if and only if either $r = 1$ or $\underline{a}$ is constant.

□ Let $A = A^{(r)}$, be the set of n-tuples defined as the set

$$\{\underline{\alpha};\ \underline{\alpha} \ge 0,\ \text{decreasing}, \alpha_i = 0,\ r+1 \le i \le n,\ \alpha_i\ \text{ an integer},\ 1 \le i \le r, |\underline{\alpha}| = r\}.$$

The A has n elements and in particular $\underline{\alpha}_0 = (\overbrace{1,\ldots,1}^{r\text{ terms}},0,\ldots,0) \in A$. Further if $\underline{\alpha} \in A$, $\underline{\alpha}_0 \prec \underline{\alpha}$, and so by Corollary 2(a) $\mathfrak{A}_{n,\underline{\alpha}_0}(\underline{a}) \le \mathfrak{A}_{n,\underline{\alpha}}(\underline{a})$.
Now $\mathfrak{Q}_n^{[r]}(\underline{a})$ is an rth power mean of the $\{\mathfrak{A}_{n,\underline{\alpha}}\}_{\underline{\alpha}\in A}$, see Example (ii); and, from Example (i), $\mathfrak{A}_{n,\underline{\alpha}_0}(\underline{a}) = \mathfrak{S}_n^{[r]}(\underline{a})$; so the result is immediate by the internality of the power means, III 1(2). □

REMARK (vii) This result, which in the notation of 5.2 is $\mathfrak{W}_n^{[k,1]}(\underline{a}) \le \mathfrak{W}_n^{[k,-1]}(\underline{a})$, was stated without proof in [*Zappa 1939*]. In the same paper Zappa states a much more general inequality using means derived from $t_n^{[k;\underline{\sigma}]}$, see 5.3. This inequality reduces, in the case of constant $\underline{\sigma}$, s say, to $\mathfrak{W}_n^{[k,s]}(\underline{a}) \le \mathfrak{W}_n^{[k,-s]}(\underline{a})$. No proof was given, although in the case of constant $\underline{\sigma}$ the proof of Corollary 6 can be used.

The following theorem generalizes 4 Corollary 2.

THEOREM 7 *If $\underline{a}, \underline{b}$, and $\underline{\alpha}$ are positive n-tuples with $0 \le |\underline{\alpha}| \le 1$, then*

$$\mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) + \mathfrak{A}_{n,\underline{\alpha}}(\underline{b}) \le \mathfrak{A}_{n,\underline{\alpha}}(\underline{a}+\underline{b}). \tag{10}$$

□

First we assume that $\mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) = \mathfrak{A}_{n,\underline{\alpha}}(\underline{b}) = 1$, when by homogeneity it suffices to show that if $0 < \lambda < 1$ then $\mathfrak{A}_{n,\underline{\alpha}}(\lambda\underline{a} + \overline{1-\lambda}\,\underline{b}) \ge 1$.

$$\begin{aligned}
\mathfrak{A}_{n,\underline{\alpha}}(\lambda\underline{a} + \overline{1-\lambda}\,\underline{b}) &= \Big(\frac{1}{n!}\sum! \prod_{j=1}^{n}(\lambda a_{i_j} + \overline{1-\lambda} b_{i_j})^{\alpha_j}\Big)^{\frac{1}{|\underline{\alpha}|}} \\
&\ge \left(\frac{1}{n!}\Big(\sum!\Big(\lambda\prod_{j=1}^{n} a_{i_j}^{\alpha_j/|\underline{\alpha}|} + (1-\lambda)\prod_{j=1}^{n} b_{i_j}^{\alpha_j/|\underline{\alpha}|}\Big)^{|\underline{\alpha}|}\right)^{\frac{1}{|\underline{\alpha}|}}, \quad \text{by III 2(2), (H)}, \\
&\ge \Big(\frac{1}{n!}\Big(\sum!\lambda\prod_{j=1}^{n} a_{i_j}^{\alpha_j} + (1-\lambda)\prod_{j=1}^{n} b_{i_j}^{\alpha_j}\Big)^{\frac{1}{|\underline{\alpha}|}} = 1, \quad \text{by (r;s)}.
\end{aligned}$$

In general,

$$\frac{\mathfrak{A}_{n,\underline{\alpha}}(\underline{a}+\underline{b})}{\mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) + \mathfrak{A}_{n,\underline{\alpha}}(\underline{b})} = \mathfrak{A}_{n,\underline{\alpha}}(\lambda\underline{a}' + \overline{1-\lambda}\,\underline{b}')$$

where $\underline{a}' = \dfrac{1}{\mathfrak{A}_{n,\underline{\alpha}}(\underline{a})}$; $\underline{b} = \dfrac{1}{\mathfrak{A}_{n,\underline{\alpha}}(\underline{b})}$; $\lambda = \dfrac{\mathfrak{A}_{n,\underline{\alpha}}(\underline{a})}{\mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) + \mathfrak{A}_{n,\underline{\alpha}}(\underline{b})}$. This shows that the argument above is sufficient. □

By analogy with definition (2) Bartoš & Znám, [*Bartoš & Znám*], have defined, for $\underline{\alpha}$ with $|\underline{\alpha}| = 1$, a geometric mean analogue of (2):

$$\mathfrak{G}_{n,\underline{\alpha}}(\underline{a}) = \Big(\prod_n ! \sum_{j=1}^{n} \alpha_j a_{i_j}\Big)^{1/n!}.$$

and proved an inequality similar to (5).

THEOREM 8 *With the above notations,*

$$\mathfrak{G}_n(\underline{a}) \le \mathfrak{G}_{n,\underline{\alpha}}(\underline{a}) \le \mathfrak{A}_n(\underline{a}). \tag{11}$$

□ The left-hand inequality in (11) follows from (6) by multiplying all these inequalities over all permutations. Also by (GA)

$$\mathfrak{G}_{n,\underline{\alpha}}(\underline{a}) \le \frac{1}{n!} \sum_n ! \sum_{j=1}^{n} \alpha_j a_{i_j} = \mathfrak{A}_n(\underline{a}),$$

which gives the right-hand inequality and completes the proof. □

REMARK (viii) In the same reference Bartoš & Znám have defined, for $\underline{\alpha}$ with $|\underline{\alpha}| = 1$,

$$\mathfrak{G}_n^{\underline{\alpha}}(\underline{a}) = \Big(\prod_{j=1}^{n} \sum_{i=0}^{n-1} \alpha_i a_{j+i-1}\Big)^{1/n}; \qquad \mathfrak{A}_n^{\underline{\alpha}}(\underline{a}) = \frac{1}{n} \sum_{j=1}^{n} \prod_{i=0}^{n-1} a_{j+i-1}^{\alpha_i};$$

where for $k > n$, $a_k = a_{k-n}$. Using arguments similar to those in the previous remark they have proved that

$$\mathfrak{G}_n(\underline{a}) \le \mathfrak{A}_n^{\underline{\alpha}}(\underline{a}) \le \mathfrak{A}_n(\underline{a}), \text{ and } \mathfrak{G}_n(\underline{a}) \le \mathfrak{G}_n^{\underline{\alpha}}(\underline{a}) \le \mathfrak{A}_n(\underline{a}). \tag{12}$$

However in this case some of the inequalities can become equalities even when $\underline{a}$ is not constant.

EXAMPLE (iii) Consider the case: $a_i = i$, $1 \le i \le 3, a_4 = 2$, $2\alpha_1 = \alpha_2 = 2\alpha_3 = \alpha_4 = \frac{1}{3}$ when $\mathfrak{G}_4^{\underline{\alpha}}(\underline{a}) = \mathfrak{A}_4(\underline{a}) = 2$; while with the same $\underline{\alpha}$ with $2a_1 = a_2 = a_3 = 2a_4 = 2$, $1 \le i \le 3, a_4 = 2$ we get $\mathfrak{A}_4^{\underline{\alpha}}(\underline{a}) = \mathfrak{G}_4(\underline{a}) = \sqrt{2}$.

EXAMPLE (iv) Letting $a_i = i$, $1 \le i \le n$, $\alpha_1 = \alpha_2 = \frac{1}{2}, \alpha_i = 0$, $3 \le i \le n$ in the first inequality in (12) leads to:

$$\frac{2^{2n}}{n+1} \le \binom{2n}{n} \le \frac{(2n+2)^n}{(n+1)!}.$$

REMARK (ix) A further quantity similar to one in Remark (viii) has been studied in [*Djoković & Mitrović*]; see also [*AI pp 284–285,3.6.46*], [*Mitrinović & Djoković*], [*Mijalković & Mitrović*].

REMARK (x) An interesting generalization of Muirhead means occurs in [*Rado*]. Other generalizations have been discussed; for instance we could consider:

$$\mathfrak{m}_n^{[r]}(\underline{a};\underline{\alpha}) = \frac{1}{r!}\sum_r ! \prod_{j=1}^{r} a_{i_j}^{\alpha_j}, \qquad \mathfrak{A}_{n,\underline{\alpha}}^{[r]}(\underline{a}) = \left(\frac{1}{\binom{n}{r}}\mathfrak{m}_n^{[r]}(\underline{a};\underline{\alpha})\right)^{\frac{1}{|\underline{\alpha}|}}.$$

See [*Beck 1977; Bekišev; Brenner 1978; Hering 1972,1973; Schönwald*].

7 Further Generalizations

7.1 THE HAMY MEANS The following slightly different means have been considered by Hamy and others; see [*Smith p.440, ex.38*], [*Hamy; Hara, Uchiyama & Takahasi; Ku, Ku & Zhang 1997; Ness 1964*].
Let $\underline{a}$ be an n-tuple and r an integer, $1 \le r \le n$, then the *Hamy mean of order r of $\underline{a}$* is:

$$(\mathfrak{H}\mathfrak{a})_n^{[r]}(\underline{a}) = \frac{1}{r!\binom{n}{r}}\sum_r ! \big(\prod_{j=1}^{r} a_{i_j}\big)^{1/r}.$$

Since $(\mathfrak{H}\mathfrak{a})_n^{[n]}(\underline{a}) = \mathfrak{G}_n(\underline{a})$, and $(\mathfrak{H}\mathfrak{a})_n^{[1]}(\underline{a}) = \mathfrak{A}_n(\underline{a})$ the following result that contains an analogue of S(r;s) shows that these means form yet another scale of comparable means, between the geometric and arithmetic means.

THEOREM 1 *(a) Let $\underline{a}$ be an n-tuple and r an integer, $1 \le r \le n$, then*

$$\mathfrak{S}_n^{[r]}(\underline{a}) \le (\mathfrak{H}\mathfrak{a})_n^{[r]}(\underline{a}),$$

with equality if and only if either $r = 1$, or $r = n$ or $\underline{a}$ is constant.

(b) Let $\underline{a}$ be an n-tuple and r and s integers, $1 \le r < s \le n$, then

$$(\mathfrak{H}\mathfrak{a})_n^{[s]}(\underline{a}) \le (\mathfrak{H}\mathfrak{a})_n^{[r]}(\underline{a}),$$

with equality if and only if $\underline{a}$ is constant.

□ (a) Immediate from (r;s).
(b) (*i*) The first proof is simple and direct and is that of Hamy.
If we assume that $\underline{a}$ is not constant it is sufficient to prove

$$\sum_{q+1} ! \big(\prod_{j=1}^{q+1} a_{i_j}\big)^{1/(q+1)} < (n-q)\sum_q ! \big(\prod_{j=1}^{q} a_{i_j}\big)^{1/q}. \tag{1}$$

Consider a typical term in the left-hand side sum.

$$\begin{aligned}
&(a_1a_2\ldots a_{q+1})^{1/(q+1)}\\
&= \left((a_2a_3\ldots a_{q+1})^{1/q}(a_1a_3\ldots a_{q+1})^{1/q}\ldots(a_1a_2\ldots a_q)^{1/q}\right)^{1/(q+1)}\\
&< \frac{(a_2a_3\ldots a_{q+1})^{1/q}+(a_1a_3\ldots a_{q+1})^{1/q}+\ldots+(a_1a_2\ldots a_q)^{1/q}}{q+1}, \text{by (GA)}.
\end{aligned}$$

Applying this argument to each term of the sum on the left-hand side of (1) we find that this sum is less than $(q+1)(q+1)!\binom{n}{q+1}$ terms of the type $\frac{1}{q+1}(\prod_{j=1}^{q} a_{i_j})^{1/q}$. However there are only $q!\binom{n}{q}$ different terms of this type, and by symmetry each occurs equally often— that is $(q+1)(n-q)$ times. This gives (1).

(*ii*) An alternative proof was given in [*Dunkel 1909/10*].

By (r;s), assuming that $\underline{a}$ is not constant,

$$\left((\mathfrak{H}\mathfrak{a})_n^{[q]}(\underline{a})\right)^q = \left(\frac{1}{q!\binom{n}{q}}\sum_q !(\prod_{j=1}^{q} a_{i_j})^{1/q}\right)^q > \left(\frac{1}{q!\binom{n}{q}}\sum_q !(\prod_{j=1}^{q} a_{i_j})^{1/(q+1)}\right)^{q+1}.$$

By S(r;s) applied to $\underline{a}^{1/(q+1)}$, we have that

$$\left(\frac{1}{q!\binom{n}{q}}\sum_q !(\prod_{j=1}^{q} a_{i_j})^{1/(q+1)}\right)^{1/q} > \left(\frac{1}{(q+1)!\binom{n}{q+1}}\sum_{q+1} !(\prod_{j=1}^{q+1} a_{i_j})^{1/(q+1)}\right)^{1/(q+1)}.$$

Combining these two inequalities we get the required result. □

The paper by Ku, Ku & Zhang contains some interesting inequalities including the fact that $\left((\mathfrak{H}\mathfrak{a})_n^{[r]}(\underline{a})\right)^r$ is log-concave.

7.2 THE HAYASHI MEANS Interchanging products and sums in 1(2) lead to the means defined by Hayashi; [*Hayashi*].

$$\mathfrak{T}_n^{[r]}(\underline{a}) = \frac{1}{r}\left(\prod_r !(\sum_{j=1}^{r} a_{i_j})\right)^{\binom{n}{r}}, \ 1 \le r \le n.$$

THEOREM 2 *If* $1 \le r \le s$ *then*

$$\mathfrak{T}_n^{[r]}(\underline{a}) \le \mathfrak{T}_n^{[s]}(\underline{a}),$$

with equality if and only if $\underline{a}$ *is constant.*

□ For a proof the reader is referred to the original paper. □

Since $\mathfrak{T}_n^{[1]}(\underline{a}) = \mathfrak{G}_n(\underline{a})$ and $\mathfrak{T}_n^{[n]}(\underline{a}) = \mathfrak{A}_n(\underline{a})$ this is yet another generalization of (GA).

7.3 THE BIPLANAR MEANS Using the ideas of III 5.2.1 we can follow Gini and others to define what they called the *biplanar combinatorial (p,q) power mean of order (c,d)*; here c, d are integers, $1 \le c, d \le n$; see [*Gini et al*], [*Castellano 1948,1950; Gatti 1956b,1957; Gini 1938; Zappa 1939*].

$$\mathfrak{B}_n^{p,c;q,d}(\underline{a}) = \left(\frac{\binom{n}{d} \sum_c! \prod_{j=1}^c a_{i_j}^p}{\binom{n}{c} \sum_d! \prod_{j=1}^d c a_{i_j}^q} \right)^{1/(pc-qd)} = \left(\frac{\mathfrak{s}_n^{[c]}(\underline{a}^p)}{\mathfrak{s}_n^{[d]}(\underline{a}^q)} \right)^{1/(pc-qd)}.$$

EXAMPLE (i) Clearly $\mathfrak{B}_n^{1,c;0,d} = \mathfrak{S}_n^{[c]}$, and $\mathfrak{B}_n^{p,1;q,1} = \mathfrak{G}_n^{p,q}$.

The following simple theorem is in [*Gini & Zappa*].

THEOREM 3 *With the above notations $\mathfrak{B}_n^{1,d+m;1,d}$ is a decreasing function of d, and $\mathfrak{B}_n^{1,c;1,d}$ is a decreasing function of c*

□ The first part is an immediate consequence of 2 Remark (iii).
For the second part first note that $\mathfrak{B}_n^{1,c;1,d} = \left(\prod_{u=d+1}^c \mathfrak{B}_n^{1,u;1,u-1}\right)^{1/(c-d)}$. So, by the first part $\mathfrak{B}_n^{1,c;1,d}$ is a geometric mean of an increasing sequence.
Hence $\mathfrak{B}_n^{1,c;1,d} \ge \mathfrak{B}_n^{1,c;1,c-1}$.
Further $\mathfrak{B}_n^{1,c;1,d} = (\mathfrak{B}_n^{1,c;1,c-1})^{1/(c-d)} (\mathfrak{B}_n^{1,c-1;11,d})^{1-1/(c-d)}$ showing that $\mathfrak{B}_n^{1,c;1,d}$ is a geometric mean of the two means on the right-hand side. As we have just seen it is not less than the first of these two means and so, by the internality of the geometric mean, II 1.2 Theorem 5, it cannot exceed the second.

□

REMARK (i) The proof of the second part of Theorem 3 shows that

$$\mathfrak{B}_n^{1,c;1,c-1} \le \mathfrak{B}_n^{1,c;1,d} \le \mathfrak{B}_n^{1,c-1;1,d},$$

and it is easily deduced that if $k > 0$, $c < d$ then $\mathfrak{B}_n^{1,c+k;1,c-1} \le \mathfrak{B}_n^{1,c;1,d} \le \mathfrak{B}_n^{1,c-1;1,d}$.

REMARK (ii) A similar extension that is related to the Gini means, III 5.2.1, has been discussed in [*Jecklin 1948b; Jecklin & Eisenring*].

7.4 THE HYPERGEOMETRIC MEAN Let $\underline{a}$ and $\underline{b}$ be n-tuples, $\underline{w}'$ an $(n-1)$-tuple; let E be the set of $\underline{w}'$ with the sum of the elements, W_{n-1}, less than 1. For each $\underline{w}' \in E$ let $\underline{w} = (\underline{w}', 1-W_{n-1})$, so that $W_n = 1$. Then for $p \in \mathbb{R}$ the *hypergeometric R-function* is

$$R(p, \underline{b}, \underline{a}) = \int_E \Big(\sum_{i=1}^n w_i a_i \Big)^{-p} P(\underline{b}, \underline{w})\, \mathrm{d}\underline{w}';$$

$P(\underline{b}, \underline{w}) = \dfrac{(B_n - 1)!}{\prod_{i=1}^{n}(b_i - 1)!} \displaystyle\prod_{i=1}^{n} w_i^{b_i - 1}$, is the *weight function*, and $\int_E P(\underline{b}, \underline{w})\, \mathrm{d}\underline{w}' = 1$.
The hypergeometric R-function has a close connection with the theory of means. A *(homogeneous) hypergeometric mean of* $\underline{a}$ is constructed as follows; see [*Carlson 1965,1966*].
If $c > 0$,

$$\mathfrak{C}(p, c; \underline{a}, \underline{w}) = \begin{cases} \left(R(-p, c\underline{w}, \underline{a})\right)^{1/p}, & \text{if } p \neq 0, \\ \lim_{p \to 0} \mathfrak{C}(p, c; \underline{a}, \underline{w}), & \text{if } p = 0. \end{cases}$$

The following result is in [*Carlson & Tobey*].

THEOREM 4 *(a)* $\lim_{c \to 0} \mathfrak{C}(p, c; \underline{a}, \underline{w}) = \mathfrak{M}_n^{[p]}(\underline{a}; \underline{w})$.
(b) If $\underline{a}$ *is not constant and* $p > 1, c > 0$ *then*

$$\mathfrak{C}(p, c; \underline{a}, \underline{w}) < \mathfrak{M}_n^{[p]}(\underline{a}; \underline{w}). \tag{2}$$

(c) If $\underline{a}$ *is not constant and* $p > 1$ *and* $c < c'$ *then*

$$\mathfrak{C}(p, c; \underline{a}, \underline{w}) < \mathfrak{C}(p, c'; \underline{a}, \underline{w}). \tag{3}$$

If $p < 1$ *then inequalities (~2) and (~3) hold.*

REMARK (i) Carlson has pointed out that the Whiteley means are special cases of the hypergeometric mean, and that the following generating relation is valid.

$$\prod_{i=1}^{n} (1 - ta_i)^{-cw_i} = \sum_{n=0}^{\infty} \frac{c(c+1)\dots(c+n-1)}{n!} R(-n, c\underline{w}, \underline{a}) t^n.$$

REMARK (ii) For further extensions see [*Tobey*].

VI OTHER TOPICS

This chapter will cover a variety of topics that do not fit into the previous discussions. In particular there are two variable means, means defined for pairs of numbers and which do not not readily generalize to n-tuples. There is an elementary introduction to integral means and to matrix analogues of mean inequalities. The topic of axiomatization of means is discussed but only briefly as the topic leads away from the interest of this book into the theory of functional equations and functional inequalities; [*Aczél 1966*].

1 Integral Means and Their Inequalities

1.1 GENERALITIES The extension of the concept of a mean to functions by the use of integrals is a natural one. In most calculus texts the quantity $\frac{1}{b-a}\int_a^b f$ is called the *average, or arithmetic mean, of f on the interval* $[a, b]$, and this is justified by noting that the approximating Riemann sums are the arithmetic means of values of f at points distributed across $[a, b]$. In the usual notation,

$$\frac{1}{b-a}\sum_{i=1}^{n} f(y_i)(x_i - x_{i-1}) = \mathfrak{A}_n(\underline{a}; \underline{w}), \tag{1}$$

where $a_i = f(y_i)$, $w_i = (x_i - x_{i-1})$, $1 \le i \le n$.

Further developments of this idea use concepts outside the scope of this book as the natural setting is that of general measure spaces. However while most books on measure theory discuss certain inequalities, in particular integral analogues of (H), (J) and (M). they do not usually consider means.

If we restrict ourselves to real-valued functions on an interval the approach in (1) can be used to extend (J). Then the definition of a quasi-arithmetic mean is obtained by analogy with the discrete case, and this leads to the basic mean inequalities. If the integrals used are of the Stieltjes type these integral results may imply the discrete results as special cases. This method does not yield the cases of equality, but these can be obtained under wide assumptions by the methods in [*HLP p.151*]; see also [*Julia*], [*Mitrinović & Vasić 1968a*], and 1.3.1 below.

REMARK (i) Another approach that incudes both the integral and discrete inequalities as special cases is the use of *time scales*; see [*Agarwal, Bohner & Peterson*].

More precisely: if $\mu : I = [a, b] \mapsto \mathbb{R}$ is an increasing, left-continuous function[1] with $\mu(I) = \mu(b) - \mu(a) > 0$[2] then we say that $f : [a, b] \mapsto \mathbb{R}$ is *μ-integrable on* $[a, b]$ if there is an $\mathrm{I} \in \mathbb{R}$ such that for all $\epsilon > 0$ there is a $\delta : [a, b] \mapsto \mathbb{R}_+^*$ such that for all δ-fine partitions $\varpi = (a_0, a_1, \ldots, a_n; y_1, \ldots, y_n)$ such that

$$|\sum_{\varpi} f \mathrm{d}\mu - \mathrm{I}| < \epsilon, \text{ where } \sum_{\varpi} f \mathrm{d}\mu = \sum_{i=1}^{n} f(y_i)(\mu(a_i) - \mu(a_{i-1})). \tag{2}$$

Then $\mathrm{I} = \mathrm{I}(f)$ is the *μ-integral of f on $I = [a, b]$*. written $\mathrm{I} = \int_I f \,\mathrm{d}\mu = \int_a^b f \,\mathrm{d}\mu$, or $\int_a^b f(x)\mu(\mathrm{d}x)$.[3]

The exact integral defined depends on what is meant by a δ-fine partition; in all cases $a = a_0 < a_1 < \cdots < a_n = b$; if then $[a_{i-1}, a_i] \subset]y_1 - \delta(y_i), y_1 + \delta(y_i)[$, $1 \le i \le n$ the integral is the *Lebesgue-Stieltjes integral*, if in addition $y_i \in [a_{i-1}, a_i]$, $1 \le i \le n$, it is the *Perron-Stieltjes integral*, and if in this case δ is a constant, or just continuous, the integral is the *Riemann-Stieltjes integral*.

CONVENTION In the case $\mu(x) = x$, $a \le x \le b$, when the measure is Lebesgue measure, the integral is written $\int_a^b f$, or $\int_a^b f(x)\,\mathrm{d}x$, and references to both μ and Stieltjes are omitted from all associated notations.

If $\mu(I) = 1$ then the measure is called a *probability measure (on I)*.

For details see [*Bartle; Lee & Výborný*] where it is shown that the ideas extend easily to unbounded intervals.

Which form of integration used does not matter, and in the case of μ-almost everywhere non-negative functions the more general Perron-Stieltjes integral is equivalent to the Lebesgue-Stieltjes integral. Since in considering inequalities we always require that f be μ-integrable and also that $|f|$ be μ-integrable this will mean that our μ-integrals will be Lebesgue-Stieltjes integrals[4], and we write $f \in \mathcal{L}_\mu(a, b)$.

If $p \in \mathbb{R}_+^*$ we write $f \in \mathcal{L}_\mu^p(a, b)$ if $|f|^p \in \mathcal{L}_\mu(a, b)$ and $\mathcal{L}_\mu^1(a, b) = \mathcal{L}_\mu(a, b)$. We complete this definition by introducing the following quantities: *the μ–essential*

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[1] These restrictions are convenient but are not necessary; see [*Bartle pp.391–399*]. In fact much more general measures can be considered; [*Rudin pp.60–63*], [*Roselli & Willem*].

[2] Note that this implies that $0 \le \mu < \infty$. This corresponds to the assumption of non-negative weights in previous chapters and is only waived for 1.2.1 Theorem 2 below.

[3] It will be convenient to allow an integral to be infinite when the integrand is non-negative almost everywhere; see [*Lieb & Loss p.15*].

[4] However more general integrals have been used even though the hypotheses are shown to imply Lebesgue integrability; see [*Ostaszewski & Sochacki*].

upper bound of $f = \inf\{\beta;\ \mu\{x;\ f(x) > \beta\} = 0\}$, and *the* μ*–essential lower bound of* $f = \sup\{\beta;\ \mu\{x;\ f(x) < \beta\} = 0\}$; f is said to be μ*-essentially bounded*, $f \in \mathcal{L}^{\infty}_{\mu}(a,b)$, if both of these quantities are finite.
In all these classes the functions are finite μ-almost everywhere.
As usual if $p \in \mathbb{R}^*$ then p' is the conjugate index of p; Notations 4. Further we can, by considering the function $1/f$, allow $p < 0$, but we now must assume that $f \neq 0$ μ-almost everywhere.

1.2 Basic Theorems

1.2.1 Jensen, Hölder, Cauchy and Minkowski Inequalities The following inequality is an integral analogue of (J), and will be referred to as (J)-$\int$. There are many proofs in the standard literature; the one below is from [*Lieb & Loss pp.38-39*]; see also [*HLP pp.150–151; PPT pp.45–47*].

Theorem 1 [Jensen's Inequality] *If $f \in \mathcal{L}_{\mu}(a,b)$, and if Φ is convex and μ-integrable on some interval $]c,d[$ containing the range of f then*

$$\Phi\Big(\frac{1}{\mu(I)}\int_a^b f\,\mathrm{d}\mu\Big) \le \frac{1}{\mu(I)}\int_a^b \Phi\circ f\,\mathrm{d}\mu. \qquad (\ J)\text{-}\textstyle\int$$

If Φ is strictly convex then (J)-$\int$ is strict unless f is μ-almost everywhere constant.

□ That $A = \dfrac{1}{\mu(I)}\displaystyle\int_a^b f\,\mathrm{d}\mu$ is in the domain of Φ is a simple consequence of obvious properties of integrals, and the fact that the domain of Φ is as stated. Further since Φ is continuous the function $\Phi\circ f$ is μ-measurable.
By I 4.1 Corollary 5 Φ has a support at A; that is for some constant m,

$$\Phi(y) \ge \Phi(A) + m(y - A),\ c < y < d. \qquad (3)$$

From (3) we see that $(\Phi\circ f)^-(x) \le |\Phi(A)| + |m||A| + |m||f(x)|$, so $(\Phi\circ f)^- \in \mathcal{L}_{\mu}(a,b)$.
If right-hand side in (J)-$\int$ is infinite the result is trivial so we can assume that $\Phi\circ f \in \mathcal{L}_{\mu}(a,b)$.
Now take $y = f(x)$ in (3) to get

$$\Phi\circ f(x) \ge \Phi\Big(\frac{1}{\mu(I)}\int_a^b f\,\mathrm{d}\mu\Big) + m\bigg(f(x) - \frac{1}{\mu(I)}\int_a^b f\,\mathrm{d}\mu\bigg), \qquad (4)$$

which on integrating gives (J)-$\int$.
If Φ is strictly convex then (3) is strict except when $y = A$. However if f is not constant μ-almost everywhere then (4) must be strict on a set of positive μ-measure, and so (J)-$\int$ is strict. □

REMARK (i) The result also follows by applying (J) to the Riemann sums in (2), and taking limits, as in the discussion of (1) but this does not give the case of equality. An example of this procedure is given below in 1.3.1 Theorem 9.

REMARK (ii) As was suggested in the preliminary remarks Theorem 1 can be considered in more general settings; see [*PPT pp.43–65*], [*Abramovich, Mond & Pečarić 1997; Beesack & Pečarić; Roselli & Willem*].

REMARK (iii) If Φ'' 0 then (J)-$\int$ is strict unless is -almost everywhere constant; see [*HLP p.51*].

An integral analogue of the Jensen-Steffensen inequality, I 4.3 Theorem 20, can also be proved; [*AI p.109;, MPF p.13*]; [*Boas 1970; Fink & Jodeit 1990*].
A measure , not necessarily non-negative, on [ ] is said to be *end-positive* if

$$([\quad]) \quad 0 \quad \text{and for all} \quad \leq \ \leq \quad ([\quad]) \geq 0 \quad \text{and} \quad ([\quad]) \geq 0$$

This is a generalization of I 4.3 Remark(i).

THEOREM 2 [JENSEN-STEFFENSEN INEQUALITY] *If is an end-positive measure on* [ ], $\in \mathcal{L}_\mu($ $)$ *is monotonic, and if* Φ *is convex and -integrable on some interval containing the range of then (J)-$\int$ holds.*

THEOREM 3 *(a)* [HÖLDER'S INEQUALITY] *If* 1 : [ ] $\mapsto \mathbb{R}$ *are -almost everywhere non-negative, with* $\in \mathcal{L}^p_\mu($ $)$, *and* $\in \mathcal{L}^{p'}_\mu($ $)$, *then* $\in \mathcal{L}_\mu($ $)$, *and*

$$\int_a^b \quad \mathrm{d} \ \leq \Big(\int_a^b \ {}^p\mathrm{d} \ \Big)^{1/p}\Big(\int_a^b \ {}^{p'}\mathrm{d} \ \Big)^{1/p'}; \tag{(H)-$\int$}$$

if is not zero -almost everywhere there is equality if and only if for some $\in \mathbb{R}$ *we have that* ${}^p = \quad {}^{p'}$ *-almost everywhere.*
(b) [CAUCHY'S INEQUALITY] *If* : [ ] $\mapsto \mathbb{R}$ *are -almost everywhere non-negative, with* $\in \mathcal{L}^2_\mu($ $)$, *then* $\in \mathcal{L}_\mu($ $)$, *and*

$$\int_a^b \quad \mathrm{d} \ \leq \Big(\int_a^b \ {}^2\mathrm{d} \ \Big)^{1/2}\Big(\int_a^b \ {}^2\mathrm{d} \ \Big)^{1/2}; \tag{(C)-$\int$}$$

if is not zero -almost everywhere there is equality if and only if for some $\in \mathbb{R}$ *we have that* $=$ *-almost everywhere.*
(c) [MINKOWSKI'S INEQUALITY] *If* 1, *and if* $\in \mathcal{L}^p_\mu($ $)$ *are -almost everywhere non-negative, then* $(\ + \ ) \in \mathcal{L}^p_\mu($ $)$, *and*

$$\Big(\int_a^b (\ + \ )^p\mathrm{d} \ \Big)^{1/p} \leq \Big(\int_a^b \ {}^p\mathrm{d} \ \Big)^{1/p} + \Big(\int_a^b \ {}^p\mathrm{d} \ \Big)^{1/p}; \tag{(M)-$\int$}$$

if f is not zero μ-almost everywhere there is equality if and only if for some $\lambda \in \mathbb{R}$ we have that $f = \lambda g$ μ-almost everywhere.

□ (a) Since the result is trivial if either f or g is μ-almost everywhere zero we may suppose that both are μ-almost everywhere positive. Then we can use proof (i) of III 2.1 Theorem 1 by considering,

$$\begin{aligned}\frac{fg}{\left(\int_a^b f^p \,\mathrm{d}\mu\right)^{1/p}\left(\int_a^b g^{p'} \,\mathrm{d}\mu\right)^{1/p'}} &= \Big(\frac{f^p}{\int_a^b f^p \,\mathrm{d}\mu}\Big)^{1/p}\Big(\frac{g^{p'}}{\int_a^b g^{p'} \,\mathrm{d}\mu}\Big)^{1/p'}\\ &\le \frac{1}{p}\Big(\frac{f^p}{\int_a^b f^p \,\mathrm{d}\mu}\Big) + \frac{1}{p'}\Big(\frac{g^{p'}}{\int_a^b g^{p'} \,\mathrm{d}\mu}\Big) \text{ by (GA).}\end{aligned}$$

This implies that fg is μ-integrable, and integrating both sides gives (H)-$\int$. The case of equality follows from that of (GA).

(b) This is just the case $p = p' = 2$ of (a); the integral form of (C) is due to Bunyakovskiĭ, [*Bunyakovskiĭ*].

(c) The deduction of (M) from (H), III 2.4 Theorem 9 proof (i), can be adapted using (H)-$\int$. □

Remark (iv) If $p < 1$ then we have ($\sim$H)-$\int$, but the conditions of integrability have to be restated; [*HLP pp.140–141*].

Remark (v) Using (H)-$\int$ it is easy to show that if $-\infty \le r < s \le \infty$ then $f \in \mathcal{L}^s_\mu(a,b)$ implies that $f \in \mathcal{L}^r_\mu(a,b)$; again care must be taken when we allow negative indices.

As in the case of (M) an important use of (M)-$\int$ is the proof of the triangle inequality; see III 2.4 (T). If $f, g, h \in \mathcal{L}^p_\mu(a,b)$, $p > 1$ then,

$$\Big(\int_a^b |f-g|^p \,\mathrm{d}\mu\Big)^{1/p} \le \Big(\int_a^b |f-h|^p \,\mathrm{d}\mu\Big)^{1/p} + \Big(\int_a^b |h-g|^p \,\mathrm{d}\mu\Big)^{1/p}. \qquad \text{(T)-}\textstyle\int$$

This is the essential property for showing that $||f||_{\mu,p} = \Big(\int_a^b |f|^p \,\mathrm{d}\mu\Big)^{1/p}$, $p \ge 1$, is a norm, on the class of functions $\mathcal{L}^p_\mu(a,b)$.[5]

Example (i) The following is a simple inequality for the factorial function is application of (H)-$\int$; [*de la Vallée Poussin*]. If $p > 1$, $[0 < p < 1]$,

$$\Big(\frac{1}{p}\Big)! = \int_0^\infty e^{-x}x^{1/p}\,\mathrm{d}x <, [>], \Big(\int_0^\infty e^{-x}x\,\mathrm{d}x\Big)^{1/p}\Big(\int_0^\infty e^{-x}\,\mathrm{d}x\Big)^{1/p'} = 1.$$

[5] More precisely: a norm on the equivalence classes of μ-almost everywhere equal functions of $\mathcal{L}^p_\mu(a,b)$; [*Lieb & Loss pp.36–37; Rudin p.65*].

1.2.2 MEAN INEQUALITIES Suppose now that $f : I = [a,b] \mapsto \mathbb{R}$, and that $\mathcal{M}$ is a strictly monotonic function defined on an interval containing the range of f, with $\mathcal{M} \circ f \in \mathcal{L}_\mu(a,b)$ then the *quasi-arithmetic $\mathcal{M}$-mean, or just quasi-arithmetic mean, of f on $[a,b]$ with weight μ* is:

$$\mathfrak{M}_{[a,b]}(f;\mu) = \mathcal{M}^{-1}\Big(\frac{1}{\mu(I)}\int_a^b \mathcal{M}\circ f \,\mathrm{d}\mu\Big); \tag{5}$$

and when convenient we will define $\mathfrak{M}_{[a,a]}(f;\mu) = f(a)$. There is no loss in generality in assuming $\mathcal{M}$ to be strictly increasing; see IV 1.2 Remark (ii)

REMARK (i) The various usages associated with IV 1.1 Definition 1 will be used without further comments; in particular if μ is Lebesgue measure it is omitted from the notation. In addition if $\mu(\mathrm{d}x) = m(x)\mathrm{d}x$ we will write $\mathfrak{M}_{[a,b]}(f;\mu) = \mathfrak{M}_{[a,b]}(f;m)$.

REMARK (ii) These means can be defined on more general measure spaces, in particular for non-compact intervals, but that will not be pursued here; see most of the references and in particular [*Losonczi 1977*]. In addition the Bajraktarević means, IV 7.1, have been studied in their integral form; [*Gigante 1995b*].

REMARK (iii) The quasi-arithmetic $\mathcal{M}$-mean sometimes occurs with a different notation; see [*HLP p.158*] where $f : \mathbb{R} \mapsto \mathbb{R}$, $\int_{-\infty}^{\infty} \mathrm{d}\mu = 1$, and $\mathfrak{M}(f;\mu) = \mathcal{M}^{-1}\Big(\int_{-\infty}^{\infty}\mathcal{M}\circ f\,\mathrm{d}\mu\Big)$. .

Using the methods of IV 2 Theorem 5 the following result follows from Theorem 1.

THEOREM 4 *Let $\mathcal{M}$ and $\mathcal{N}$ be strictly monotonic functions on an interval that contains the range of $f : I = [a,b] \mapsto \mathbb{R}$, $\mathcal{N}$ strictly increasing, [respectively decreasing], then for all f with $\mathcal{M}\circ f$, $\mathcal{N}\circ f \in \mathcal{L}_\mu(a,b)$,*

$$\mathfrak{M}_{[a,b]}(f;\mu) \le \mathfrak{N}_{[a,b]}(f;\mu) \tag{6}$$

if and only if $\mathcal{N}$ is convex, [respectively concave], with respect to $\mathcal{M}$. Further if $\mathcal{N}$ is strictly convex, [respectively, strictly concave], with respect to $\mathcal{M}$ then (6) is strict unless f is μ-almost everywhere constant.
If $\mathcal{N}$ is decreasing, [respectively, increasing], and $\mathcal{N}$ is convex, [respectively, concave], with respect to M then ($\sim$6) holds.

REMARK (iv) If $(\mathcal{N}\circ\mathcal{M}^{-1})'' > 0$ then (6) is strict unless f is μ-almost everywhere constant; see 1.2.1 Remark (iii).

Taking $\mathcal{M}(x) = x^r$, $r \in \mathbb{R}^*$, assuming that $f \geq 0$ μ-almost everywhere, $f > 0$ μ-almost everywhere if $r < 0$, and $f \in \mathcal{L}^r_\mu(a,b)$, (5) defines the *$r$-th power mean of f on $[a,b]$ with weight μ*

$$\mathfrak{M}^{[r]}_{[a,b]}(f;\mu) = \Big(\frac{1}{\mu(I)}\int_a^b f^r \,\mathrm{d}\mu\Big)^{1/r};$$

If $r = 0$ and $\log\circ f \in \mathcal{L}_\mu(a,b)$ then (5) defines the mean $\mathfrak{M}^{[0]}_{[a,b]}(f;\mu)$. The case $r = 0$ is also the limit of the general case on letting $r \to 0$; see[*HLP 136–139*]. The particular cases $r = -1, 0, 1, 2$ are as expected called the *harmonic, geometric, arithmetic, and quadratic means of f on $[a,b]$, with weight μ* respectively, written:

$$\mathfrak{H}_{[a,b]}(f;\mu) = \frac{\mu(I)}{\int_a^b 1/f \,\mathrm{d}\mu}; \quad \mathfrak{G}_{[a,b]}(f;\mu) = \exp\Big(\frac{1}{\mu(I)}\int_a^b \log\circ f \,\mathrm{d}\mu\Big);$$

$$\mathfrak{A}_{[a,b]}(f;\mu) = \Big(\frac{1}{\mu(I)}\int_a^b f \,\mathrm{d}\mu\Big); \quad \mathfrak{Q}_{[a,b]}(f;\mu) = \sqrt{\frac{1}{\mu(I)}\int_a^b f^2 \,\mathrm{d}\mu};$$

reference should be made to [*HLP pp.136–137*] for the exact interpretation of $\mathfrak{G}_{[a,b]}(f;\mu)$. Letting $r \to \pm\infty$ we get

$$\mathfrak{M}^{[-\infty]}_{[a,b]}(f;\mu) = \mu\text{–essential lower bound of } f \text{ on } [a,b];$$
$$\mathfrak{M}^{[\infty]}_{[a,b]}(f;\mu) = \mu\text{–essential upper bound of } f \text{ on } [a,b].$$

If μ is Lebesgue measure we drop it from the notations, in accordance with the Convention in 1 above; this corresponds to the equal weight case for discrete means.

Remark (v) These means have most of the properties of the analogous discrete means, as is easily seen. For instant the arithmetic mean has the appropriate properties, (Re), (Ho), (Ad), listed in II 1.1 Theorem 2. The properties (In) and (Mo) are given below in Theorem 6 (a), (b).

Remark (vi) Another useful property is the analogue of III 2.3 Lemma 6(b); the function $m(r) = m_f(r) = \big(\mathfrak{M}^{[r]}_{[a,b]}(f;\mu)\big)^r$, $r \in \mathbb{R}$, is log-convex, strictly unless f is μ-almost everywhere constant; [*HLP pp.145–146*].

The following simple example of invariance property, see II 5.2, V 2 Corollary 5, is often useful; see [*AI p.9*]

Theorem 5 *If f is strictly increasing on $[a,b]$ then $\mathfrak{A}_{[x,y]}(f;\mu)$, $a \leq x < y \leq b$, is strictly increasing in both x and y.*

□ Suppose that $a \leq x < y < z \leq b$, and let $I = [x,y], J = [y,z], K = [x,z]$, then simple calculations show that

$$\mathfrak{A}_{[x,z]}(f;\mu) = \frac{\mu(I)}{\mu(K)}\mathfrak{A}_{[x,y]}(f;\mu) + \frac{\mu(J)}{\mu(K)}\mathfrak{A}_{[y,z]}(f;\mu).$$

Thus the integral arithmetic mean on the left-hand side is also an arithmetic mean of the two integral arithmetic means on the right-hand side. So the integral arithmetic mean on the left lies between the two integral arithmetic means on the right, and if f is non-negative and increasing we get from the internality of the discrete arithmetic mean, that,

$$\mathfrak{A}_{[x,y]}(f;\mu) \le \mathfrak{A}_{[x,z]}(f;\mu) \le \mathfrak{A}_{[y,z]}(f;\mu).$$

This implies the required result. □

REMARK (vii) A simple corollary of this result is that integral power means have the same property.

THEOREM 6 *(a)* [IN] *If* $-\infty < r < \infty$ *then*

$$\mu\text{–essential lower bound of } f \le \mathfrak{M}^{[r]}_{[a,b]}(f;\mu) \le \mu\text{–essential lower bound of } f. \quad (7)$$

(b) [MO] *If* $f \le g$ *μ-almost everywhere then*

$$\mathfrak{M}^{[r]}_{[a,b]}(f;\mu) \le \mathfrak{M}^{[r]}_{[a,b]}(g;\mu).$$

(c)[(r;s)] If $-\infty \le r < s \le \infty$ *then,*

$$\mathfrak{M}^{[r]}_{[a,b]}(f;\mu) \le \mathfrak{M}^{[s]}_{[a,b]}(f;\mu), \qquad (r;\ s)\text{-}\textstyle\int$$

(d) [(GA)]

$$\mathfrak{H}_{[a,b]}(f;\mu) \le \mathfrak{G}_{[a,b]}(f;\mu) \le \mathfrak{A}_{[a,b]}(f;\mu), \qquad \text{(GA)-}\textstyle\int$$

In (a), (c) and (d) there is equality if and only if f is μ-almost everywhere constant.

□ (a) and (b) follow from simple properties of the integral.
(c) (*i*) Taking particular functions for $\mathcal{M}$ and $\mathcal{N}$ we get (r;s)-$\int$ for finite r and s from (6); this is completely analogous to the discrete case, see IV 2 Example (v). The other cases are included in (7).
(*ii*) Of course there are many independent proofs of (r;s)-$\int$, $-\infty < r < s < \infty$. Consider for instance the following; [*Qi 1999b; Qi, Mei, Xia & Xu*]. Let $I = [a,b]$ and define, for $t \ne 0$, $\psi(t) = \log\left(\mathfrak{M}^{[t]}_{[a,b]}(f;\mu)\right)$. Then

$$\begin{aligned}\psi(t) =& \frac{\log\left(\int_a^b f^t\,\mathrm{d}\mu\right) - \log(\mu(I))}{t} = \frac{\left(\log\int_a^b f^t\,\mathrm{d}\mu\right) - \log\left(\int_a^b f^0\,\mathrm{d}\mu\right)}{t}\\ =& \frac{1}{t}\int_0^t \left(\frac{\int_a^b f^s \log\circ f\,\mathrm{d}\mu}{\int_a^b f^s\,\mathrm{d}\mu}\right)\mathrm{d}s,\end{aligned}$$

by interchanging differentiation and integration, [*McShane pp.216–217*].

By Theorem 6(b) the result follows if we can show that the integrand in the last integral, $\xi(s) = \dfrac{\int_a^b f^s \log\circ f \,\mathrm{d}\mu}{\int_a^b f^s \,\mathrm{d}\mu}$, is increasing. Now, again interchanging differentiation and integration,

$$\xi'(s) = \frac{\int_a^b f^s (\log\circ f)^2 \,\mathrm{d}\mu \int_a^b f^s \,\mathrm{d}\mu - \left(\int_a^b f^s \log\circ f \,\mathrm{d}\mu\right)^2}{\left(\int_a^b f^s \,\mathrm{d}\mu\right)^2},$$

which is non-negative by (C)-$\int$. This shows that ξ is increasing and completes the proof.

(d) Clearly (GA)-$\int$ is a special case of (r; s)-$\int$. □

REMARK (viii) Inequality (7) can be generalized in a manner due to Cauchy, see II 1.2 Lemma 4 (a).

$$\mathfrak{M}^{[-\infty]}\Big(\frac{f}{g};\mu\Big) \le \frac{\mathfrak{M}(f;\mu)}{\mathfrak{M}(g;\mu)} \le \mathfrak{M}^{[\infty]}\Big(\frac{f}{g};\mu\Big). \tag{8}$$

These inequalities have been studied and generalized by several authors; [*Karamata; Pólya & Szegö 1972 pp.80,90*], [*Godunova 1972; Karamata; Kwon & Shon; Lukkassen; Lupaş 1975,1978; Pečarić & Savić; Sándor & Toader; Winckler 1860*].

REMARK (ix) The proof (*ii*) of (c) can be modified to give a proof of the discrete version of (r;s).

An interesting inequality compares the geometric means of the arithmetic means of the derivatives of two functions with the arithmetic mean of the derivative of their geometric mean.

THEOREM 7 *Let $f, g, h : [a,b] \to \mathbb{R}_+^*$, with f increasing, g, h continuously differentiable and $g(a) = h(a), g(b) = h(b)$ then*

$$\mathfrak{G}_2\big(\mathfrak{A}_{[a,b]}(g';f), \mathfrak{A}_{[a,b]}(h';f)\big) \le \mathfrak{A}_{[a,b]}\big(\mathfrak{G}'(g,h);f\big).$$

REMARK (x) The case $a = 0, b = 1$, $g(x) = x^{2p+1}$, $h(x) = x^{2q+1}$ is due to Pólya; the generalization is by Alzer. Further generalizations in which the geometric mean has been replaced by other means such as the Gini or Extended means, see 2.1.3 below, have also been given; [*DI p.206*][6], [*Pólya & Szegö p.72*], [*Alzer 1990p; Pearce & Pečarić 1998*].

[6] This reference has a mistake; the derivative is taken inside the square root, instead of outside.

REMARK (xi) A study of the iteration of the arithmetic mean has been made by Bückner; [*Bückner*]. Nanjundiah and Ky Fan type inequalities have also been studied; [*Rassias pp.27–50*], [*Čižmešija & Pečarić; Kwon 1996*].

An extremely important extension of the integral power means to analytic functions is due to Hardy. If f is analytic in $\{z;\ |z| = |\rho e^{i\theta}| < 1\}$ and if $0 < \rho < 1, 0 \leq r \leq \infty$, then we define[7]

$$\mathfrak{M}^{[r]}(f;\rho) = \begin{cases} \exp\left(\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \log^+ |f(\rho e^{i\theta})|\,\mathrm{d}\theta\right), & \text{if } r = 0, \\ \left(\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |f(\rho e^{i\theta})|^r\,\mathrm{d}\theta\right)^{1/r}, & \text{if } 0 < r < \infty, \\ \sup_{-\pi\leq\theta\leq\pi} |f(\rho e^{i\theta})|, & \text{if } r = \infty. \end{cases}$$

These means clearly have all the properties of the power means above, having analogues of (H)-$\int$, (M)-$\int$, and (r;s)-$\int$. In addition the following important theorem can be proved.

THEOREM 8 *$\mathfrak{M}^{[r]}(f;\rho)$ is an increasing function of ρ, $0 \leq \rho < 1$.*

□ The case $r = \infty$ follows from the maximum modulus theorem for analytic functions; for the other cases see [*Rudin p.330*]. □

These properties form the basis of an important area of mathematics known as *Hardy spaces*; see [*EM4, pp.366–369*].

1.3 FURTHER RESULTS

1.3.1 A GENERAL RESULT

A procedure whereby integral inequalities are deduced from discrete inequalities can be stated in a fairly general way.

THEOREM 9 *Let $\psi : \mathbb{R}^3 \mapsto \mathbb{R}$ be continuous, J an interval, $J \subseteq \mathbb{R}, F, G : J \mapsto \mathbb{R}$, $H : J \times J \mapsto \mathbb{R}$, $\underline{a}, \underline{b}$ n-tuples with elements in J, $\underline{w}$ an n-tuple, $\lambda \in \mathbb{R}$. If for all such $\underline{a}, \underline{b}, \underline{w}$,*

$$\psi\left(\sum_{i=1}^{n} w_i F(a_i), \sum_{i=1}^{n} w_i G(b_i), \sum_{i=1}^{n} w_i H(a_i, b_i)\right) \geq \lambda, \tag{9}$$

then for all $f, g : [a,b] \mapsto J$ with $F \circ f, G \circ g, H(f,g)$ μ-integrable

$$\psi\left(\int_a^b F \circ f\,\mathrm{d}\mu, \int_a^b G \circ g\,\mathrm{d}\mu, \int_a^b H(f,g)\,\mathrm{d}\mu\right) \geq \lambda. \tag{10}$$

[7] Using the notation of Notations 8; $\log^+ = \max\{\log, 0\}$.

□ If $\eta > 0$, and $\underline{u} \in \mathbb{R}^3$ then there is an $\epsilon > 0$ such that if $|\underline{u} - \underline{v}| < \epsilon$, then $|\psi(\underline{u}) - \psi(\underline{v})| < \eta$.

Now there is a $\delta : [a,b] \mapsto \mathbb{R}_+^*$ such that for all δ-fine partitions ϖ of $[a,b]$,

$$\left|\sum_{\varpi} F \circ f \mathrm{d}\mu - \int_a^b F \circ f \,\mathrm{d}\mu\right| < \frac{\epsilon}{\sqrt{3}}, \quad \left|\sum_{\varpi} G \circ g \mathrm{d}\mu - \int_a^b G \circ g \,\mathrm{d}\mu\right| < \frac{\epsilon}{\sqrt{3}},$$

$$\left|\sum_{\varpi} H(f,g) \mathrm{d}\mu - \int_a^b H(f,g) \,\mathrm{d}\mu\right| < \frac{\epsilon}{\sqrt{3}}.$$

Hence

$$\begin{aligned}\psi\Big(\int_a^b F \circ f \,\mathrm{d}\mu, \int_a^b G \circ g \,\mathrm{d}\mu, \int_a^b H(f,g) \,\mathrm{d}\mu\Big) \\ &\geq \psi\left(\sum_{i=1}^n w_i F(a_i), \sum_{i=1}^n w_i G(b_i), \sum_{i=1}^n w_i H(a_i,b_i)\right) - \eta \\ &\geq \lambda - \eta,\end{aligned}$$

which completes the proof. □

Remark (i) This result can be given several obvious variants and extensions to functions ψ of two variables, or of more than three variables.

Example (i) Taking $\psi(x,y,z) = x^{1/p}y^{1/p'} - z$, $F(x) = x^p$, $G(x) = x^{p'}$, $H(x,y) = xy, \lambda = 0$, then (9) is (H) and (10) is (H)-$\int$.

Example (ii) Taking $\psi(x,y,z) = xy$, $F(x) = x^p$, $G(x) = \dfrac{1}{x}, f = g$ and $\lambda = ((M+m)^2)/4Mm$, where $0 < m \leq f \leq M$; then ($\sim$ 9) is the Kantorovič, or Schweitzer, inequality, III 4.1 (11), (12), and ($\sim$10) is the integral analogue

$$\Big(\int_a^b f \,\mathrm{d}\mu\Big)\Big(\int_a^b \frac{1}{f} \,\mathrm{d}\mu\Big) \leq \frac{(M+m)^2}{4Mm}. \tag{11}$$

Remark (ii) A completely different proof of (11), also a special case of a general method, can be found in the interesting paper [*Rennie 1963*].

Remark (iii) A general integral approach to the Kantorovič inequality can be found in [*Clausing 1982*]; see also [*Diaz & Metcalf 1964b; Lupaş & Hidaka*].

1.3.2 Beckenbach's Inequality; Beckenbach-Lorentz Inequality Analogues of these inequalities, III 2.5.5 Theorem 17 and III 2.5.7 Theorem 19 (c), have been proved directly; [*Mond & Pečarić 1995b; Zhuang 1993*].

THEOREM 10 *Suppose $p > 1$, $f \in \mathcal{L}^p_\mu(a,b)$, with f μ-almost everywhere non-negative, and $g \in \mathcal{L}^{p'}_\mu(a,b)$, with g μ-almost everywhere non-negative, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and define h by $h = (\alpha g/\beta)^{p'/p}$. Then*

$$\frac{\left(\alpha+\gamma\int_a^b f^p\,\mathrm{d}\mu\right)^{1/p}}{\beta+\gamma\int_a^b fg\,\mathrm{d}\mu} \ge \frac{\left(\alpha+\gamma\int_a^b h^p\,\mathrm{d}\mu\right)^{1/p}}{\beta+\gamma\int_a^b hg\,\mathrm{d}\mu}, \tag{12}$$

with equality if and only if $f = h$, μ almost everywhere. If $0 < p < 1$ then (~12) holds.

□ Obviously $h \in \mathcal{L}^p_\mu(a,b)$, and the right-hand side of (12) is just:

$$\frac{(\alpha/\beta)^{p'/p}\Big(\alpha(\beta/\alpha)^{p'}+\gamma\int_a^b g^{p'}\,\mathrm{d}\mu\Big)^{1/p}}{(\alpha/\beta)^{p'/p}\Big(\alpha(\beta/\alpha)^{p'}+\gamma\int_a^b g^{p'}\,\mathrm{d}\mu\Big)} = \Big(\alpha^{-p'/p}\beta^{p'}+\gamma\int_a^b g^{p'}\,\mathrm{d}\mu\Big)^{-1/p'}. \tag{13}$$

Now:

$$\begin{aligned}\beta+\gamma\int_a^b fg\,\mathrm{d}\mu \le&\beta+\gamma\Big(\int_a^b f^p\,\mathrm{d}\mu\Big)^{1/p}\Big(\int_a^b g^{p'}\,\mathrm{d}\mu\Big)^{1/p'}, \text{ by (H)-}\textstyle\int,\\ =&\alpha^{1/p}(\beta\alpha^{-1/p})+\Big(\gamma\int_a^b f^p\,\mathrm{d}\mu\Big)^{1/p}\Big(\gamma\int_a^b g^{p'}\,\mathrm{d}\mu\Big)^{1/p'}\\ \le&\Big(\alpha+\gamma\int_a^b f^p\,\mathrm{d}\mu\Big)^{1/p}\Big(\alpha^{-p'/p}\beta^{p'}+\gamma\int_a^b g^{p'}\,\mathrm{d}\mu\Big)^{1/p'}, \text{ by (H)},\end{aligned}$$

which by (13) completes the proof of (11).
The cases of equality follow from the cases of equality for (H)-$\int$ and (H). □

THEOREM 11 *Let $p, f, g, h, \alpha, \beta, \gamma$ be as in the previous theorem . Further assume that $\alpha-\gamma\int_a^b f^p\,\mathrm{d}\mu > 0$ and $\alpha-\gamma\int_a^b h^p\,\mathrm{d}\mu > 0$. Then*

$$\frac{\left(\alpha-\gamma\int_a^b f^p\,\mathrm{d}\mu\right)^{1/p}}{\beta-\gamma\int_a^b fg\,\mathrm{d}\mu} \le \frac{\left(\alpha-\gamma\int_a^b h^p\,\mathrm{d}\mu\right)^{1/p}}{\beta-\gamma\int_a^b hg\,\mathrm{d}\mu}, \tag{14}$$

with equality if and only if $f = h$, μ almost everywhere.

□ As in the proof of Theorem 10, $h \in \mathcal{L}^p_\mu(a,b)$, and we have that the right-hand side of (14) is just $\Big(\alpha^{-p'/p}\beta^{p'}-\gamma\int_a^b g^{p'}\,\mathrm{d}\mu\Big)^{-1/p'}$.
By III 2.5.7 Theorem 19(a), the Hölder-Lorentz inequality, with $n = 2$, $a_i = \alpha^{1/p}$, $a_2 = \left(\gamma\int_a^b f^p\,\mathrm{d}\mu\right)^{1/p}$, $b_1 = \alpha^{-1/p}\beta$, $b_2 = \left(\gamma\int_a^b g^{p'}\,\mathrm{d}\mu\right)^{1/p'}$ we have

$$\begin{aligned}&\Big(\alpha-\gamma\int_a^b f^p\,\mathrm{d}\mu\Big)^{1/p}\Big(\alpha^{p'/p}\beta^{p'}-\gamma\int_a^b g^{p'}\,\mathrm{d}\mu\Big)^{1/p'}\\ &\qquad\le\beta-\gamma\Big(\int_a^b f^p\,\mathrm{d}\mu\Big)^{1/p}\Big(\int_a^b g^{p'}\,\mathrm{d}\mu\Big)^{1/p'}\\ &\qquad\le\beta-\gamma\int_a^b fg\,\mathrm{d}\mu, \quad \text{by (H)-}\textstyle\int.\end{aligned}$$

As in Theorem 10 this completes the proof. □

1.3.3 CONVERSE INEQUALITIES Following the general remarks above converse inequalities can be obtained from the discrete analogues but considerable work has been done on direct approaches; [*Alzer 1991h; Barnes 1969; Choi; D'Apuzzo & Sbordone; He L; Kwon 1995; Mond & Pečarić 1994; Nehari; Pečarić & Pearce; Yang & Teng; Zhuang 1991*].

The following converse of (J)-$\int$ goes back to Knopp and should be compared to I 4.4 Theorem 28; [*Popoviciu pp.33–35*], [*Knopp 1935; Pečarić & Beesack 1987b*].

THEOREM 12 *Let $\Phi : [m, M] \mapsto \mathbb{R}$ be convex and strictly monotonic, and let $g : [0,1] \mapsto [m, M]$ then*

$$\int_0^1 \Phi \circ g - \Phi\Big(\int_0^1 g\Big) \le (1 - t_0)\Phi(m) + t_0\Phi(M) - \Phi(\overline{1 - t_0 m} + t_0 M),$$

where $\overline{1 - t_0 m} + t_0 M$ is the mean-value point for Φ on $[m, M]$.

A particularly simple example of a converse Hölder inequality is given in the next theorem; [*Zhuang 1993*].

THEOREM 13 *If $f, g : [a, b] \mapsto \mathbb{R}$ with $0 \le m_1 \le f \le M_1$, $0 \le m_2 \le g \le M_2$, with f^p and $g^{p'}$ μ-integrable, $p > 1$, then*

$$\Big(\int_a^b f^p \,\mathrm{d}\mu\Big)^{1/p}\Big(\int_a^b g^{p'} \,\mathrm{d}\mu\Big)^{1/p'} \le K \int_a^b fg \,\mathrm{d}\mu, \tag{15}$$

where $K = \max\left\{\dfrac{m_1^p/p + M_2^{p'}/p'}{m_1 M_2}, \dfrac{M_1^p/p + m_2^{p'}/p'}{M_1 m_2}\right\}$.

□

$$\begin{aligned} K\int_a^b fg\,\mathrm{d}\mu \ge& \int_a^b \Big(\frac{1}{p}f^p + \frac{1}{p'}g^{p'}\Big)\,\mathrm{d}\mu, \text{ by II 4.2 Theorem 7,}\\ =&\frac{1}{p}\int_a^b f^p\,\mathrm{d}\mu + \frac{1}{p'}\int_a^b g^{p'}\,\mathrm{d}\mu \ge \text{the left-hand side of (15), by (GA).}\end{aligned}$$

□

A converse of Beckenbach's inequality, 1.3.2 Theorem 10, has also been given by Zhuang.

1.3.4 RYFF 'S INEQUALITY In a very interesting paper Ryff, [*Ryff*], has discussed an integral analogue of Muirhead's theorem, V 6 Theorem 1, and of V 6 Corollary 2 (a).

For simplicity we consider real-valued functions on $[0,1]$, and first we define an integral analogue to the order $\prec$ defined in I 3.3; see [*AI pp.162–168; DI pp.9,198; HLP pp.276–279; MO p.15; PPT pp.324–325*].
If f and g are two decreasing functions we say the $f \prec g$ when

$$\int_0^x f \le \int_0^x g,\ 0 \le x \le 1, \qquad \text{and} \qquad \int_0^1 f = \int_0^1 g. \tag{16}$$

In general $f \prec g$ if (16) holds between the decreasing rearrangements of f and g, where the *decreasing rearrangement of* a function f, written f^*, is defined by requiring: (i) f^* is decreasing and right continuous, (ii) the sets $\{x;\ f(x) > y\}$ and $\{x;\ f^*(x) > y\}$ have the same measure for all y.
Various of the results for the order defined in I.3.3 have analogues in this situation; in particular I 4.1 Theorem 10(a), and the following result of Ryff that is an analogue of Muirhead's theorem.

THEOREM 14 [RYFF'S INEQUALITY] *Let f, g be bounded and measurable on $[0,1]$. If $f \prec g$, and if u is a positive function with u^p integrable for all $p \in \mathbb{R}$, then*

$$\int_0^1 \log\Big(\int_0^1 u(y)^{f(x)}\,\mathrm{d}y\Big)\,\mathrm{d}x \le \int_0^1 \log\Big(\int_0^1 u(y)^{g(x)}\,\mathrm{d}y\Big)\,\mathrm{d}y. \tag{17}$$

Conversely if (17) holds for all such u then $f \prec g$.

REMARK (i) The argument that (17) is the correct analogue of V 6(3) is given in detail in the paper of Ryff. If the order of integration in (17) is reversed the integrals may fail to exist.

REMARK (ii) The question of the cases of equality in (17) remains open. Ryff conjectures that there is equality if and only if either u is constant almost everywhere, or $f^* = g^*$.

1.3.5 BEST POSSIBLE INEQUALITIES An inequality between two quantities P, Q of the form $P \le KQ$ is usually understood to be best possible when K, a constant depending on the parameters in P and Q, has been evaluated, and for no smaller constant does the inequality hold in general. In interesting papers Fink and Jodheit have suggested an alternative way of considering an inequality to be best possible; [*Fink 1994; Fink & Jodheit 1984,1990*].
This idea is best explained by the inequality of Karamata, I 4.1 Theorem 10(a): $\sum_{i=1}^n f(b_i) \le \sum_{i=1}^n f(a_i)$ *for all functions f convex on I if and only if* $\underline{b} \prec \underline{a}$. Fink pointed out that the following is also true: $\sum_{i=1}^n f(b_i) \le \sum_{i=1}^n f(a_i)$ *holds for all* $\underline{a}, \underline{b}$ *with* $\underline{b} \prec \underline{a}$ *if and only if f is convex*; [*Fink 1994*],

In other words Karamata's theorem not only characterizes the sequences for which the inequality holds for all convex functions, but also it characterizes the class of convex functions. We illustrate this idea further with two results, an integral analogue of Čebišev's inequality, see II 5.3, and (J)-$\int$.

We say that *two functions defined on the interval* $[a, b]$ *are similarly ordered when* $\big(f(x)-f(y)\big)\big(g(x)-g(y)\big) \geq 0$ for all $x, y \in [a, b]$; this is analogous to I 3.3 Definition 15.

In the next two theorems μ is a regular Borel measure[8], and the notation in Theorem 16 is that of 1.2.1 Theorem 1.

THEOREM 15 [ČEBIŠEV'S INEQUALITY] *If f, g are piecewise continuous and non-negative μ-almost everywhere on $[a, b]$, then the inequality*

$$\int_a^b fg\,\mathrm{d}\mu \int_a^b \mathrm{d}\mu \geq \int_a^b f\,\mathrm{d}\mu \int_a^b g\,\mathrm{d}\mu$$

holds for all similarly ordered pairs of functions f and g if and only if $\mu \geq 0$. Further this inequality holds for all $\mu \geq 0$. if and only if the functions f and g are similarly ordered.

THEOREM 16 [JENSEN'S INEQUALITY] *(J)-$\int$ holds for all $f \in \mathcal{L}_\mu^\infty(a, b)$ and Φ convex if and only if $\mu \geq 0$. Further (J)-$\int$ holds for all $f \in \mathcal{L}_\mu^\infty(a, b)$ and $\mu \geq 0$ if and only if Φ is convex.*

□ If Φ is convex and $\mu \geq 0$ then we have (J)-$\int$, 1.2.1 Theorem 1. Now assume (J)-$\int$, for all Φ is convex and $f \in \mathcal{L}_\mu^\infty(a, b)$. Take $f = 1_{[c,d]}$, $[c, d] \subseteq [a, b]$ and $\Phi(x) = x^2$ then by (J)-$\int$, $\mu\big([c, d]\big) \geq 0$, which shows that $\mu \geq 0$.

Finally taking $f(x) = x$ and μ a convex combination of point masses at x and y, then by (J)-$\int$ we find that Φ is convex.□

REMARK (i) As we have seen, 1.2.1 Theorem 2, the Jensen-Steffensen inequality, (J)-$\int$ will hold for other measures if the class of functions is changed to the class of monotonic functions.

1.3.6 OTHER RESULTS (a) [AN INEQUALITY OF OSTROWSKI] If $f : [a, b] \mapsto \mathbb{R}$ is Lipschitz with Lipschitz constant M, then Ostrowski has proved that

$$|f(x) - \mathfrak{A}_{[a,b]}(f)| \leq M(b-a)\left(\frac{1}{4} + \left(\frac{x-(a+b)/2}{(b-a)}\right)^2\right),$$

8 A measure is a *Borel measure* if all Borel sets are measurable; it is *regular* if the measure of every set can be approximated from the inside by the measures of its closed subsets. If the function μ of 1.1 above is continuous the measure is both regular and Borel; [*EM1 p.437*].

and many authors have given extensions; [*PPT p.209*], [*Dragomir & Rassias*], [*Lupaş 1972; Milovanović & Milovanović 1979; Milovanović & Pečarić; Ostrowski; Pečaric & Savić*].

(b) [GAUSS'S INEQUALITY]If $r > 0$ and μ is a non-negative measure then the *rth absolute moment of the measure* μ, $r > 0$, $\nu_r = \int_{-\infty}^{\infty} |x-a|^r \mu(\mathrm{d}x)$. So the classical inequality $\nu_r^{1/r} \leq \nu_s^{1/s}, 0 < r < s$, is a particular case of (r,s)-$\int$. Gauss stated the following improvement, in the case $r = 2, s = 4$,

$$\big((r+1)\nu_r\big)^{1/r} \leq \big((s+1)\nu_s\big)^{1/s}.$$

The first proof of the general inequality was given by Winckler although his proof contained an error that was corrected by Faber; see [*MPF pp.53–55; PPT pp.219–220*], [*Gauss*], [*Bernstein & Kraft; Fujiwara; Mitrinović & Pečarić 1986; Winckler 1866*].

The following closely related result can be found in [*BB pp.43–44; PPT p.216*].

THEOREM 17 *If* $f : [a,b] \mapsto \mathbb{R}^+$ *is convex with* $f(a) = 0$, *and if* $0 < r < s$, *then* $(r+1)^{1/r}\mathfrak{M}_{[a,b]}^{[r]}(f) \leq (s+1)^{1/s}\mathfrak{M}_{[a,b]}^{[s]}(f)$.

(c) [STEFFENSEN'S INEQUALITY] The following is a very well known result, [*MPF pp.311–331*], [*Steffensen 1918*].

THEOREM 18 *If* $f, g : [a,b] \mapsto \mathbb{R}$ *are integrable,* f *decreasing,* $0 \leq g \leq 1$ *and if* $\lambda = \int_a^b g$ *then,*

$$\int_{b-\lambda}^{a} f \leq \int_a^b fg \leq \int_{b-\lambda}^{a+\lambda} f.$$

This inequality, [*AI p.107; DI pp.239–240; PPT pp.181–182*], has been the subject of much study and perhaps the most extensive generalization is the following theorem of Mitrinović & Pečarić, [*Mitrinović & Pečarić 1988a*]; see also [*PPT pp.182–195*], [*Pečarić 1989*].

THEOREM 19 *Let* $\lambda > 0, \int_a^b g > 0$, *then the inequalities*

$$\frac{1}{\lambda}\int_{b-\lambda}^{b} f \leq \frac{\int_a^b fg}{\int_a^b g} \leq \frac{1}{\lambda}\int_a^{a+\lambda} f \tag{18}$$

are valid for all decreasing f *if and only if for every* x, $a \leq x \leq b$,

$$0 \leq \lambda \int_x^b g \leq (b-x)\int_a^b g, \quad \textit{and} \quad 0 \leq \lambda \int_a^x g \leq (x-a)\int_a^b g.$$

Further the second inequality in (18) holds if and only if for every x, $a \leq x \leq b$, $0 \leq \lambda \int_a^x g \leq (x-a)\int_a^b g$, *and* $0 \leq \int_x^b g$.

The above authors have noted that the above theorem implies the following result; [*Godunova, Levin & Čebaevskaya*].

COROLLARY 20 *Let f, g be non-negative in $[a,b]$, f decreasing, g increasing; let ϕ be convex and increasing on $[0,\infty[$, with $\phi(0)=0$. Define the function g_1 by $g_1\phi(g/g_1)=1$, with $g_1\ge 0$, and suppose that $\int_a^b g_1 \le 1$. Then if $\lambda=\phi(\int_a^b g)$,*

$$\frac{1}{\int_a^b g}\int_a^b fg \le \phi^{-1}\Big(\frac{1}{\lambda}\int_a^\lambda \phi\circ f\Big).$$

(d) The following inequality is due to Prékopa; [*Prékopa*].

$$\Big(\int_{-\infty}^{\infty} f^2\Big)^{1/2}\Big(\int_{-\infty}^{\infty} g^2\Big)^{1/2} \le \frac{1}{2}\int_{-\infty}^{\infty} \sup_{x+y=t}\big(f(x)g(x)\big)\,\mathrm{d}t. \tag{19}$$

REMARK (i) Leindler has noted that the discrete analogue of (19) is false, but is valid if the factor $1/2$ is omitted. He has also proved an integral analogue of his result III 4.3 Remark (viii), as well as various generalizations of (19); [*Leindler 1971,1972b,1973a*].

(e) THEOREM 21 *If f is convex on $[a,b]$ then $A(x,y)=\mathfrak{A}_{[x,y]}(f)$, $a\le x,y\le b$, is Schur convex on $[a,b]\times[a,b]$.*

□ This is an immediate consequence of I 4.8 Theorem 48 and the right-hand side of Hadamard-Hermite inequality, I 4.1 (4). □

REMARK (ii) The converse of this theorem holds in that the Schur convexity of the function A implies that f is convex. Further the left-hand side of I 4.1(4) is an easy corollary of this result; [*Elezović & Pečarić 2000a*].

(f) Finally we mention a result of Seiffert and Alzer connecting the arithmetic mean of a function on an interval and the function values at the identric and logarithmic means of the end-points of the interval, [*Alzer 1989ℓ; Seiffert 1989*].

THEOREM 22 *If $0<a<b$ and $f:[a,b]\mapsto\mathbb{R}$ the following inequalities hold,*

$$f\big(\mathfrak{L}(a,b)\big)\le \mathfrak{A}_{[a,b]}(f)\le f\big(\mathfrak{I}(a,b)\big),$$

provided: f is strictly increasing and (a) for the left inequality $1/f^{-1}$ is convex, (b) for the right inequality $f^{-1}$is log-convex.

2 Two Variable Means

There are many means that are defined for 2-tuples that have no obvious extensions to general n-tuples. Further a case has been made that amongst the many means on 2-tuples that we will consider below the only ones with natural extensions to

n-tuples are the classical power means; [*Lorenzen*]. Finally it might be mentioned that two-variable cases of means defined earlier have many special applications; see for instance a use of two variable quasi-arithmetic means in [*Wimp 1986*].

2.1 The Generalized Logarithmic and Extended Means

2.1.1 The Generalized Logarithmic Means

If $p \in \overline{\mathbb{R}}$, $a > 0, b > 0$, $a \neq b$, the *generalized logarithmic mean of order p of a and b*, is:

$$\mathfrak{L}^{[p]}(a,b) = \begin{cases} \left(\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{1/p}, & \text{if } p \neq -1, 0, \pm\infty, \\ \dfrac{b-a}{\log b - \log a}, & \text{if } p = -1, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & \text{if } p = 0, \\ \max\{a,b\}, & \text{if } p = \infty, \\ \min\{a,b\}, & \text{if } p = -\infty; \end{cases} \tag{1}$$

the definition is completed by defining $\mathfrak{L}^{[p]}(a,a) = a, a > 0$, $p \in \overline{\mathbb{R}}$. Clearly these means are homogeneous and symmetric; and so in particular there is no loss in generality in assuming $0 < a < b$.

Example (i) It is easily checked that the special cases in the above definition are limits of the general case. That is: if $p_0 = -\infty, -1, 0$, or ∞ then $\mathfrak{L}^{[p_0]}(a,b) = \lim_{p\to p_0} \mathfrak{L}^{[p]}(a,b)$; and for all p, $\mathfrak{L}^{[p]}(a,a) = \lim_{b\to a} \mathfrak{L}^{[p]}(a,b)$.

Example (ii) Special cases of these means are means introduced earlier, II 1.1, 1.2, 5.5 and III 1: $\mathfrak{L}^{[-2]}(a,b) = \mathfrak{G}(a,b)$, $\mathfrak{L}^{[-1]}(a,b) = \mathfrak{L}(a,b)$, $\mathfrak{L}^{[0]}(a,b) = \mathfrak{I}(a,b)$, $\mathfrak{L}^{[-1/2]}(a,b) = \mathfrak{M}^{[1/2]}(a,b)$, $\mathfrak{L}^{[1]}(a,b) = \mathfrak{A}(a,b)$.

Remark (i) In some references $\mathfrak{L}^{[p]}(a,b)$ is called $\mathfrak{L}^{[p+1]}(a,b)$; the present usage seems more natural because of (2).

Remark (ii) These means have been redefined many times; see [*Cisbani; Dodd 1932; Galvani*].

Various identities exist between certain generalized logarithmic means and other means.

Example (iii) $\mathfrak{L}^{[-3]}(a,b) = \sqrt[3]{\mathfrak{H}(a,b)\mathfrak{G}^2(a,b)}$, $\quad \mathfrak{L}^{[-1/2]}(a,b) = \dfrac{\mathfrak{A}(a,b) + \mathfrak{G}(a,b)}{2}$, and $\mathfrak{L}(a,b) = \mathfrak{G}^2(a,b)\mathfrak{L}\left(\dfrac{1}{a}, \dfrac{1}{b}\right)$.

Example (iv) For no value of p do we have that $\mathfrak{L}^{[p]}(a,b) = \mathfrak{H}(a,b)$ as it is easily seen that if $\mathfrak{L}^{[p]}(1,2) = \mathfrak{H}(1,2)$ then $\mathfrak{L}^{[p]}(1,3) \neq \mathfrak{H}(1,3)$; see 2.1.3 Corollary 20(c).

These means have several useful integral representations.

LEMMA 1 *(a) If $p \in \mathbb{R}$, $0 < a < b$ and $\iota(x) = x$ then*

$$\mathfrak{L}^{[p]}(a,b) = \mathfrak{M}^{[p]}_{[a,b]}(\iota). \tag{2}$$

(b) If $p \in \mathbb{R}^*$, then

$$\mathfrak{L}^{[p]}(a,b) = \exp\left(\frac{1}{p}\int_1^{p+1} \frac{1}{t}\log\left(\mathfrak{I}(a^t,b^t)\right) \mathrm{d}t\right).$$

(c)

$$\mathfrak{L}(a,b) = \left(\int_0^1 \frac{1}{(1-t)a+tb}\,\mathrm{d}t\right)^{-1}. \tag{3}$$

(d)

$$\mathfrak{L}(a,b) = \int_0^1 a^{1-t}b^t\,\mathrm{d}t. \tag{4}$$

□ All of these can be verified by simple integrations. □

REMARK (iii) The central role of the identric mean is seen from (b); see [*Leach & Sholander 1983; Stolarsky 1975*]. The useful integral representation (c) is due to Carlson; see [*Pittenger 1985*]. The representation (d) is due to Neuman, [*Neuman 1994*]; see also [*Bullen 1994a*]

The following theorem generalizes identity (2); [*Yang Z; Cao*]

THEOREM 2 *If $f \in \mathcal{C}^2(a,b)$ is positive and strictly convex on $[a,b]$ then for all $p \in \mathbb{R}$,*

$$\mathfrak{M}^{[p]}_{[a,b]}(f) < \mathfrak{L}^{[p]}\big(f(a), f(b)\big);$$

if f is strictly concave this inequality is reversed.

REMARK (iv) The proof is based on an extension, by Yang, of the Hadamard-Hermite inequality, I 4.1 (4); [*MPF p.12*]. Some errors in the last reference, and in Yang, are corrected in [*Guo & Qi*].

THEOREM 3 *(a) If $0 < a \le b$, and $-\infty \le r < s \le \infty$, then*

$$a \le \mathfrak{L}^{[r]}(a,b) \le \mathfrak{L}^{[s]}(a,b) \le b, \tag{5}$$

with equality if and only if $a = b$. In particular the generalized logarithmic means are strictly internal

(b) For all $p \in \overline{\mathbb{R}}$ the generalized logarithmic mean, $\mathfrak{L}^{[p]}(a,b)$, is strictly increasing both as function of a and b.

□ (a) follows from (r;s)-$\int$ and (2), while (b) is a simple result of (2), 1.2.2 Theorem 5 and 1.2.2 Remark (vii). □

REMARK (v) The cases $r=-2, s=-1; r=-1, s=0; r=0, s=1$ of (a) have been proved earlier; see II 5.5 Theorem 15

REMARK (vi) The special cases $p=-1,0$ of (b) have been given earlier, see II 5.5 Theorem 14.

Using Examples (ii) and (iii) special cases of (5) give various inequalities between the generalized logarithmic means and various means defined earlier.

COROLLARY 4 *(a) If $0 < a \le b$ and $-2 < p < -1/2$ then*

$$\mathfrak{G}(a,b) \le \mathfrak{L}^{[p]}(a,b) \le \mathfrak{M}^{[1/2]}(a,b). \tag{6}$$

(b) If $0 < a \le b$ and $-2 < p < 1$ then

$$\mathfrak{G}(a,b) \le \mathfrak{L}^{[p]}(a,b) \le \mathfrak{A}(a,b). \tag{7}$$

(c) If $0 < a < b$ then

$$\mathfrak{L}(a,b) < \frac{\mathfrak{G}(a,b)+\mathfrak{A}(a,b)}{2} < \mathfrak{I}(a,b).$$

In both (6) and (7) there is equality if and only if $a=b$.

REMARK (vii) In the case $p=-1$ the left-hand inequality in (6) is due to Ostle & Terwillinger; the case of $p=-1$ of (7) was proved by Králik; see [*AI p.273*], [*Mitrinović 1964 p.15,1965 p.192*], [*Králik; Ostle & Terwillinger*].

REMARK (viii) Many proofs of (7) have been given in the special case $p=-1$, the logarithmic mean case; see II 5.5, [*College Math. J., 14 (1983), 353–356; Yang Y 1987*].

REMARK (ix) The result in (c) should be compared with that in II 5.5 Remark (v); see also 2.1.3 Remark (v).

The following lemma shows that when $p=-1$ inequality (7) implies the stronger inequality (6); [*Carlson 1971*]; see also [*Pearce & Pečarić 1997; Sándor 1990c,d*].

LEMMA 5

$$\mathfrak{G}(a,b) \le \mathfrak{L}(a,b) \le \mathfrak{A}(a,b) \Longrightarrow \mathfrak{G}(a,b) \le \mathfrak{L}(a,b) \le \mathfrak{M}^{[1/2]}(a,b)$$

□ Take (7), with $p = -1$, replace a, b by $\sqrt{a}, \sqrt{b}$ respectively, and multiply the resulting inequality by $(\sqrt{a}+\sqrt{b})/2$. This gives

$$\frac{1}{2}\sqrt[4]{ab}\,(\sqrt{a}+\sqrt{b}) \le \mathfrak{L}(a,b) \le \mathfrak{M}^{[1/2]}(a,b). \tag{8}$$

Using (GA) on the left-hand side, (8) implies (6) in the case $p = -1$. □

The idea used in this proof can be iterated to give another definition of the logarithmic mean.

$$(ab)^{1/2^{n+1}} \prod_{i=1}^{n}\left(\frac{a^{1/2^i}+b^{1/2^i}}{2}\right) \le \mathfrak{L}(a,b) \le \frac{a^{1/2^n}+a^{1/2^n}}{2}\prod_{i=1}^{n}\left(\frac{a^{1/2^i}+b^{1/2^i}}{2}\right). \tag{9}$$

it is readily seen that the left-hand term in (9) increases with n, while the right-hand term decreases, and that they have the same limit. Hence

$$\mathfrak{L}(a,b) = \prod_{i=1}^{\infty}\left(\frac{a^{1/2^i}+b^{1/2^i}}{2}\right).$$

The following result can be found in [*Elezović & Pečarić 2000a*]; see also [*Qi, Sándor, Dragomir & Sofo*].

THEOREM 6 *$\mathfrak{L}^{[p]}(a,b)$ is Schur convex if $p > 1$ and Schur concave if $p < 1$.*

□ The function $f(x) = x^p$, $x > 0$ is convex if $p > 1$ or $p < 0$ and so by 1.3.6 Theorem 21 the function $\dfrac{y^{p+1}-x^{p+1}}{(p+1)(y-x)}$ is Schur convex for those ranges of p. Hence since $g(x) = x^{1/p}$ is increasing if $p > 0$ we get, using I 4.8 Theorem 48, that $\mathfrak{L}^{[p]}(a,b)$ is Schur convex if $p > 1$. The other cases, $p \ne 0$, have a similar proof, and the case $p = 0$ follows using limits. □

Inequalities (6) or (7) are quite elementary so it is natural to ask if they can be improved by answering the following question. What are the best possible power mean bounds for a generalized logarithmic mean? The following theorem of Pittenger gives a complete answer; [*Pittenger 1980b*].

THEOREM 7 *Let $0 < a \le b$ and $p \in \mathbb{R}$, and define*

$$p_1 = \begin{cases} \min\{(p+2)/3,\, p\log 2/\log(p+1)\}, & \textit{if } p > -1,\ p \ne 0,\\ \min\{2/3, \log 2\}, & \textit{if } p = 0,\\ \min\{(p+2)/3, 0\}, & \textit{if } p \le -1;\end{cases}$$

and define p_2 *as* p_1 *but with* min *replaced by* max. *Then*

$$\mathfrak{M}^{[p_1]}(a,b) \le \mathfrak{L}^{[p]}(a,b) \le \mathfrak{M}^{[p_2]}(a,b). \tag{10}$$

There is equality in (10) if and only if either $a = b$, *or* $p = 1, -1/2$, *or* -2. *The values of* p_1, p_2 *are sharp.*

REMARK (x) If $-2 < p < -1/2$, or $p > 1$ then $p_2 = (p+2)/3$ and the right-hand inequality in (10) is : $\mathfrak{L}^{[p]}(a,b) \le \mathfrak{M}^{[(p+2)/3]}(a,b)$, a result due to Stolarsky; [*Stolarsky 1980*].

REMARK (xi) The case $p = -1$ of (10),

$$\mathfrak{G}(a,b) \le \mathfrak{L}(a,b) \le \mathfrak{M}^{[1/3]}(a,b), \tag{11}$$

is due to Lin, see [*Lin T; Yan*]. The substitution $b = e^x$, $a = e^{-x}$ in (11) leads the equivalent inequality $(\cosh x/3)^3 > \dfrac{\sinh x}{x} > 1$, which is a sharpening of the left inequality in I 2.2(14). For an interesting approach to Lin's result and the more general inequality of Pittenger see [*Hästö*].

REMARK (xii) If $p \ne -1$ using the substitution of the preceding remark in (10) leads the equivalent inequality:

$$\big(\cosh p_1 x\big)^{1/p_1} \le \left(\frac{\sinh(p+1)x}{(p+1)\sinh x}\right)^{1/p} \le \big(\cosh p_2 x\big)^{1/p_2}.$$

REMARK (xiii) A probabilistic proof of (10) can be found in [*Székely 1974*].

The above result of Pittinger can be expressed in terms of the following concept due to Székeley; [*Székely 1975/6*]. Given a mean, call it $\mathfrak{M}_n(\underline{a})$, define:

$$r = \sup\{s;\ \mathfrak{M}_n^{[s]}(\underline{a}) \le \mathfrak{M}_n(\underline{a}), \text{ for all } n\text{-tuples } \underline{a}\},$$
$$R = \inf\{s;\ \mathfrak{M}_n^{[s]}(\underline{a}) \ge \mathfrak{M}_n(\underline{a}), \text{ for all } n\text{-tuples } \underline{a}\}.$$

Then we say that $\mathfrak{M}_n(\underline{a})$ *is a distance* $R - r$ *from the power means.*;

COROLLARY 8 *The generalized logarithmic mean* $\mathfrak{L}^{[p]}(a,b)$ *is a distance* $p_2 - p_1$ *from the power means.*

REMARK (xiv) For further generalizations of these results see [*MPF pp.40–48*], [*Carlson*], [*Chen & Wang 1990; Imoru 1982; Pearce & Pečarić 1994,1995b; Pittenger 1987*].

Other results have been obtained particularly in the cases of the logarithmic and identric means; [*Allasia, Giodarno & Pečarić; Alzer 1986a,b,1987a,1989j,1993e; Sándor 1991b,1993,1995c,d; Sándor & Raşa*].

THEOREM 9 *(a) If $0 < a < b$ then*

$$\sqrt{\mathfrak{A}(a,b)\mathfrak{G}(a,b)} < \sqrt{\mathfrak{L}(a,b)\mathfrak{I}(a,b)} < \mathfrak{M}^{[1/2]}(a,b);$$
$$\mathfrak{L}(a,b) + \mathfrak{I}(a,b) < \mathfrak{A}(a,b) + \mathfrak{G}(a,b);$$
$$\sqrt{\mathfrak{I}(a,b)\mathfrak{G}(a,b)} < \mathfrak{L}(a,b) < \frac{\mathfrak{G}(a,b)+\mathfrak{I}(a,b)}{2}.$$

(b) If $0 < r < s$ and $0 < a < b$ then $\max_{a\le x,y\le b}\Big(\dfrac{\mathfrak{L}^{[s]}(x,y)}{\mathfrak{L}^{[r]}(x,y)}\Big) = \dfrac{\mathfrak{L}^{[s]}(a,b)}{\mathfrak{L}^{[r]}(a,b)}$.

REMARK (xv) That $\max_{a\le x,y\le b}\big(\mathfrak{L}^{[s]}(x,y) - \mathfrak{L}^{[r]}(x,y)\big) = \mathfrak{L}^{[s]}(a,b) - \mathfrak{L}^{[r]}(a,b)$ is a conjecture of Alzer.

A relation between the logarithmic mean and the Muirhead means has also been investigated.

THEOREM 10 *If $a, b > 0$ and if $\underline{\alpha} = \big((1+\sqrt{\delta})/2, (1-\sqrt{\delta})/2\big)$, $0 \le \delta \le 1/3$ then*

$$\mathfrak{A}_{2,\underline{\alpha}}(a,b) \le \mathfrak{L}(a,b). \tag{12}$$

If $1/3 < \delta < 1$ then (12) fails for some pair a, b.

The case $\delta = 1/4$ is due to Carlson, but the general result was given by Pittenger; [*Carlson 1965; Pittenger 1980a*].

REMARK (xvi) The substitution of Remark (x) gives the following inequality that is equivalent to (12): $x\cosh(t\sqrt{x\delta}) < \sinh x$.

REMARK (xvii) Connections with the arithmetico-geometric mean, see 3.2 below, have been obtained; [*MPF pp.47–48*], [*Carlson & Vuorinen; Sándor 1995a,1996*].

Inequalities of a Ky Fan type have also been proved; [*Alzer 1987d; Chen & Hu; Wang, Chen & Li*].

THEOREM 11 *If $0 < a < b \le \frac{1}{2}$ and if $p_1 = 2/3$, $p_2 = \log(1+\sqrt{5}/2)/\log 2$ then*

$$\frac{\mathfrak{M}^{[p_1]}(a,b)}{\mathfrak{M}^{[p_1]}(1-a,1-b)} < \frac{\mathfrak{I}(a,b)}{\mathfrak{I}(1-a,1-b)} < \frac{\mathfrak{M}^{[p_2]}(a,b)}{\mathfrak{M}^{[p_2]}(1-a,1-b)}.$$

The values p_1, p_2 are best possible.

Some means that can be considered variants of the logarithmic mean have been considered by Seiffert, [*Seiffert 1987,1993,1995a,b,c*]. They have been called, *Seiffert means*, [*Toader 1999*]. If $0 < a < b$ we define:

$$\mathfrak{S}^{(1)}(a,b) = \frac{b-a}{2\arctan\big((b-a)/(b+a)\big)}, \quad \mathfrak{S}^{(2)}(a,b) = \frac{b-a}{2\arcsin\big((b-a)/(b+a)\big)},$$

and

$$\mathfrak{S}^{(3)}(a,b) = \frac{b-a}{4(\arctan\sqrt{ba}-\pi)}$$

THEOREM 12 *If* $0 < a < b$ *are then*

(*a*) $$\mathfrak{G}(a,b) < \mathfrak{S}^{(2)}(a,b) < \mathfrak{A}(a,b) < \mathfrak{S}^{(1)}(a,b) < \mathfrak{Q}(a,b);$$

(*b*) $$\mathfrak{L}(a,b) < \mathfrak{S}^{(3)}(a,b) < \mathfrak{I}(a,b).$$

□ (a) The inequality I 2.2 (13) implies that

$$x \ge \arctan x \ge \sin\circ\arctan x = \frac{x}{\sqrt{1+x^2}},\ x \ge 0.$$

from which the two right-hand inequalities follow.
Inequality I 2.2 (13) also implies that

$$\arcsin x \le \tan\circ\arcsin x = \frac{x}{\sqrt{1-x^2}},\ 0 \le x < 1$$

which gives the inequality on the left.
(b) See the references. □

REMARK (xviii) These means have been generalized; [*Toader 1999*]. Further (b) has been improved to: $\mathfrak{M}^{[1/2]}(a,b) < \mathfrak{S}^{(3)}(a,b) < \mathfrak{M}^{[2/3]}(a,b)$; [*Jager*].

2.1.2 WEIGHTED LOGARITHMIC MEANS OF n-TUPLES Several authors have suggested extensions of the generalized logarithmic means to n-tuples, $n > 2$, even though 2.1.1(1) does not readily suggest such a generalization; [*Dodd 1941b; Horwitz 2000; Králik; Stolarsky 1975*]. However perhaps the most natural generalization is that of the logarithmic mean given by Pittenger based on 2.1.1(3); [*Pittenger 1985*].
Consider the non-negative n-tuples $\underline{w}$ with $w_n = 1 - W_{n-1}$ and let A_{n-1} be the simplex $\{\underline{w}'_n; W_{n-1} \le 1\}$. Define the *logarithmic mean of an n-tuple $\underline{a}$* as:

$$\mathfrak{L}_n(\underline{a}) = \left((n-1)!\int_{A_{n-1}} \frac{1}{\underline{a}.\underline{w}}\,\mathrm{d}\underline{w}\right)^{-1}.$$

It is easy to check that this mean is internal, $\min\underline{a} \le \mathfrak{L}(\underline{a}) \le \max\underline{a}$, and Pittenger shows that:

$$\mathfrak{G}_n(\underline{a}) \le \mathfrak{L}_n(\underline{a}) \le \mathfrak{M}_n^{[1/2]}(\underline{a}), \tag{13}$$

a generalization of the $p=-1$ case of 2.1.1 (6). Further by replacing the Lebesgue measure in this integral by a general probability measure μ we can define a *logarithmic mean of the n-tuple $\underline{a}$ with weight $\underline{w}$, the natural weight of the measure μ:*

$$\mathfrak{L}_n(\underline{a};\underline{w}) = \left(\int_{A_{n-1}} \frac{1}{\underline{a}.\underline{x}}\mu(\mathrm{d}\underline{x})\right)^{-1}, \quad \text{where } w_i = \int_{A_{n-1}} x_i\mu(\mathrm{d}\underline{x}),\ 1 \le i \le n;$$

[*Brenner & Carlson; Pečarić & Šimić*]. A similar extension has been given for the identric mean, [*Sándor & Trif*]; $\mathfrak{I}_n(\underline{a};\underline{w}) = \exp\left(\int_{A_{n-1}} \log(\underline{a}.\underline{x})\mu(\mathrm{d}\underline{x})\right)$.
The following inequalities of Ky Fan type have been proved; [*Gavrea & Trif; Neuman & Sándor; Sándor & Trif*]. (The notation is defined in IV 4.4.)

THEOREM 13 *If μ is a probability measure on A_{n-1} and if the n-tuple $\underline{w}$ is the natural weight of μ and if $\underline{a} \in]0,1/2]^n$ then*

$$\frac{\mathfrak{L}_n(\underline{a};\underline{w})}{\mathfrak{L}'_n(\underline{a};\underline{w})} \le \frac{\mathfrak{I}_n(\underline{a};\underline{w})}{\mathfrak{I}'_n(\underline{a};\mu)},$$

$$\frac{\mathfrak{G}_n(\underline{a};\underline{w})}{\mathfrak{G}'_n(\underline{a};\underline{w})} \le \frac{\mathfrak{I}_n(\underline{a};\underline{w})}{\mathfrak{I}'_n(\underline{a};\underline{w})} \le \frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{A}'_n(\underline{a};\underline{w})},$$

$$\frac{1}{\mathfrak{H}'_n(\underline{a};\underline{w})} - \frac{1}{\mathfrak{H}_n(\underline{a};\underline{w})} \le \frac{1}{\mathfrak{L}'_n(\underline{a};\underline{w})} - \frac{1}{\mathfrak{L}_n(\underline{a};\underline{w})} \le \frac{1}{\mathfrak{A}'_n(\underline{a};\underline{w})} - \frac{1}{\mathfrak{A}_n(\underline{a};\underline{w})}$$

the inequalities being strict unless $\underline{a}$ is constant.

It is not known if the following inequality holds:

$$\frac{1}{\mathfrak{G}'_n(\underline{a};\underline{w})} - \frac{1}{\mathfrak{G}_n(\underline{a};\underline{w})} \le \frac{1}{\mathfrak{L}'_n(\underline{a};\underline{w})} - \frac{1}{\mathfrak{L}_n(\underline{a};\underline{w})},$$

but the inequality $\dfrac{\mathfrak{G}_n(\underline{a};\underline{w})}{\mathfrak{G}'_n(\underline{a};\underline{w})} \le \dfrac{\mathfrak{L}_n(\underline{a};\underline{w})}{\mathfrak{L}'_n(\underline{a};\underline{w})}$ does not hold for all choices of weights, see [*Gavrea & Trif*].
Reference should be made to IV 4.4 Theorem 15 and 4.4 Remark(vii).

Another natural extension of the logarithmic mean has been given by Neuman using the representation 2.1.1(4); :

$$\tilde{\mathfrak{L}}_n(\underline{a};\underline{w}) = \int_{A_{n-1}} \prod_{i=1}^{n} a_i^{x_i}\mu(\mathrm{d}\underline{x}),$$

Neuman proves an analogue of inequality (13), $\tilde{\mathfrak{L}}_n(\underline{a})$ replacing $\mathfrak{L}_n(\underline{a})$; [*Neuman 1994; Pearce, Pečarić & Šimić 1998b*].

THEOREM 14 *If μ is a probability measure on A_{n-1} and if the n-tuple $\underline{w}$ is the natural weight of μ then*

$$\mathfrak{G}_n(\underline{a};\underline{w}) \le \tilde{\mathfrak{L}}(\underline{a};\underline{w}) \le \mathfrak{A}_n(\underline{a};\underline{w}).$$

The inequality is strict if $\underline{a}$ is not constant.

2.1.3 THE EXTENDED MEANS If $r, s \in \mathbb{R}$ and $a, b > 0$, $a \neq b$, then the *extended mean of order r, s of a and b*, is given by:

$$\mathfrak{E}^{r,s}(a,b) = \begin{cases} \left(\dfrac{r(b^s - a^s)}{s(b^r - a^r)}\right)^{1/(s-r)}, & \text{if } r \neq s,\ s \neq 0, \\ \left(\dfrac{b^r - a^r}{r(\log b - \log a)}\right)^{1/r}, & \text{if } r \neq 0, s = 0, \\ \dfrac{1}{e^{1/r}}\left(\dfrac{b^{b^r}}{a^{a^r}}\right)^{1/(b^r - a^r)}, & \text{if } r = s \neq 0, \\ \sqrt{ab}, & \text{if } r = s = 0; \end{cases} \tag{14}$$

the definition is completed by defining $\mathfrak{E}^{[p]}(a,a) = a, a > 0, p \in \mathbb{R}$. These means are readily seen to be homogeneous and symmetric; and further $\mathfrak{E}^{r,s}(a,b) = \mathfrak{E}^{s,r}(a,b)$. So we can always, without loss in generality, assume that $a \le b$, and, or, $r \le s$. The special cases in the above definition follow from the general case by taking appropriate limits.

Extended means, which are sometime called *difference means*, [*Losonczi & Páles 1998*] were introduced by Stolarsky and so are also called *Stolarsky means*[9]; [*Alzer 1990n*]. They were studied later by Leach & Sholander; [*MPF pp.45–46*], [*Kim Y 2000a; Leach & Sholander 1978,1983,1984; Páles 1990a; Qi 2001a,b,2002b; Stolarsky 1975; Yang & Cao*].

There are various simple relations between these means and others introduced earlier.

EXAMPLE (i) $\mathfrak{E}^{r,2r}(a,b) = \mathfrak{M}^{[r]}(a,b), \quad \mathfrak{E}^{-r,r}(a,b) = \mathfrak{G}(a,b), \quad r \in \mathbb{R};$

$$\mathfrak{E}^{1,p+1}(a,b) = \mathfrak{L}^{[p]}(a,b), \quad p \in \mathbb{R};$$

$$\mathfrak{E}^{0,r}(a,b) = \big(\mathfrak{L}(a^r,b^r)\big)^{1/r}, \quad \mathfrak{E}^{r,r}(a,b) = \big(\mathfrak{I}(a^r,b^r)\big)^{1/r}, \quad r \in \mathbb{R}^*.$$

In particular

$$\mathfrak{E}^{1,2}(a,b) = \mathfrak{A}(a,b); \ \mathfrak{E}^{-2,-1}(a,b) = \mathfrak{H}(a,b); \ \mathfrak{E}^{2,4}(a,b) = \mathfrak{Q}(a.b).$$

EXAMPLE (ii) $\mathfrak{E}^{r,s}(a,b)\mathfrak{E}^{-r,-s}(a,b) = \mathfrak{G}^2(a,b), \quad r, s \in \mathbb{R};$

---

[9] Another name is *extended mean-value mean* .

$$\mathfrak{E}^{2r,2s}(a,b) = \sqrt{\mathfrak{E}^{r,s}(a,b)\mathfrak{G}^{r,s}(a,b)},\ r \neq s, r, s \in \mathbb{R}^*.$$

where the last mean is a Gini mean with equal weights, see III 5.2.1. In particular taking $s = r+1$ the last identity gives the Lehmer means, III 5.2.1, in terms of the extended means:

$$\mathfrak{G}^{r,r+1}(a,b) = \mathfrak{H}^{[r+1]}(a,b) = \frac{\left(\mathfrak{E}^{2r,2r+2}(a,b)\right)^2}{\mathfrak{E}^{r,r+1}(a,b)}.$$

LEMMA 15 *(a) If $r, s \in \mathbb{R}$ then*

$$\mathfrak{E}^{r,s}(a,b) = \mathfrak{M}^{[s-r]}_{[0,1]}\left(m^{[r]}\right), \tag{15}$$

where

$$m^{[r]}(t) = \mathfrak{M}^{[r]}(a,b;1-t,t),\ 0 \le t \le 1. \tag{16}$$

(b) If $0 < a, b$ and $r, s \in \mathbb{R}$ then

$$\mathfrak{E}^{r,s}(a,b) = \begin{cases} \left(\dfrac{e(s)}{e(r)}\right)^{1/(s-r)}, & \text{if } r \neq s, \\ \exp\left(\dfrac{e'(r)}{e(r)}\right), & \text{if } r = s, \end{cases}$$

where $e(u) = e(u;a,b) = \int_a^b t^{u-1}\,dt$, when $e'(u) = \int_a^b t^{u-1}\log t\,du$.
(c) If $0 < a < b$ and $r \neq s$,

$$\log\left(\mathfrak{E}^{r,s}(a,b)\right) = \frac{1}{s-r}\int_r^s \frac{e'(u)}{e(u)}\,du, \tag{17}$$

where the function e is as in (b); if $r = s$ then

$$\log\circ\log\left(\mathfrak{E}^{r,r}(a,b)\right) = \int_0^1 \frac{f'(u)}{f(u)}\,du,$$

where $f(v) = \int_a^b t^{r-1}\log^v t\,dt$, when $f'(v) = f(v+1)$.

□ (a) and (b): both of these results are obtained by simple integrations.
(c) If $r \neq s$ we have from (b),

$$\log\left(\mathfrak{E}^{r,s}(a,b)\right) = \frac{1}{s-r}\left(\log e(s) - \log e(r)\right) = \frac{1}{s-r}\int_r^s (\log e(u))'\,du.$$

If $r = s$ again from (b)

$$\log\circ\log\left(\mathfrak{E}^{r,r}(a,b)\right) = \log f(1) - \log f(0) = \int_0^1 (\log f(u))'\,du.$$

□

Remark (i) (a) and (b) are extensions of 2.1.1(2), and formula 2.1.1(4) is a special case of (b); [*Bullen 1994a; Pearce, Pečarić & Šimić 1998a*].

Remark (ii) The representation (15) has been used to generalize these means to define what could be called *extended quasi-arithmetic means*; see [*Pearce, Pečarić & Šimić 1999*].

Remark (iii) Generalizations of the integral form in (b) have been studied; [*Guo & Qi; Qi 1998a,b,2000b; Qi & Luo 1998; Qi, Mei & Xu; Qi, Xu & Debnath; Qi & Zhang ; Sun M*]. For instance if $r \neq s$ and $0 < a < b$, $f > 0$ and $w \geq 0$, but not zero almost everywhere:

$$\mathfrak{E}^{r,s}_{[a,b]}(f; w) = \left(\frac{\int_a^b w(t) f^s(t)\, \mathrm{d}t}{\int_a^b w(t) f^r(t)\, \mathrm{d}t}\right)^{1/(s-r)}.$$

Theorem 16 *(a) The extended mean $\mathfrak{E}^{r,s}(a,b)$ a strictly increasing function of a, b, r and s.*
(b) If $s > 0$, respectively $s < 0$, then $\mathfrak{E}^{r,s}(a,b)$ is strictly log-concave, respectively log-convex, as a function of r. In particular the generalized logarithmic means $\mathfrak{L}^{[p]}(a,b)$ are strictly log-concave functions of p, $p \geq -1$.
(c) If $r \geq 1$ then

$$\mathfrak{E}^{r,r+1}(a+a', b+b') \leq \mathfrak{E}^{r,r+1}(a,b) + \mathfrak{E}^{r,r+1}(a',b').$$

the inequality being reversed if $r \leq 1$.

□ (a) (*i*) We first consider the case of $\mathfrak{E}^{r,s}(a,b)$ as a function of a and b; by symmetry we can restrict attention to b and assume that $b > a$. In addition we assume that $r, s \in \mathbb{R}^*$, $s > r$. Then

$$\left(\frac{\partial (\mathfrak{E}^{r,s})^{s-r}(a,b)}{\partial b}\right)\frac{s(b^r - a^r)^2}{r} = b^{r-1}a^s\big((s-r)(b/a)^s - s(b/a)^{s-r} + r\big)$$

Further putting $\theta = b/a$, $\theta > 1$, consider $f(\theta) = (s-r)\theta^s - s\theta^{s-r} + r$. Clearly $f(1) = 0$ and $f'(\theta) = s(s-r)\theta^{s-r-1}(\theta^r - 1) > 0$.
This shows that $\dfrac{\partial (\mathfrak{E}^{r,s})^{s-r}(a,b)}{\partial b} > 0$, and completes the proof in this case.
The other cases can be handled similarly, or be reduced, by Example (i) to the similar properties for $\mathfrak{L}$ and $\mathfrak{I}$ which have been noted in 2.1.1 Theorem 3(b).
(*ii*) We now consider the case of $\mathfrak{E}^{r,s}(a,b)$ as a function of r and s. If $r_1, s_1, r_2, s_2 \in \mathbb{R}$ with $r_1 \neq s_1$ and $s_1 \neq s_2$ and $r_1 \leq r_2, s_1 \leq s_2$ then

$$\mathfrak{E}^{r_1,s_1}(a,b) \leq \mathfrak{E}^{r_2,s_2}(a,b) \tag{18}$$

by (17) and 1.2.2 Theorem 5. Further if either $r_1 \neq r_2$ or $s_1 \neq s_2$ the inequality is strict.

A very simple proof can also be given noting the fact that by Lemma 15(c)

$$\log \circ \mathfrak{E}^{r,s}(a,b) = \frac{\log \circ m_\iota(s-1) - \log \circ m_\iota(r-1))}{s-r},$$

where $\iota(t) = t$ and the function m_ι is defined in 1.2.2 Remark (vi). By that remark $\log \circ m_\iota$ is strictly convex and (18) is immediate from a basic property of convex functions, I 4.1 Remark (v), and the fact that the logarithmic function is strictly increasing; see [*PPT pp.119–120*].

The other cases follow by taking limits.

Alternatively the case when $r_1 = s_1 = r$ and $s_1 = s_2 = s$, $r < s$, follows from (15) and (r;s)-$\int$.

(b) This property of the extended means is due to Qi; [*Qi 2001*]. The proof depends on showing that the function e'/e, see Lemma 15(b), is strictly concave, or convex. The proof of this is quite complicated and uses techniques of Gould & Mays, see below in Theorem 19. The last remark follows from Example (i).

(c) This inequality is due to Alzer and a proof can be found in [*Alzer 1988d*]. □

Remark (iv) The proof (a)(*ii*) using the log-convexity of m_ι is similar to the proof of the analogous result for Gini means, III 5.2.1 Theorem 6(b); both proofs are found in [*PPT pp.119–120*].

Remark (v) Since log-concave functions are concave, see I 4.5.2 Lemma 32, (b) implies that if $0 < a < b$ then

$$\frac{\mathfrak{L}(a,b) + \mathfrak{A}(a,b)}{2} < \mathfrak{I}(a,b), \text{ and } \mathfrak{L}(a,b) < \frac{2}{3}\mathfrak{G}(a,b) + \frac{1}{3}\mathfrak{A}(a,b)$$

a strengthening of the right-hand inequality of 2.1.1 Corollary 4(c), and a proof of II 5.5 Remark (v).

Remark (vi) The above inequality of Alzer can be regarded as an analogue of (M). In the same paper the author has also given analogues of the Čebišev and Liapunov inequalities; see also [*Alzer 1987c; Czinder & Páles; Losonczi 2002*]. A different generalization can be found in [*Páles 1988d*].

Inequality (18) has been improved by Leach & Sholander, [*MPF pp. 44–45*], [*Alzer 1989j; Czinder & Páles; Leach & Sholander 1983; Losonczi 2002; Páles 1988d*].

Theorem 17 *If r_1, s_1, r_2, s_2 are all of the same sign, with $r_1 \neq s_1$ and $s_1 \neq s_2$ then*

$$\mathfrak{E}^{r_1,s_1}(a,b) \le \mathfrak{E}^{r_2,s_2}(a,b)$$

for all positive a and b if and only if $r_1 + s_1 \le s_1 + s_2$ and $\ell(r_1, s_1) \le \ell(r_2, s_2)$, where

$$\ell(u,v) = \begin{cases} \mathfrak{L}(u,v), & \text{if } u > 0 \text{ and } v > 0,\ u \neq v, \\ 0. & \text{if } uv = 0. \end{cases}$$

If $\min\{r_1, s_1, r_2, s_2\} < 0 < \max\{r_1, s_1, r_2, s_2\}$ *the same result holds if* $\ell(u,v) = (|u| - |v|)/(u - v)$, $u \neq v$.

Example (iii) An easy deduction from the analogous theorem for the generalizations in Remark (ii) is that $(s!/r!)^{1/s-r}$ increases with both r and s; [*Qi 2002a*].

The following is a generalization of the right-hand side of the Hadamard-Hermite inequality, I 4.1(4); [*Pearce & Pečarić 1996; Pearce, Pečarić & Šimić 1998b*].

Theorem 18 *If the function f is positive and r-mean convex on $[a,b]$ then*

$$\mathfrak{M}^{[p]}_{[a,b]}(f) \le \mathfrak{E}^{r,p+r}\big(f(a), f(b)\big)$$

□ Assume that $p \neq 0$ then

$$\begin{aligned}\mathfrak{M}^{[p]}_{[a,b]}(f) =&\Big(\frac{1}{b-a}\int_a^b f^p\Big)^{1/p} = \Big(\int_0^1 f^p(tb + \overline{1-t}a)\,\mathrm{d}t\Big)^{1/p}\\ \le&\Big(\int_0^1 \Big(\mathfrak{M}^r\big(f(a), f(b); 1-t, t\big)\Big)^p \mathrm{d}t\Big)^{1/p}, \text{ by } r\text{-mean convexity, III 6.3,}\\ =&\mathfrak{E}^{r,p+r}\big(f(a), f(b)\big), \text{ by (15) and (16).}\end{aligned}$$

The case $p = 0$ can be proved in a similar manner. □

Remark (vii) The case $p = 2, r = 1$ is due to Sándor; [*Sándor 1990c*]; and the original inequality, the right-hand side of I 4.1(4), is just the case $r = p = 1$.

In Example (i) particular extended means were shown to be cases of other means introduced. In general deciding if a given family of means includes a particular mean, or another family of means, is not easy. An interesting method for doing just that has been developed by Gould & Mays; [B^2 *pp.263–266*], [*Gould & Mays*]; see also [*Ume & Kim*]. Note that if $\mathfrak{M}$ is any homogeneous symmetric two-variable mean and if $0 < a < b$ then $\mathfrak{M}(b,a) = \mathfrak{M}(a,b) = b\mathfrak{M}(a/b, 1) = b\mathfrak{M}(1-t, 1)$ where $0 \le t = (b-a)/b < 1$.

THEOREM 19 *If $p, q, r, s \in \mathbb{R}$ and $0 \le t < 1$, then*

$$\mathfrak{M}^{[p]}(1-t,1) = 1 - \frac{1}{2}t + \frac{p-1}{8}t^2 + \frac{p-1}{16}t^3 - \frac{(p-1)(p-3)(2p+5)}{384}t^4 + \cdots;$$

$$\begin{aligned}\mathfrak{H}^{[q]}(1-t,1) = 1 - \frac{1}{2}t + \frac{q-1}{4}t^2 + \frac{q-1}{8}t^3 - \frac{(q-1)(q+1)(q-3)}{48}t^4 \\ - \frac{(q-1)(q^2-2q-1)}{32}t^5 + \frac{(q-1)(q+3)(q-5)(2q^2-4q-1)}{960}t^6 + \cdots;\end{aligned}$$

$$\begin{aligned}\mathfrak{E}^{r,s}(1-t,1) = 1 - \frac{1}{2}t + \frac{r+s-3}{24}t^2 + \frac{r+s-3}{48}t^3 \\ - \frac{2(r^3+r^2s+rs^2+s^3) - 5(r+s)^2 - 70(r+s) + 225}{5760}t^4 \\ - \frac{2(r^3+r^2s+rs^2+s^3) - 5(r+s)^2 - 30(r+s) + 105}{3840}t^5 + \frac{E}{2903040}t^6 + \cdots;\end{aligned}$$

where $E = 16(r^5+r^4s+r^3s^2+r^2s^3+rs^4+s^5) - 42(r^4+2r^3s+2r^2s^2+2rs^3+s^4) - 1687(r^3+s^3) - 1617(r+s)rs + 4305(r+s)^2 + 15519(r+s) - 59535.$

□ See the reference. □

COROLLARY 20 *(a) The only two-variable means that are both power means and counter-harmonic means are the arithmetic, geometric and harmonic means.*
(b) The only two-variable means that are both extended means and counter-harmonic means are the arithmetic, geometric and harmonic means.
(c) The only two-variable means that are both power means and generalized logarithmic means are the arithmetic, quadratic and geometric means.

□ (a) Comparing the coefficients of the first four terms in the above expansions for $\mathfrak{M}^{[p]}(1-t,1)$ and $\mathfrak{H}^{[q]}(1-t,1)$ shows that we need $q = 2p-1$. Substituting this in the coefficients of the fifth terms leads to $p = -1, 0,$ or 1 when $q = 0, \frac{1}{2}, 1$ respectively. Simple calculations then show that for these values of p and q, $\mathfrak{M}^{[p]}(a,b) = \mathfrak{H}^{[q]}(a,b)$ for all positive a, b; see also III 5.1.
(b) This follows similarly by comparing the coefficients, up to those of t^6, in the above expansions for $\mathfrak{E}^{r,s}(1-t,1)$ and $\mathfrak{H}^{[q]}(1-t,1)$, see [*Gould & Mays*].
(c) From the above expansion for $\mathfrak{E}^{r,s}(1-t,1)$ and Example (i)

$$\begin{aligned}\mathfrak{L}^{[q]}(1-t,1) =& 1 - \frac{1}{2}t + \frac{q-1}{24}t^2 + \frac{q-1}{48}t^3 \\ &- \frac{2(q+2)\big((q+1)^2+1\big) - 5(q+2)^2 + 70(q+2) + 225}{5760} + \cdots,\end{aligned}$$

and the result follows as above by comparing coefficients. □

The extended means of Stolarsky, (14), are special cases of means of n-tuples; [*Stolarsky 1975*]. If $t \in \mathbb{R}^*$ we write $\iota^t(x) = x^t$, $x > 0$, then the *extended mean of*

order r, s, $r \neq s$, $r, s \in \mathbb{R}^*$, *of the n-tuple of distinct positive numbers* $\underline{a}$ is

$$\mathfrak{E}_n^{r,s}(\underline{a}) = \left(\frac{(r-n+1)_{n-1}[\underline{a}; \iota^s]_{n-1}}{(s-n+1)_{n-1}[\underline{a}; \iota^r]_{n-1}} \right)^{1/(s-r)}$$

where $(a)_b = (a+b)!/a!$, the so-called *Pochammer symbol*. The values when either some of the entries in $\underline{a}$ are equal, or $r = s$, or $r = 0$, or $s = 0$ are obtained by taking limits; it can be shown that these limits exist.

EXAMPLE (iv) When $n = 2$ this definition reduces to (14)

EXAMPLE (v) $\mathfrak{E}_n^{n-1,n}(\underline{a}) = \mathfrak{A}_n(\underline{a})$; $\mathfrak{E}_n^{n-1,-1}(\underline{a}) = \mathfrak{G}_n(\underline{a})$; $\mathfrak{E}_n^{-1,-2}(\underline{a}) = \mathfrak{H}_n(\underline{a})$.

The following theorem extends Corollary 20; [*Alzer & Ruscheweyh 2001*].

THEOREM 21 *The only means that are both extended means and equal weighted Gini means are (a) the power means if* $n = 2$, *(b) the arithmetic, geometric and harmonic means if* $n \geq 3$. *In particular if* $n \geq 3$ *the only power means that are extended means are the arithmetic, geometric, and harmonic means.*

REMARK (viii) The last part of the theorem was conjectured earlier in [*Leach & Sholander 1983*]. See also [*Leach & Sholander 1984; Losconzi & Páles 1998; Pearce, Pečarić & Šimić 1999; Tobey*].

2.1.4 HERONIAN, CENTROIDAL AND NEO-PYTHAGOREAN MEANS Some particular cases of the extended means, 2.1.3 (14), have a long history. In particular we have *the Heronian mean*

$$\mathfrak{H}\mathfrak{e}(a,b) = \mathfrak{E}^{1/2,3/2}(a,b) = \frac{a+\sqrt{ab}+b}{3};$$

and the *the centroidal mean*

$$\mathfrak{C}(a,b) = \mathfrak{E}^{2,3}(a,b) = \frac{2}{3}\left(\frac{a^2+ab+b^2}{a+b}\right).$$

REMARK (i) If in I 1.3.1 Figure 1 a line is drawn through the centroid parallel to AB its length is $\mathfrak{C}(a,b)$; [*Eves 1983 p.168*]. The Heronian mean occurs in an Egyptian manuscript dated 1850 B.C., the Moscow papyrus, in a formula of Heron for the volume, V, for the frustum of a pyramid of height h, lower base area a, and upper base area b:

$$V = h\mathfrak{H}\mathfrak{e}(a,b);$$

see [*Eves 1980 pp.11–14, Heath vol.II, pp.332–334*][10].

An elaboration of proof (*ii*) of II 2.2.1 Lemma 3 gives the following simple inequality

10 For the frustum of a rectangular pyramid, with bases of sides c,d and c',d' this can be written $\frac{h}{3}(cd+c'd'+\mathfrak{G}(cd',c'd))$. If the geometric mean is replaced by the arithmetic mean we get the volume of a rectangular prismoid, a formula also given by Heron; [*Legendre pp.124–125*],[*Bradley*].

THEOREM 22 *If a and b are distinct positive numbers then*

$$\mathfrak{G}(a,b) < \mathfrak{He}(a,b) < \mathfrak{A}(a,b).$$

□ From the obvious $|\sqrt{b}-\sqrt{a}| > 0$ we get that $b+a > 2\sqrt{ab}$. Then $b+a+\sqrt{ab} > 3\sqrt{ab}$ which is the left inequality; and $3(b+a) > 2(b+a)+2\sqrt{ab}$ which is the right inequality. □

A result similar to the theorem of Pittenger, 2.1.1 Theorem 7, has been obtained for the Heronian mean; [*Alzer & Janous*].

THEOREM 23

$$\mathfrak{M}^{[\log 2/\log 3]}(a,b) < \mathfrak{He}(a,b) < \mathfrak{M}^{[2/3]}(a,b),$$

and the values $\log 2/\log 3, 2/3$ *are best possible in that the first cannot be increased and the second cannot be decreased.*

Noting that $\mathfrak{He}(a,b) = \dfrac{1}{3}\mathfrak{G}(a,b) + \dfrac{2}{3}\mathfrak{A}(a,b)$ it is natural to define the *generalized Heronian mean*, [*Janous*].

$$\mathfrak{He}^{[t]}(a,b) = (1-t)\mathfrak{G}(a,b) + t\mathfrak{A}(a,b), \quad 0 \le t \le 1;$$

or as

$$\mathfrak{He}^{(w)}(a,b) = \mathfrak{He}^{[2/(2+w)]}(a,b) = \begin{cases} \dfrac{a+w\sqrt{ab}+b}{2+w}, & \text{if } 0 \le w < \infty, \\ \sqrt{ab}, & \text{if } w = \infty. \end{cases}$$

EXAMPLE (i) $\mathfrak{He}^{[0]}(a,b) = \mathfrak{G}(a,b)$, $\mathfrak{He}^{[1/2]}(a,b) = \mathfrak{M}^{[1/2]}(a,b)$, $\mathfrak{He}^{[2/3]}(a,b) = \mathfrak{He}(a,b)$, $\mathfrak{He}^{[1]}(a,b) = \mathfrak{A}(a,b)$.

EXAMPLE (ii) If $0 \le t_1 < t_2 \le 1$, $a \ne b$ then, using (GA), $\mathfrak{He}^{[t_1]}(a,b) < \mathfrak{He}^{[t_2]}(a,b)$.

THEOREM 24 *(a) If* $1/2 \le t < 1$ *then*

$$\mathfrak{M}^{[(\log 2(/(\log 2-\log t)]}(a,b) < \mathfrak{He}^{[t]}(a,b) < \mathfrak{M}^{[t]}(a,b);$$

if $0 < t < 1/2$ *the exponents in the power means are reversed. In both cases the exponents are best possible*

(b)

$$\mathfrak{He}^{[0]}(a,b) < \mathfrak{L}(a,b) < \mathfrak{He}^{[1/3]}(a,b), \tag{18}$$

the value 0 *cannot be increased, and the value* $1/3$ *cannot be decreased.*

(c)

$$\mathfrak{He}^{[1/3]}(a,b) < \mathfrak{I}(a,b) < \mathfrak{He}^{[2/e]}(a,b);$$

again the number 1/3 cannot be increased, nor the value (2/e) decreased.

REMARK (ii) Inequality (18) should be compared with inequality 2.1.1 (10) in the case $p = 0$.

REMARK (iii) Another generalization of the Heronian mean is the extended mean $\mathfrak{E}^{1/n,1+1/n}(a,b)$, and some of its properties are discussed in [*Chen & Wang 1993*]; see also [*Guo & Qi*].

If $0 < a < b$ then $\mathfrak{A}(a,b), \mathfrak{G}(a,b)$, $\mathfrak{H}(a,b)$ and the contraharmonic mean $\mathfrak{H}^{[2]}(a,b)$ are respectively the solutions, for x of the following proportions, see II 1.1, 1.20, III 5.2.1 Footnote 3: $x - a : b - x :: 1 : 1$; $x - a : b - x :: a : x$, (or $x : b$); $x - a : b - x :: a : b$; $x - a : b - x :: b : a$. These means are the first four of the *neo-Pythagorean means*, ${}^{i}\mathfrak{P}(a,b)$, $1 \le i \le 10$, obtained by Greek mathematicians on considering all possible proportions; [B^2 *pp.255–256*]. ${}^{1}\mathfrak{P}(a,b) = \mathfrak{A}(a,b), {}^{2}\mathfrak{P}(a,b) = \mathfrak{G}(a,b), {}^{3}\mathfrak{P}(a,b) = \mathfrak{H}(a,b), {}^{4}\mathfrak{P}(a,b) = \mathfrak{H}^{[2]}(a,b)$; and the remaining six are given below.

Proportion	*Solution*
$x - a : b - x :: x : a$	${}^{5}\mathfrak{P}(a,b) = \dfrac{1}{2}\big((b-a) + \sqrt{(b-a)^2 + 4a^2}\big);$
$x - a : b - x :: b : x$	${}^{6}\mathfrak{P}(a,b) = \dfrac{1}{2}\big(-(b-a) + \sqrt{(b-a)^2 + 4b^2}\big);$
$b - a : b - x :: b : a$	${}^{7}\mathfrak{P}(a,b) = a + \dfrac{(b-a)^2}{b};$
$b - a : x - a :: b : a$	${}^{8}\mathfrak{P}(a,b) = b - \dfrac{(b-a)^2}{b};$
$b - a : b - x :: b : x$	${}^{9}\mathfrak{P}(a,b) = \dfrac{b^2}{\big((b-a)+b\big)};$
$b - a : x - a :: x : a$	${}^{10}\mathfrak{P}(a,b) = \dfrac{1}{2}\big(a + \sqrt{a^2 + 4a(b-a)}\big).$

The definitions are completed by: ${}^{i}\mathfrak{P}(a,b) = {}^{i}\mathfrak{P}(b,a)$, ${}^{i}\mathfrak{P}(a,a) = a$, $1 \le i \le 10$. All these means are strictly internal, homogeneous and symmetric.

REMARK (iv) The means obtained by inverting the proportions on the right-hand side are called *sub-contrary*; so ${}^{4}\mathfrak{P}(a,b)$ is sub-contrary to the harmonic mean, ${}^{3}\mathfrak{P}(a,b)$, and ${}^{5}\mathfrak{P}(a,b)$ to the geometric mean, the mean ${}^{2}\mathfrak{P}(a,b)$, has two sub-contrary means. ${}^{5}\mathfrak{P}(a,b)$ and ${}^{6}\mathfrak{P}(a,b)$; [*Heath I pp.84–89*], [*Wassell*].

2.1.5 SOME MEANS OF HARUKI AND RASSIAS Let p be a strictly monotonic function with continuous second derivatives defined on $\mathbb{R}_+^*$ and write

$$r_1(\theta) = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta},$$

and

$$r_2(\theta) = a\sin^2\theta + b\cos^2\theta, \quad r_3(\theta) = (\sin^2\theta/a + \cos^2\theta/b)^{-1},$$

where $0 < a < b$ and define the three means [11]

$$\mathfrak{R}_i(a,b;p) = p^{-1}\Big(\frac{1}{2\pi}\int_0^{2\pi} p\big(r_i(\theta)\big)\,\mathrm{d}\theta\Big),\ 1 \le i \le 3.$$

It is easily checked that all of these are symmetric, reflexive and internal means. They include several well known means as special cases.

THEOREM 25 *(a)* $\mathfrak{R}_1(a,b;p) = \mathfrak{A}(a,b)$ *if and only if for some real* α *and* β, $\alpha \ne 0$, $p(x) = \alpha\log x + \beta$; *(b)* $\mathfrak{R}_1(a,b;p) = \mathfrak{G}(a,b)$ *if and only if for some real* α *and* β, $\alpha \ne 0$, $p(x) = \alpha x^{-2} + \beta$; *(c)* $\mathfrak{R}_1(a,b;p) = \mathfrak{Q}(a,b)$ *if and only if for some real* α *and* β, $\alpha \ne 0$, $p(x) = \alpha x^2 + \beta$; *(d)* $\mathfrak{R}_1(a,b;p) = \mathfrak{G}\otimes\mathfrak{A}(a,b)$ *if and only if for some real* α *and* β, $\alpha \ne 0$, $p(x) = \alpha x^{-1} + \beta$.

□ See [*Haruki*]; the mean $\mathfrak{G}\otimes\mathfrak{A}(a,b)$ in (d) is the arithmetico-geometric mean of Gauss defined below in 3.1. □

Two other means can be defined by analogy with the equality in II 2.2.1 Remark (i), $\mathfrak{G}(a,b) = \sqrt{\mathfrak{A}(a,b)\mathfrak{H}(a,b)}$:

$$\mathfrak{J}(a,b) = \sqrt{\mathfrak{A}(a,b)\mathfrak{G}(a,b)}, \qquad \mathfrak{K}(a,b) = \sqrt{\mathfrak{H}(a,b)\mathfrak{G}(a,b)}.$$

It is easily checked that these are also symmetric, reflexive and internal means and that $\mathfrak{G}(a,b) = \sqrt{\mathfrak{J}(a,b)\mathfrak{K}(a,b)}$.

THEOREM 26 *(a)* $\mathfrak{R}_2(a,b;p) = \mathfrak{A}(a,b)$ *if and only if for some real* α *and* β, $\alpha \ne 0$, $p(x) = \alpha x + \beta$; *(b)* $\mathfrak{R}_2(a,b;p) = \mathfrak{G}(a,b)$ *if and only if for some real* α *and* β, $\alpha \ne 0$, $p(x) = \alpha x^{-1} + \beta$; *(c)* $\mathfrak{R}_2(a,b;p) = \mathfrak{M}^{[1/2]}(a,b)$ *if and only if for some real* α *and* β, $\alpha \ne 0$, $p(x) = \alpha\log x + \beta$; *(d)* $\mathfrak{R}_2(a,b;p) = \mathfrak{K}(a,b)$ *if and only if for some real* α *and* β, $\alpha \ne 0$, $p(x) = \alpha x^{-2} + \beta$.

THEOREM 27 *(a)* $\mathfrak{R}_3(a,b;p) = \mathfrak{G}(a,b)$ *if and only if for some real* α *and* β, $\alpha \ne 0$, $p(x) = \alpha x + \beta$; *(b)* $\mathfrak{R}_3(a,b;p) = \mathfrak{H}(a,b)$ *if and only if for some real* α *and* β, $\alpha \ne 0$, $p(x) = \alpha x^{-1} + \beta$; *(c)* $\mathfrak{R}_3(a,b;p) = \mathfrak{M}^{[-1/2]}(a,b)$ *if and only if for some real* α *and* β, $\alpha \ne 0$, $p(x) = \alpha\log x + \beta$; *(d)* $\mathfrak{R}_3(a,b;p) = \mathfrak{J}(a,b)$ *if and only if for some real* α *and* β, $\alpha \ne 0$, $p(x) = \alpha x^2 + \beta$.

The proofs can be found in [*Haruki & Rassias*] and use the following lemma of some independent interest.

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[11] These definitions should be compared with the identity below, 3.2 (8).

LEMMA 28

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{a\sin^2\theta+b\cos^2\theta}\,\mathrm{d}\theta=\frac{1}{\mathfrak{G}(a,b)};$$
$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{(a\sin^2\theta+b\cos^2\theta)^2}\,\mathrm{d}\theta=\frac{1}{\mathfrak{K}^2(a,b)};$$
$$\frac{1}{2\pi}\int_0^{2\pi}\log\bigl(a\sin^2\theta+b\cos^2\theta\bigr)\,\mathrm{d}\theta=\log\bigl(\mathfrak{M}^{[1/2]}(a,b)\bigr).$$

REMARK (i) Other references for these means are [*Kim Y 1999; Kim & Ume; Toader 2002*].

REMARK (ii) A generalization in which the expression $\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}$ is replaced by $\sqrt[n]{a^n\cos^2\theta+b^n\sin^2\theta}$ can be found in [*Haruki; Haruki & Rassias; Toader 1998; Toader & Rassias*].

2.2 MEAN-VALUE MEANS

2.2.1 LAGRANGIAN MEANS If $M:\mathbb{R}\mapsto\mathbb{R}$ is a strictly convex, or strictly concave, continuously differentiable function then the Lagrangian mean-value theorem of differentiation[12], ensures that for all $a,b\in\mathbb{R}$, $a<b$, there is a unique point c with $a<c<b$, the mean-value point of M on $[a,b]$, such that

$$M'(c)=\frac{M(b)-M(a)}{b-a},\ \text{equivalently}\ c=(M')^{-1}\left(\frac{M(b)-M(a)}{b-a}\right). \tag{20}$$

This number c defines a mean called the *mean-value mean by M of a and b*; following Berrone & Morro, [*Berrone & Moro 1998*], we will call it the *Lagrangian mean of a and b by M*, written $\mathfrak{M}_{\mathfrak{L}}^{[M]}(a,b)$; the definition is completed by defining $\mathfrak{M}_{\mathfrak{L}}^{[M]}(b,a)=\mathfrak{M}_{\mathfrak{L}}^{[M]}(a,b)$, and $\mathfrak{M}_{\mathfrak{L}}^{[M]}(a,a)=a$.

REMARK (i) These means have most of the properties of means; (Re) and (Sy), by the above definition; strict internality, from the mean-value theorem since $a<c<b$; and (Mo) from a basic property of convex functions, I 4.1 Remark (v). As for (Ho) see Theorem 30 below.

EXAMPLE (i) The generalized logarithmic means, $\mathfrak{L}^{[p]}$, 2.1.1, are examples of Lagrangian means. This is seen by taking $M(x)=x^{p+1}$, $p\neq-1,0$, $M(x)=\log x$, $p=-1$, and $M(x)=x\log x$, $p=0$.

Lagrangian means can be expressed as the quasi-arithmetic mean of a function, see 1.2.2; for if $M'=\mathcal{M}$ then clearly from (20) and 1.2.2 (5),

[12] See I 2.1 Footnote 1.

$$\mathfrak{M}_{\mathfrak{L}}^{[M]}(a,b) = \mathcal{M}^{-1}\Big(\frac{1}{b-a}\int_a^b \mathcal{M}\Big) = \mathfrak{M}_{[a,b]}(\iota). \tag{21}$$

The equivalence of these approaches is particularly useful as theorems on quasi-arithmetic means of functions, see for instance 1.2.2 Theorem 4, can be used to obtain results for Lagrangian means; [*Berrone & Moro 1998; Mays; Vota*]. In addition results have been obtained that compare Lagrangian means with quasi-arithmetic means; [*Elezović & Pečarić 2000b*].

THEOREM 29 *$\mathfrak{M}_{\mathfrak{L}}^{[M]}(a,b)$ is equal to the $\mathfrak{M}_{\mathfrak{L}}^{[N]}(a,b)$ for all a,b, if and only if for some $\alpha,\beta,\gamma,\alpha\neq 0$,*

$$M(x) = \alpha N(x) + \beta x + \gamma, \quad x \in \mathbb{R}. \tag{22}$$

□ From (21) we see that (22) is equivalent to $\mathcal{M}(x) = \alpha\mathcal{N}(x) + \beta$, and the result is an immediate consequence of the integral analogue of IV 1.2 Theorem 5; see [*HLP Theorem 243*]. □

REMARK (ii) Because of this theorem we can without loss in generality assume that in (20) the function M is convex.

THEOREM 30 *If a Lagrangian mean $\mathfrak{M}_{\mathfrak{L}}^{[M]}$ is homogeneous then for some finite p, and all a and b, $\mathfrak{M}_{\mathfrak{L}}^{[M]}(a,b) = \mathfrak{L}^{[p]}(a,b)$.*

□ This is a consequence of the integral analogue of IV 1.2 Theorem 6; see [*Jessen 1931b*]. □

COROLLARY 31 *For no choice of M is $\mathfrak{M}_{\mathfrak{L}}^{[M]}$ the harmonic mean.*

□ Since the harmonic mean is homogeneous we need, by Theorem 30, only look at the means $\mathfrak{L}^{[p]}(a,b)$. However by 2.1.1 Example (iv), or 2.1.3 Corollary 20(c), the harmonic mean is not a generalized logarithmic mean. □

The above definition has been generalized using confluent divided differences, see I 4.7. If $M : \mathbb{R} \mapsto \mathbb{R}$ is strictly $2n$-convex, or strictly $2n$- concave, with a continuous derivative of order $2n-1$ then for any $2n$-tuple $\underline{a}$ with distinct entries there is a unique c, $\min \underline{a} < c < \max \underline{a}$ such that

$$[\underline{a}; M]_{2n-1} = \frac{M^{(2n-1)}(c)}{(2n-1)!}, \quad \text{equivalently} \quad c = (2n-1)!(M^{(2n-1)})^{-1}\big([\underline{a}; M]_{2n-1}\big);$$

see [*Aumann p.274*]. This result remains valid for confluent divided differences, [*Horwitz 1995*]. So if $a < b$ write

$$[a,b;M]_n = [\overbrace{a,\ldots,a}^{n\,terms},\overbrace{b,\ldots,b}^{n\,terms};M]_{2n-1},$$

and define the *nth order Lagrangian mean of a and b* by M, $\mathfrak{M}_{n,\mathfrak{L}}^{[M]}(a,b)$, as the unique c such that

$$\frac{M^{(2n-1)}(c)}{(2n-1)!} = [a,b;M]_n, \quad \text{or} \quad \mathfrak{M}_{n,\mathfrak{L}}^{[M]}(a,b) = (2n-1)!(M^{(2n-1)})^{-1}\big([a,b;M]_n\big).$$

Using various results from the theory of divided differences it is possible to give a generalization of (21); [*Horwitz 1995*]. Let $M^{(2n-1)} = \mathcal{M}$ then

$$\mathfrak{M}_{n,\mathfrak{L}}^{[M]}(a,b) = \frac{(2n-1)!}{\big((n-1)!\big)^2}\mathcal{M}^{-1}\bigg(\frac{1}{(b-a)^{2n-1}}\int_a^b \mathcal{M}(t)\big((b-t)(t-a)\big)^{n-1}\,\mathrm{d}t\bigg).$$

Example (ii) If $M(x) = x^{2n}$and $\underline{a}$ is a distinct $2n$-tuple, then by I 4.7 Example (ii) $[\underline{a};M]_{2n-1} = a_1 + \cdots a_{2n}$; so $[a,b;M]_n = n(a+b)$, Further $\mathcal{M}(x) = (2n)!x$ so we easily get that $\mathfrak{M}_{n,\mathfrak{L}}^{[M]}(a,b) = \mathfrak{A}(a,b)$.

Example (iii) If $M(x) = x^{-1}$ and if $\underline{a}$ is a distinct $2n$-tuple then, again by I 4.7 Example (ii), $[\underline{a};M]_{2n-1} = (-1)^{2n-1}/(a_1a_2\ldots a_{2n})$; so $[a,b;M]_n = -1/a^nb^n$. Further $\mathcal{M}(x) = -(2n-1)!x^{-2n}$ so in this case $\mathfrak{M}_{n,\mathfrak{L}}^{[M]}(a,b) = \mathfrak{G}(a,b)$.

Example (iv) If $M(x) = x^{(2n-1)/n}$ then $\mathfrak{M}_{n,\mathfrak{L}}^{[M]}(a,b) = \mathfrak{M}^{1/2}(a,b)$; see [*Horwitz 1995*].

Remark (iii) The idea of mean-value means has been used for various extensions of the quasi-arithmetic means, in particular some of the means in III 5; see [*Cisbani; Galvani; Pearce, Pečarić & Šimić 1999; Pizzetti 1939; Zappa 1939*].

2.2.2 Cauchy means The Lagrangian means can be generalized by the use of the Cauchy mean-value theorem[13]. Let $M, N : \mathbb{R} \mapsto \mathbb{R}$ be two differentiable functions with $N' > 0$, then if $a, b \in \mathbb{R}$ with $a < b$, there is a point c, $a < c < b$, such that

$$\frac{M'(c)}{N''(c)} = \frac{M(b)-M(a)}{N(b)-N(a)}. \tag{23}$$

If then M, N satisfy conditions that ensure that c is unique, it is called the *Cauchy mean of a and b by M and N*, written $\mathfrak{M}_{\mathfrak{C}}^{[M,N]}(a,b)$; again the name was suggested by Berrone & Moro, [*Berrone & Moro 1999*]. Conditions for this definition to be a proper one have been investigated by various authors; see [*Aumann 1936,1937; Dieulefait; Losonczi 2000*].

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13 See II 2.2.2 Footnote 10.

In particular if $\mathcal{R} = M'/N'$ has an inverse then

$$\mathfrak{M}_{\mathfrak{C}}^{[M,N]}(a,b) = \mathcal{R}^{-1}\left(\frac{M(b)-M(a)}{N(b)-N(a)}\right).$$

In this case we can write the Cauchy mean in a form that generalizes (21):

$$\mathfrak{M}_{\mathfrak{C}}^{[M,N]}(a,b) = \mathcal{R}^{-1}\left(\frac{1}{N(b)-N(a)}\int_a^b \mathcal{R}\,\mathrm{d}N\right) = \mathfrak{R}_{[a,b]}(\iota;\mathcal{N}); \tag{24}$$

see [*Berrone & Moro 1999*].

EXAMPLE (i) The extended means, $\mathfrak{E}^{r,s}$, are examples of Cauchy means. This is seen by taking $M(x) = x^s$, $N(x) = x^r, r \neq s, s \neq 0$, $M(x) = x^s, N(x) = \log x$ if $r \neq 0, s = 0$, $M(x) = x^r \log x, N(x) = x^r, r = s \neq 0$, $M(x) = \log^2 x$, $N(x) = 2\log x$ if $r = s = 0$.

EXAMPLE (ii) Taking $\mathcal{R} = N = \mathcal{F}$, say we see from (23), or (24), that the Cauchy mean is an equal weight quasi-arithmetic $\mathcal{F}$-mean: that is $\mathfrak{M}_{\mathfrak{C}}^{[\frac{1}{2}\mathcal{F}^2,\mathcal{F}]}(a,b) = \mathfrak{F}_2(a,b)$; [*Berron & Moro 1999*].

REMARK (iv) As with Lagrangian means the Cauchy means can be generalized using divided differences, thus defining Cauchy means of n-tuples; see [*Leach & Sholander 1984; Losonczi 2002*].

REMARK (v) Another mean defined using integrals has been given in [*Porta & Stolarsky*].

2.3 MEANS AND GRAPHS

2.3.1 ALIGNMENT CHART MEANS Let $f : \mathbb{R}_+^* \mapsto \mathbb{R}_+^*$, then the *Beckenbach-Gini mean of the numbers* $a, b \in I$ is:

$$\mathfrak{B}^f(a,b) = \frac{af(a) + bf(b)}{f(a) + f(b)}.$$

The function f *is said to generate, or be the generator of, the mean* $\mathfrak{B}^f$. The same mean was introduced by Moskovitz and given the name *alignment chart f-mean of a and b* by Mays; see [*Matkowski 2001; Mays; Moskovitz*].
Simple calculations show that $\mathfrak{B}^f(a,b)$ is the unique c, $a < c < b$, such that (c, 0) is on the line from $(a, f(a))$ to $(b, -f(b))$.

REMARK (i) The domain of f, $\mathbb{R}_+^*$, can be any interval in $\mathbb{R}_+^*$.

EXAMPLE (i) If $f(x) = x^r$, $r \in \mathbb{R}$ then $\mathfrak{B}^f(a,b)$ is the counter-harmonic mean $\mathfrak{H}^{[r+1]}(a,b)$. Using some identifications made in III 5.1, or by direct computation,

we have in particular that: if f is constant then $\mathfrak{B}^f(a,b) = \mathfrak{A}(a,b)$, if $f(x) = 1/x$ then $\mathfrak{B}^f(a,b) = \mathfrak{H}(a,b)$ and if $f(x) = 1/\sqrt{x}$ then $\mathfrak{B}^f(a,b) = \mathfrak{G}(a,b)$.

REMARK (ii) The means of Example (i) are the only means of this type that are homogeneous; see [*Mays*].

THEOREM 32 *The two functions $f, g : I \mapsto \mathbb{R}_+^*$ generate the same Beckenbach-Gini mean if and only if for some $\alpha > 0$, $f = \alpha g$.*

□ The one direction is trivial and the other is almost immediate on simplifying the identity
$$\frac{af(a) + bf(b)}{f(a) + f(b)} = \frac{ag(a) + bg(b)}{g(a) + g(b)}.$$
□

The following interesting theorem showing when a Beckenbach-Gini mean is a quasi-arithmetic mean is due to Matkowski; [*Matkowski 2001*].

THEOREM 33 *If I is an open interval in $\mathbb{R}$, and suppose that $f, \mathcal{M} : I \mapsto \mathbb{R}_+^*$, where $\mathcal{M}$ is continuously differentiable, then the Beckenbach-Gini mean $\mathfrak{B}^f$ is the quasi-arithmetic $\mathcal{M}$-mean $\mathfrak{M}$ if and only if for some $\alpha, \beta, \gamma, \lambda \in \mathbb{R}, \gamma, \lambda \neq 0$,*
$$\mathcal{M}' = \lambda f^2, \quad \textit{and} \quad f(x) = \frac{1}{\sqrt{|\alpha x^2 + \beta x + \gamma|}}.$$

REMARK (iii) The above definition easily extends to general n-tuples defining $\mathfrak{B}_n^f(\underline{a})$; but if $n \geq 3$, this mean is a quasi-arithmetic mean if and only if either f is constant when $\mathfrak{B}_n^f(\underline{a}) = \mathfrak{A}_n(\underline{a})$ or, $f(x) = \lambda/(x + \alpha)$, $\lambda \neq 0$, when $\mathfrak{B}_n^f(\underline{a}) = \mathfrak{H}_n(\underline{a} + \alpha) - \alpha$; [*Matkowski 2001; Savage*].

2.3.2 FUNCTIONALLY RELATED MEANS Another use of graphs with means has been given in [*Häntzsche & Wendt*]. Consider the graph of the strictly monotonic function f, $y = f(x)$, and suppose that (x_1, y_1) and (x_2, y_2) both lie on the graph with $0 < x_1 < x_2$, $y_1, y_2 > 0$. Suppose further that $\mathfrak{M}$ and $\mathfrak{N}$ are two given means and that x is the $\mathfrak{M}$ mean of x_1 and x_2, that is $x = \mathfrak{M}(x_1, x_2)$, $x_1 < x < x_2$; when is it the case that $y = f(x)$ is the $\mathfrak{N}$ mean of y_1 and y_2, that is $y = \mathfrak{N}(y_1, y_2)$? In this case clearly the following functional equation must be satisfied:
$$f\big(\mathfrak{M}(x_1, x_2)\big) = \mathfrak{N}\big(f(x_1), f(x_2)\big),$$
or (25)
$$\mathfrak{M}(x_1, x_2)) = f^{-1}\Big(\mathfrak{N}\big(f(x_1), f(x_2)\big)\Big).$$

The last expression, given f and $\mathfrak{N}$, can under reasonable restrictions be calculated and defines a mean, $\mathfrak{N}_f$ say, as it is strictly internal and has the properties (Sy), (Mo) if the original mean $\mathfrak{N}$ has these properties; [B^2 *pp.230–232*].
A particular case occurs if we take f to be a power function ι^p, $p \in \mathbb{R}^*$, when we write $\mathfrak{N}_f$ as $\mathfrak{N}_p$, sometimes called the *p-modification of the mean* $\mathfrak{N}$, [*Vamanamurthy & Vuorinen*].

EXAMPLE (i) $\mathfrak{A}_p = \mathfrak{M}^{[p]}$; $\mathfrak{L}_p^{[r]} = \mathfrak{E}^{pr+p,p}$, $r \neq -1, 0$; $\mathfrak{L}_{p+1} = \mathfrak{L}^{[p]}$, $p \neq -1$.

If we require that $\mathfrak{N}_f$ be homogeneous then it is not difficult to show that $\mathfrak{N}(a,b) = \phi_t^{-1}\Big(\mathfrak{N}\big(\phi_t(a), \phi_t(b)\big)\Big)$, where $\phi_t(x) = f\big(tf^{-1}(x)\big)$. Using this it can be shown that if $\mathfrak{N}$ is the arithmetic mean and $\mathfrak{N}_f$ is homogeneous then it must be an $\mathfrak{M}^{[p]}$ for some $p \in \mathbb{R}^*$; while if $\mathfrak{N}$ is the geometric mean then $\mathfrak{N}_f$ must be an $\mathfrak{M}^{[p]}$ for some $p \in \mathbb{R}$; [B^2 *pp.239–230*].

If $\mathfrak{M}$ and $\mathfrak{N}$ are two quasi-arithmetic means on n-tuples the first equality in (25) becomes:

$$f\big(\mathfrak{M}_n(\underline{a}; \underline{u})\big) = \mathfrak{N}_n\big(f(\underline{a}); \underline{v}\big), \tag{26}$$

and the following result of Pečarić includes the results of Häntzsche & Wendt; see also [*Aczél 1966 p.79*], [*Aczél & Fenyő 1948b*].

THEOREM 34 *The functional equation (26) has a solution only if* $\underline{u} = \underline{v}$, *and then the general continuous solution is* $f(x) = \mathcal{N}^{-1}\big(\lambda\mathcal{M}(x) + \mu\big)$, *for some real* λ *and* μ.

□ Let $U_n = V_n = 1$ and $\phi = \mathcal{N} \circ f \circ \mathcal{M}^{-1}$, $u_i M(a_i) = t_i$, and $v_i\phi(x/u_i) = \phi_i(x)$, $1 \le i \le n$. Then (26) reduces to the well known Pexeider functional equation, $\phi\big(\sum_{i=1}^n t_i\big) = \sum_{i=1}^n \phi_i(t_i)$. The general continuous solution of this equation is:

$$\phi(x) = \lambda x + \sum_{i=1}^n \mu_i, \quad \phi_i(x) = \lambda x + \mu_i,\ 1 \le i \le n.$$

In our case this implies that $f(x) = \mathcal{N}^{-1}\big(\lambda\mathcal{M}(x)\mu_i/v_i + \lambda_i/v_i\big)$, $1 \le i \le n$, from which the result follows. □

COROLLARY 35 *In the case of* $\mathfrak{M}$ *being the r-th power mean, and* $\mathfrak{N}$ *the s-th power mean the function f in (26) is :*

$$f(x) = \begin{cases} (\lambda x^r + \mu)^{1/s}, & \textit{if } rs \neq 0,\\ (\lambda \log x + \mu)^{1/s}, & \textit{if } r = 0,\ s \neq 0,\\ \mu \exp(\lambda x^r), & \textit{if } r \neq 0, s = 0,\\ \mu x^\lambda, & \textit{if } r = s = 0. \end{cases}$$

2.4 Taylor Remainder Means In this section we discuss some very interesting ideas found in [*Horwitz 1990,1993*]. If $f : [a, b] \mapsto \mathbb{R}$ has $n+1$ derivatives then by Taylor's theorem[14], $f(x) = T_{n+1}(f; a; x) + R_{n+1}(f; a; x)$ where

$$T_{n+1}(f;a;x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k, R_{n+1}(f;a;x) = \frac{1}{(n-1)!}\int_a^x (x-t)^n f^{(n+1)}(t)\,\mathrm{d}t.$$

Assume that $f^{(n+1)} > 0$, when in particular f is strictly $(n+1)$-convex, I 4.7 Lemma 45(a). Then $I_n^a(x) = R_{n+1}(f; a; x)$, $a \le x \le b$, increases strictly from zero at $x = a$, while $J_n^b(x) = (-1)^{n+1}R_{n+1}(f; b; x)$, $a \le x \le b$, is strictly decreasing to zero at $x = b$. Further $I_n^a(b) = J_n^b(a)$. Hence there is a unique c, $a < c < b$, such that $I_n^a(c) = J_n^b(c)$ called the *Taylor remainder mean of order n by f of a and b, written* $\mathfrak{R}^{[f,n]}(a, b)$.[15] The definition is completed in the usual way by defining $\mathfrak{R}^{[f,n]}(b, a) = \mathfrak{R}^{[f,n]}(a, b)$ and $\mathfrak{R}^{[f,n]}(a, a) = a$. The same definition could be made if we assume that $f^{(n+1)} < 0$, when f is strictly $(n+1)$-concave.
When n is odd c is the unique root in $]a, b[$ of

$$T_{n+1}(f; a; x) - T_{n+1}(f; b; x) = 0;$$

if n is even it is the unique root of

$$\Big(f(x) - T_{n+1}(f; a; x)\Big) + \Big(f(x) - T_{n+1}(f; b; x)\Big) = 0.$$

Remark (i) Equivalently when n is odd c is the first coordinate of the unique point of intersection of the tangents of order n[16] to the graph of f at the points where $x = a$ and $x = b$.

Example (i) If $n = 0$ and $f = \mathcal{M}$ is strictly monotonic then $\mathfrak{R}^{[f,0]}(a, b) = \mathfrak{M}(a, b)$, a quasi-arithmetic $\mathcal{M}$-mean of a and b, IV 1.1 (2).

Example (ii) If $n = 1$ and f is strictly convex, or strictly concave, then

$$\mathfrak{R}^{[f,1]}(a, b) = \frac{\big(bf'(b) - f(b)\big) - \big(af'(a) - f(a)\big)}{f'(b) - f'(a)}.$$

or, if we assume that f' is absolutely continuous then $\mathfrak{R}^{[f,1]}(a, b) = \dfrac{\int_a^b xf''(x)\,\mathrm{d}x}{\int_a^b f''(x)\,\mathrm{d}x}$.

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14 See I 2.2 Footnote 2.

15 We will write c for $\mathfrak{R}^{[f,n]}(a,b)$ when it is convenient to do so.

16 The curve $y=T_{n+1}(f;a;x)$ is the *tangent of order n to the graph of f at the point where $x=a$*; it is the graph of a polynomial of degree at most n.

EXAMPLE (iii) If $f(x) = x^r$, $r \in \mathbb{R}, r \neq 0, 1$, let us write $\mathfrak{R}^{[f,n]}(a,b) = \mathfrak{R}^{[r,n]}(a,b)$. In this case

$$I_n^a(x) = \int_a^x t^{r-n-1}(x-t)^n \,\mathrm{d}t, \quad J_n^b(x) = \int_x^b t^{r-n-1}(t-x)^n \,\mathrm{d}t. \tag{27}$$

EXAMPLE (iv) If $r \neq 0, 1$, $\mathfrak{R}^{[r,1]}(a,b) = \mathfrak{E}^{r-1,r}(a,b)$; in particular $\mathfrak{R}^{[2,1]}(a,b) = \mathfrak{A}(a,b)$.

EXAMPLE (v) If $f(x) = x \log x$ then $\mathfrak{R}^{[f,1]}(a,b) = \mathfrak{L}(a,b)$.

Two other means of order 1 have been defined as follows, [*Horwitz, 1990*]:

$${}^{[1]}\mathfrak{H}^{[f]}(a,b) = f\Big(\mathfrak{R}^{[f,1]}\big(f^{-1}(a), f^{-1}(b)\big)\Big), {}^{[2]}\mathfrak{H}^{[f]}(a,b) = \ell\Big(\mathfrak{R}^{[f,1]}\big(f^{-1}(a), f^{-1}(b)\big)\Big),$$

where $y = \ell(x)$ is the line joining $\big(f^{-1}(a), a\big)$ to $\big(f^{-1}(b), b\big)$, and f has been taken to be strictly monotonic.

EXAMPLE (vi) ${}^{[1]}\mathfrak{H}^{[\exp]}(a,b) = \mathfrak{I}(a,b)$.

EXAMPLE (vii) If, following the notation of Example (iii), we write ${}^{[1]}\mathfrak{H}^{[r]}(a,b)$ when $f(x) = x^r$ then ${}^{[1]}\mathfrak{H}^{[r]}(a,b) = \mathfrak{L}^{[-1/r]}(a,b)$, $r \neq 0, 1$. In particular then, ${}^{[1]}\mathfrak{H}^{[-1]}(a,b) = \mathfrak{A}(a,b)$ and ${}^{[1]}\mathfrak{H}^{[1/2]}(a,b) = \mathfrak{G}(a,b)$.

EXAMPLE (viii) With a notation analogous to that in the previous example: ${}^{[2]}\mathfrak{H}^{[2]}(a,b) = \mathfrak{A}(a,b)$; ${}^{[2]}\mathfrak{H}^{[1/2]}(a,b) = \mathfrak{H}(a,b)$.

THEOREM 36 *If $f :]0, \infty[\mapsto \mathbb{R}$ is both strictly monotone and strictly convex, and has a range that contains $]0, \infty[$, and if $a \neq b$ then*

$$\mathfrak{R}^{[f^{-1},1]}(a,b) < {}^{[1]}\mathfrak{H}^{[f]}(a,b) < {}^{[2]}\mathfrak{H}^{[f]}(a,b); \tag{28}$$

if f is strictly concave then ($\sim$28) holds.

□ The right inequality is an immediate consequence of the basic property of convex functions; I 4.1 Remark (iii). The first inequality follows from another property of convex functions, I 4.1 Corollary 3(a). □

REMARK (ii) Taking $f(x) = \sqrt{x}$ we see from Examples (iv), (vii) and (viii) that ($\sim$28) reduces to (GA), in the equal weight $n = 2$ case.

THEOREM 37 *(a) The means $\mathfrak{R}^{[r,n]}(a,b)$ are homogeneous.*
(b) If $r < s$ then $\mathfrak{R}^{[r,n]}(a,b) < \mathfrak{R}^{[s,n]}(a,b)$.
(c) $\lim_{r\to\infty} \mathfrak{R}^{[r,n]}(a,b) = b$; $\lim_{r\to-\infty} \mathfrak{R}^{[r,n]}(a,b) = a$.

(d) $\mathfrak{N}^{[n+1,n]}(a,b) = \mathfrak{A}(a,b)$; $\mathfrak{N}^{[-1,n]}(a,b) = \mathfrak{H}(a,b)$; $\mathfrak{N}^{[n/2,n]}(a,b) = \mathfrak{G}(a,b)$.
(e) $\lim_{n\to\infty} \mathfrak{N}^{[r,n]}(a,b) = \mathfrak{H}(a,b)$.

□ (a) Put $t = xu$ in (27) to get

$$I_n^a(x) = x^r \int_{a/x}^1 u^{r-n-1}(1-u)^n \,du, \quad J_n^b(x) = x^r \int_1^{b/x} u^{r-n-1}(1-u)^n \,du. \tag{29}$$

Then, with c as above[17], $I_n^a(c) = J_n^b(c)$, so by (29) $I_n^{\lambda a}(\lambda c) = J_n^{\lambda b}(\lambda c)$. This implies that $\mathfrak{N}^{[r,n]}(\lambda a, \lambda b) = \lambda \mathfrak{N}^{[r,n]}(a,b)$.
(b) Now $I_n^a(c) = J_n^b(c)$ and by substituting $t = c(v+1)$ in the second integral in (27)

$$J_n^b(c) = c^r \int_0^{b/c-1} (v+1)^{r-n-1} v^n \,dv. \tag{30}$$

Let $g(x,r) = \left(\int_0^{b/x-1} (v+1)^{r-n-1} v^n \,dv\right) \Big/ \left(\int_{a/x}^1 u^{r-n-1}(1-u)^n \,du\right)$, then by (29) and (30), $g(c,r) = 1$, so $\dfrac{dc}{dr} = -\dfrac{\partial g/\partial r}{\partial g/\partial x}$.
From above $I_n^a(x)$increases strictly, and $J_n^b(x)$ is strictly decreasing, so $\partial g/\partial x > 0$. From the integral representations of $I_n^a(x)$, (29), and $J_n^b(x)$, (30), we see that $I_n^a(x)/x^r$ is decreasing as a function of r, and $J_n^b(x)/x^r$ is increasing. So $\partial g/\partial r < 0$. Hence $\dfrac{dc}{dr} > 0$.
(c) From (29), $I_n^a(x)/x^r < B(r-n, n+1) \to 0$, $r \to \infty$, B being the Beta function; and from (30) $J_n^b(x)/x^r > (b/x-1)^{n+1}/(n+1)$ if $r > n+1$. Now $I_n^a(c) = J_n^b(c)$ so from the previous deductions $b/c - 1 \to 0$, $r \to \infty$. Similarly $1 - a/c \to 0$, $r \to -\infty$.
(d) Put $t = (xy+a)/(y+1)$ in the first integral in (27) and $t = (xy+b)/(y+1)$ in the second and use $I_n^a(c) = J_n^b(c)$ to get

$$\begin{aligned}(c-a)^{n+1} &\int_0^\infty (cy+a)^{r-n-1}(y+1)^{-r-1} \,dy \\ &= (b-c)^{n+1} \int_0^\infty (cy+b)^{r-n-1}(y+1)^{-r-1} \,dy. \end{aligned} \tag{31}$$

Now if $r = n+1$ we have from (30) that $c - a = b - c$, or $c = (a+b)/2$.
If $r = -1$ then (31) gives $(c-a)/a = (b-c)/b$, or $c = 2ab/(a+b)$.
Now note that putting $a = 1/b$, $c = 1$ and $r = n/2$ we have (31) satisfied; that is $\mathfrak{N}^{[n/2,n]}(b^{-1}, b) = 1$. Then by (a)

$$\mathfrak{N}^{[n/2,n]}(a,b) = \sqrt{ab}\,\mathfrak{N}^{[n/2,n]}(a/\sqrt{ab}, b/\sqrt{ab}) = \sqrt{ab}\,\mathfrak{N}^{[n/2,n]}(\sqrt{a/b}, \sqrt{b/a}) = \sqrt{ab}.$$

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[17] See Footnote 15.

(e) Rewrite (31) as $\left(\frac{c/a-1}{1-c/b}\right)^{n+1} = \left(\frac{b}{a}\right)^{r-1}\frac{K(a)}{K(b)}$, where

$$K(b) = \int_0^\infty (1+cy/b)^{r-n-1}(y+1)^{-r-1}\,\mathrm{d}y.$$

It can be shown, [*Horwitz 1993 p.407*], that if $n > r-1$ then $1 < \frac{K(a)}{K(b)} \le b^2/a^2$, or $\left(\frac{K(a)}{K(b)}\right)^{1/(n+1)} \to 1$ as $n \to \infty$. Hence $c/a + c/b \to 2$ as $n \to \infty$. □

REMARK (iii) By Remark (i) we have the following interesting geometrical interpretations of (d). When n is odd the first coordinate of the point of intersection of the tangents of order n to the graph of $f(x) = 1/x$ at the points where $x = a$ and $x = b$ is the harmonic mean of a and b; the first coordinate of the point of intersection of the tangents of order n to the graph of $f(x) = x^{n/2}$ at the points where $x = a$ and $x = b$ is the geometric mean of a and b.

REMARK (iv) It follows from (d) that for all n the means $\mathfrak{R}^{[r,n]}(a,b)$ include the arithmetic, geometric and harmonic means as particular cases, and that (b) includes (GA). However we have the following theorem.

THEOREM 38 *If $n > 1$ the only means in common the two families $\mathfrak{R}^{[r,n]}(a,b)$ and $\mathfrak{R}^{[r,1]}(a,b)$ are the arithmetic, geometric and harmonic means*

□ The proof is by the method of Gould & Mays, 2.1.3 Theorem 17 ; [*Horwitz 1990 pp.233–235*]. □

REMARK (v) These means have been extended to weighted means, and to means of n-tuples, $n > 2$; [*Horwitz 1993, 1996*].

2.5 DECOMPOSITION OF MEANS Let $\mathfrak{M}(a,b)$ be any homogeneous internal mean and define the *index function of the mean $\mathfrak{M}$ with respect to the arithmetic mean* $\mathfrak{A}$ by $\mathcal{I}^{\mathfrak{A}}_{\mathfrak{M}}(t) = \mathfrak{M}(1+t, 1-t)$, $-1 < t < 1$.

EXAMPLE (i) $\mathcal{I}^{\mathfrak{A}}_{\mathfrak{A}}(t) = 1$, $-1 < t < 1$.

EXAMPLE (ii) $\mathcal{I}^{\mathfrak{A}}_{\min}(t) = 1 - |t|$, $\mathcal{I}^{\mathfrak{A}}_{\max}(t) = 1 + |t|$, $-1 < t < 1$.

The justification for this name is in the following theorem that shows that this function decomposes $\mathfrak{M}$ into the product of $\mathfrak{A}$ and $\mathcal{I}^{\mathfrak{A}}_{\mathfrak{M}}$.

THEOREM 39 *(a) If $\mathfrak{M}(a,b)$ is a homogeneous internal mean then*

$$\mathfrak{M}(a,b) = \mathfrak{A}(a,b)\mathcal{I}^{\mathfrak{A}}_{\mathfrak{M}}\left(\frac{a-b}{a+b}\right).$$

(b) $$1 - |t| \le \mathcal{I}^{\mathfrak{A}}_{\mathfrak{M}}(t) \le 1 + |t|, \; -1 < t < 1. \tag{32}$$
(c) $\mathfrak{M}$ *is symmetric if and only if* $\mathcal{I}^{\mathfrak{A}}_{\mathfrak{M}}$ *is even.*
(d) If $\mathfrak{N}(a,b)$ *is another homogeneous internal mean then*

$$\mathfrak{M} \le \mathfrak{N} \iff \mathcal{I}^{\mathfrak{A}}_{\mathfrak{M}} \le \mathcal{I}^{\mathfrak{A}}_{\mathfrak{N}}.$$

(e) For all a_1, b_1, a_2, b_2 *we have that*

$$\mathfrak{M}(a_1 + a_2, b_1 + b_2) \le \mathfrak{M}(a_1, b_1) + \mathfrak{M}(a_2, b_2),$$

if and only if $\mathcal{I}^{\mathfrak{A}}_{\mathfrak{M}}$ *is convex.*

□ All the proofs are elementary computations. □

There is a converse to (b).

THEOREM 40 *If* $f :]-1, 1[\mapsto \mathbb{R}$ *satisfies* $1 - |t| \le f(t) \le 1 + |t|, \; -1 < t < 1,$ *then*

$$\mathfrak{M}(a,b) = \mathfrak{A}(a,b)) f\Big(\frac{a-b}{a+b}\Big)$$

is a homogeneous internal mean with $\mathcal{I}^{\mathfrak{A}}_{\mathfrak{M}} = f$.

□ Let $0 < a \le b$ then

$$\begin{aligned} a =& \frac{a+b}{2}\Big(1 - \frac{b-a}{b+a}\Big) = \frac{a+b}{2}\Big(1 - \Big|\frac{a-b}{b+a}\Big|\Big) \le \frac{a+b}{2} f\Big(\frac{a-b}{b+a}\Big) \\ \le& \frac{a+b}{2}\Big(1 + \Big|\frac{a-b}{b+a}\Big|\Big) = \frac{a+b}{2}\Big(1 + \frac{b-a}{b+a}\Big) = b. \end{aligned}$$

The rest is immediate. □

REMARK (i) The arithmetic mean can be replaced in the above by any homogeneous internal mean $\mathfrak{B}$ by defining

$$\mathcal{I}^{\mathfrak{B}}_{\mathfrak{M}}(t) = \frac{\mathfrak{M}(1+t, 1-t)}{\mathfrak{B}(1+t, 1-t)} = \frac{\mathcal{I}^{\mathfrak{A}}_{\mathfrak{M}}(t)}{\mathcal{I}^{\mathfrak{A}}_{\mathfrak{B}}(t)}.$$

The resulting theory is with suitable modifications similar to the above; [*Kahlig & Matkowski*].

3 Compounding of Means

3.1 COMPOUND MEANS The process of II 1.3.5 of using the arithmetic and harmonic means of two numbers to generate a sequence that converges to their geometric mean is capable of generalization; [B^2 *pp.243–266*], [*Lehmer*].

DEFINITION 1 *Let $\mathfrak{M}$ and $\mathfrak{N}$ be two variable means, and if $a, b > 0$, define a_n, b_n, $n \geq 0$, by $a_0 = a$, $b_0 = b$ and*

$$a_n = \mathfrak{M}(a_{n-1}, b_{n-1}), \quad b_n = \mathfrak{N}(a_{n-1}, b_{n-1}),\ n \geq 1. \tag{1}$$

If both the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and are equal, then the common value is called the compound mean of a and b by $\mathfrak{M}$ and $\mathfrak{N}$, *written $\mathfrak{M}\otimes\mathfrak{N}(a, b)$.*

EXAMPLE (i) The result in II 1.3.5 shows that $\mathfrak{H}\otimes\mathfrak{A}(a, b) = \mathfrak{G}(a, b)$. An interesting discussion of this iteration is given in [*Nowicki 1998*]; see also [*Mathieu*].

EXAMPLE (ii) If $\mathfrak{M} = \max$, $\mathfrak{N} = \min$ then $\mathfrak{M}\otimes\mathfrak{N}(a, b)$ does not exist unless $a = b$. However see Corollary 3 below.

EXAMPLE (iii) If $\mathfrak{M}$ is any two variable mean then with the above notation for a_n, b_n, $n \geq 0$, $\mathfrak{A}\otimes\mathfrak{M}(a, b) = a_0 + \frac{1}{2}\sum_{n=0}^{\infty}(b_n - a_n)$.

REMARK (i) If $\mathfrak{M}$ and $\mathfrak{N}$ are symmetric then so is $\mathfrak{M}\otimes\mathfrak{N}$ if it exists, and $\mathfrak{M}\otimes\mathfrak{N} = \mathfrak{N}\otimes\mathfrak{M}$. In general $\mathfrak{M}\otimes\mathfrak{N}(a, b) = \mathfrak{N}\otimes\mathfrak{M}(b, a)$.

REMARK (ii) Since $\mathfrak{M}\otimes(\mathfrak{N}\otimes\mathfrak{N}) = \mathfrak{M}\otimes\mathfrak{N}$, we see that compounding is not associative.

REMARK (iii) Sometimes $\mathfrak{M}\otimes\mathfrak{N}(a, b)$ is called the *Gaussian product* of the means, to distinguish it from the *Archimedean product*, $\mathfrak{M}\otimes_a\mathfrak{N}(a, b)$ defined by the iteration $a_0 = a$, $b_0 = b$ and $a_n = \mathfrak{M}(a_{n-1}, b_{n-1})$, $b_n = \mathfrak{N}(a_n, b_{n-1})$, $n \geq 1$; see [*B^2 p.246*], and below 3.2.2(a).

THEOREM 2 *(a) If $p, q \in \mathbb{R}$ then $\mathfrak{M}^{[p]}\otimes\mathfrak{M}^{[q]}$ exists; further $\mathfrak{M}^{[p]}\otimes\mathfrak{M}^{[q]}$ is a power mean $\mathfrak{M}^{[s]}$ for some $s \in \mathbb{R}$ if and only if $p + q = 0$, when $s = 0$.*
(b) If $0 < a < b$ and $p < q$ then

$$\mathfrak{M}^{[p]}(a, b) < \mathfrak{M}^{[p]}\otimes\mathfrak{M}^{[q]}(a, b) < \mathfrak{M}^{[q]}(a, b).$$

(c) If $0 < a < b$ and $p < q < r < s$ then

$$\mathfrak{M}^{[p]}\otimes\mathfrak{M}^{[q]}(a, b) < \mathfrak{M}^{[r]}\otimes\mathfrak{M}^{[s]}(a, b).$$

□ (a) Assume that $0 < a < b$ and $p < q$, then from (r;s), and using the notation of Definition 1, $a = a_0 < a_1 < a_2 < \cdots b_2 < b_1 < b_0 = b$. So the limits $\lim_{k\to\infty} a_k$, $\lim_{k\to\infty} b_k$ exist, α, β say, and:

$$a_n = \mathfrak{M}^{[p]}(a_{n-1}, b_{n-1}) < \alpha \leq \beta < b_n = \mathfrak{M}^{[q]}(a_{n-1}, b_{n-1}),\ n \geq 1.$$

Hence by continuity, $\alpha = \mathfrak{M}^{[p]}(\alpha, \beta)$, $\beta = \mathfrak{M}^{[q]}(\alpha, \beta)$, either of which implies that $\alpha = \beta$, by III 1 Theorem 2 (a).
Noting that $\mathfrak{M}^{[p]}(a, b)\mathfrak{M}^{[-p]}(a, b) = ab$, it follows that for the defining sequences of $\mathfrak{M}^{[p]} \otimes \mathfrak{M}^{[-p]}$, $a_n b_n = ab$, $n \geq 0$. Then

$$\mathfrak{M}^{[p]} \otimes \mathfrak{M}^{[-p]}(a, b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{a_n b_n} = \sqrt{ab} = \mathfrak{G}(a, b).$$

Now from 2.1.3 Theorem 19 $\mathfrak{M}^{[r]}(a, b) = b\mathfrak{M}^{[r]}(1-t, 1) = b\left(1 + \sum_{n=1}^{\infty} r_n t^n\right)$ where $a = (1-t)b$, $0 < t < 1$; the first four coefficients are $r_1 = -\dfrac{1}{2}$, $r_2 = \dfrac{r-1}{8}$, $r_3 = -\dfrac{r-1}{16}$, $r_4 = \dfrac{(r-1)(r-3)(2r+5)}{384}$. Below we will use this with r replaced by p, q and s.
If a_n, b_n, $n \geq 0$, are the terms of the defining sequence of $\mathfrak{M}^{[p]} \otimes \mathfrak{M}^{[q]}(a, b)$ then from the above with $b_{n-1} = (1-\tau)a_{n-1}$,

$$b_n - a_n = b_{n-1}\big(\mathfrak{M}^{[q]}(1-\tau, 1) - \mathfrak{M}^{[p]}(1-\tau, 1)\big) = b_{n-1}\sum_{k=2}^{\infty}(q_k - p_k)\tau^k.$$

Hence the convergence of these sequences to $\alpha = \beta = \mathfrak{M}^{[p]} \otimes \mathfrak{M}^{[q]}$ is quadratic. Assume now that $b = 1$ and that $\mathfrak{M}^{[p]} \otimes \mathfrak{M}^{[q]} = \mathfrak{M}^{[s]}$, then simple calculations yield

$$\begin{aligned} a_3 = 1 - \frac{t}{2} + \frac{p+q-2}{16}\Big(t^2 + \frac{t^3}{2}\Big) \\ - \frac{4(p^3+q^3) + (p+q)^3 - 6(p+q)^2 - 56(p+q) + 120}{3072}t^4 + \cdots \end{aligned}$$

Since the convergence is quadratic a_3 and $\mathfrak{M}^{[p]} \otimes \mathfrak{M}^{[q]}(1-t, 1) = \mathfrak{M}^{[s]}(1-t, 1)$ agree up to the term t^7. From the coefficient of t^2 that $(p+q-2)/16 = (s-1)/8$; that is $p + q = 2s$. Using this and comparing the coefficients of t^4 leads to $s^3 = s(p^2 - pq + q^2)$. Since $p \neq q$ we find that $s = 0$.
(b) and (c) are immediate. □

COROLLARY 3 *If two strictly internal continuous means are comparable then their compounds exist.*

□ It suffices to note that the existence part of the proof in Theorem 2 only uses the comparability of the strictly internal continuous power means. [B^2 *p.244*]. □

In a similar way can obtain the following theorem for Hamy means, see V 7.1[18].

[18] In part (c) note that $(\mathfrak{H}\mathfrak{a})^{[1/2]} = \mathfrak{G}$; see V 7.1.

THEOREM 4 *(a) If $p, q \in \mathbb{R}$ then $(\mathfrak{H}\mathfrak{a})^{[p]} \otimes (\mathfrak{H}\mathfrak{a})^{[q]}$ exists; further $(\mathfrak{H}\mathfrak{a})^{[p]} \otimes (\mathfrak{H}\mathfrak{a})^{[q]}$ is a Hamy mean $(\mathfrak{H}\mathfrak{a})^{[s]}$ for some $s \in \mathbb{R}$ if and only if $p+q = 0, 1,$ or 2 when $s = 0, 1/2$ and 1 respectively.*
(b) $(\mathfrak{H}\mathfrak{a})^{[p]} \otimes (\mathfrak{H}\mathfrak{a})^{[q]}$ is a power mean $\mathfrak{M}^{[s]}$ for some $s \in \mathbb{R}$ if and only if $p+q = 0, 1,$ or 2 when $s = -1, 0$ and 1 respectively.
(c) $\mathfrak{M}^{[p]} \otimes \mathfrak{M}^{[q]}$ is a Hamy mean $(\mathfrak{H}\mathfrak{a})^{[s]}$ for some $s \in \mathbb{R}$ if and only if $p+q = 0$ when $s = 1/2$.

REMARK (iv) Lehmer has considered in some detail the new mean $\mathfrak{A} \otimes (\mathfrak{H}_{\mathfrak{a}})^{[2]}$; [*Lehmer*].

A generalization of $\mathfrak{M}^{[p]} \otimes \mathfrak{M}^{[q]}$ to n-tuples, $n \geq 3$, has been given;[*Gustin 1947*]. Let $\underline{t}$ be a real n-tuple then the compound $\bigotimes_{i=1}^{n} \mathfrak{M}_n^{[t_i]}(\underline{a}; \underline{w})$ is defined as follows: $\underline{a}^{(0)} = \underline{a} = (a_1, \ldots, a_n) = (a_1^{(0)}, \ldots, a_n^{(0)})$, $\underline{a}^{(k)} = (a_1^{(k)}, \ldots, a_n^{(k)})$, $k \geq 1$, where

$$a_i^{(k)} = \mathfrak{M}_n^{[t_i]}(\underline{a}^{(k-1)}; \underline{w}), \ 1 \leq i \leq n, \ k \geq 1. \tag{2}$$

Then

$$\bigotimes_{i=1}^{n} \mathfrak{M}_n^{[t_i]}(\underline{a}; \underline{w}) = \lim_{k \to \infty} a_i^{(k)}, \ 1 \leq i \leq n.$$

THEOREM 5 *The compound mean $\bigotimes_{i=1}^{n} \mathfrak{M}_n^{[t_i]}(\underline{a}; \underline{w})$ exists and is strictly internal.*

□ It suffices to show that the n limits in (2) exist and are equal.
Assume, without loss in generality, that $t_1 \leq \cdots \leq t_n$. Then from (r;s) for all $k \geq 1$, $a_1^{(k)} \leq \cdots \leq a_n^{(k)}$. Hence

$$a_1^{(k)} \leq \mathfrak{M}_n^{[t_i]}(\underline{a}^{(k)}; \underline{w}) = a_i^{(k+1)} \leq a_n^{(k+1)} = \mathfrak{M}_n^{[t_n]}(\underline{a}^{(k)}; \underline{w}) \leq a_n^{(k)}.$$

From this we can define $\alpha_1 = \lim_{k \to \infty} a_1^{(k)} \leq \alpha_n = \lim_{k \to \infty} a_n^{(k)}$. Now let $\phi(x) = x^{t_1}$, if $t_1 \neq 0$, and $= \log x$ if $t_1 = 0$. Then

$$\phi(\alpha_1) = \lim_{k \to \infty} \phi(a_1^{(k)}) = \lim_{k \to \infty} \frac{1}{W_n} \sum_{i=1}^{n} w_i \phi(a_i^{(k)}) = \frac{1}{W_n} \sum_{i=1}^{n} w_i \lim_{k \to \infty} \phi(a_i^{(k)}).$$

Hence by the continuity of ϕ, $\lim_{k \to \infty} a_i^{(k)}$ exists with value α_i, say, $2 \leq i \leq n-1$. From III 1(2), $\alpha_1 = \cdots = \alpha_n$. □

A different proof of this result is in [*Everett & Metropolis*].

REMARK (v) This procedure can be used for more general means; see [B^2 *pp.266–273*], [*Borwein & Borwein*]. However the general topic of compounding means of

more than two numbers leads outside of our main topic, the main interest seems to be the determination of the domain of convergence in the complex plane; see [*Myrberg*], and five subsequence papers by the same author in the same journal over the next decade. See as well 3.2.2(f) below.

REMARK (vi) Other papers studying compounding and iteration of means are [*Andreoli 1957b; Carlson 1971,1972a; Ciorănescu 1936b; Heinrich; Rosenberg; Stieltjes; Toader 1991a; Todd; Tonelli; Wimp 1985*].

3.2 THE ARITHMETICO-GEOMETRIC MEAN AND VARIANTS

3.2.1 THE GAUSSIAN ITERATION The compound mean $\mathfrak{G}\otimes\mathfrak{A}(a,b)$ is called the *(Gauss) arithmetico-geometric, or just the arithmetic-geometric, mean of a and* b[19]. In this case, with $0<a<b$, the sequences in 3.1(1) become

$$0<a=a_0<b_0=b,\ a_n=\sqrt{a_{n-1}b_{n-1}},\ b_n=\frac{a_{n-1}+b_{n-1}}{2},\ n\geq 1. \qquad (3)$$

Defining $c_n=\sqrt{b_n^2-a_n^2}$, $n\geq 0$, then $c_n=\dfrac{b_{n-1}-a_{n-1}}{2}$, $n\geq 1$ and these sequences can be extended to all integers as follows. Let $n\leq -1$ and define:

$$b_n=b_{n+1}+c_{n+1}, a_n=b_{n+1}-c_{n+1}, c_n=\sqrt{b_n^2-a_n^2}=\frac{b_{n+1}-a_{n+1}}{2}. \qquad (4)$$

LEMMA 6 *(a) a_n and b_n satisfy (4) for all $n\in\mathbb{Z}$.*
(b) $a_n<a_{n+1}<b_n<b_{n+1}$ for all $n\in\mathbb{Z}$.
(c) $\lim_{n\to-\infty}a_n=0$, and $\lim_{n\to-\infty}b_n=\infty$.
(d) $\mathfrak{G}\otimes\mathfrak{A}(a,b)=\mathfrak{G}\otimes\mathfrak{A}(a_n,b_n)$ for all $n\in\mathbb{Z}$.
(e) $\mathfrak{G}\otimes\mathfrak{A}(\lambda a,\lambda b)=\lambda\mathfrak{G}\otimes\mathfrak{A}(a,b)$ if $\lambda\geq 0$.
(f) $a<\mathfrak{G}(a,b)<\mathfrak{G}\otimes\mathfrak{A}(a,b)<\mathfrak{A}(a,b)<b$.

□ All these are immediate except perhaps (c). An easy induction shows that if $n\geq 1$ then $c_{-n}^2>4^n c_0^2$. This using (b) and (4) gives that $\lim_{n\to-\infty}b_n=\infty$. Another easy induction gives $a_{-n-1}/a_{-n}<a_{-n}/n_{-n}$, $n\geq 1$. This using (b) and the limit just established completes the proof of (c). □

REMARK (i) It follows from (e) and (f) and the definition that the arithmetico-geometric mean is strictly internal and has the properties (Ho) and (Sy).

LEMMA 7 *(a) If $0<a<b$ then $\mathfrak{G}\otimes\mathfrak{A}(a,b)=a\prod_{n=0}^{\infty}\dfrac{1}{\sqrt{\cos\theta_n}}$, where $\cos\theta_n=a_n/b_n$ and $0<\theta_n<\pi/2$, $n\in\mathbb{N}$.*
(b) If $-1<t<1$ then

$$(1+t^2)\mathfrak{G}\otimes\mathfrak{A}\Big(1-\frac{2t}{1+t^2},1+\frac{2t}{1+t^2}\Big)=\mathfrak{G}\otimes\mathfrak{A}(1-t^2,1+t^2). \qquad (5)$$

[19] This is also called the *Gaussian mean* although it seems to have been first defined by Lagrange.

□ (a) Note that from (3) $a_n/b_n = (a_n/a_{n+1})^2, n \geq 0$.
(b) Use Lemma 6(e) and then Lemma 6(d) with $n = 1$. □

Identity (5), which characterizes the arithmetico-geometric mean, [*Mohr 1953*], can be used to obtain the following theorem; see [B^2*p.6*].

THEOREM 8

$$\frac{1}{\mathfrak{G}\otimes\mathfrak{A}(1-x, 1+x)} = 1 + \sum_{n=1}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 x^{2n}, \quad |x| < 1. \tag{6}$$

□ Let the left-hand side of (6) be denoted by $f(x)$, when (5) becomes

$$\frac{1}{1+t^2} f\Big(\frac{2t}{1+t^2}\Big) = f(t^2). \tag{7}$$

Since $f(0) = 1$, and f is even suppose that $f(x) = 1 + \sum_{n=1}^{\infty} \alpha_{2n+1} x^{2n}$. Then using (7) and identifying coefficients gives the recurrence $\alpha_1 = 1$, and if $n \geq 1$, $(2n)^2\alpha_{2n+1} = (2n-1)^2\alpha_{2n-1}$. This leads to (6). □

COROLLARY 9 *If* $0 < a \leq b$ *then*

$$\frac{1}{\mathfrak{G}\otimes\mathfrak{A}(a, b)} = \frac{1}{\pi}\int_0^{\pi} \frac{1}{\sqrt{b^2\sin^2\phi + a^2\cos^2\phi}}\,\mathrm{d}\phi. \tag{8}$$

□ (*i*) Putting $bx = \sqrt{b^2 - a^2}$ the integral is just $1/b\pi \int_0^{\pi} 1/(\sqrt{1 - x^2\cos^2\phi})\,\mathrm{d}\phi$, the *complete elliptic integral of the first kind*. The same change, using Lemma 6(d) with $n = -1$, gives the reciprocal of the left-hand side as $b\,\mathfrak{G}\otimes\mathfrak{A}(1-x^2, 1+x^2)$. The result is now immediate from (6).
(*ii*) A proof by van de Riet, [*van de Riet 1964*], uses the following identity due to van der Pol:

$$\int_0^{\pi/2} \frac{1}{\sqrt{(R+r)^2 - 4Rr\sin 2\theta}}\,\mathrm{d}\theta = \int_0^{\pi/2} \frac{1}{\sqrt{(R^2 - r^2\cos 2\theta}}\,\mathrm{d}\theta,$$

both sides being multiples of the potential of a uniform circular ring; see [*Kellog p.59; Whittaker & Watson p.399*]. From this identity van de Riet proves that

$$\int_0^{\pi/2} \frac{1}{\sqrt{b^2\sin^2\phi + a^2\cos^2\phi}}\,\mathrm{d}\phi = \int_0^{\pi/2} \frac{1}{\sqrt{b_n^2\sin^2\phi + a_n^2\cos^2\phi}}\,\mathrm{d}\phi, \tag{9}$$

which gives the result on letting $n \to \infty$. □

REMARK (ii) The identity (8) defines the arithmetico-geometric mean and variants of the right-hand side have been used to define other means; see above 2.1.5.

REMARK (iii) The identity (9) is due to Gauss and from (8) we see that it is just Lemma 6(d); see [*Kellogg pp.58–62; Melzak 1961, pp.68–70; Whittaker & Watson p.533*]; for a generalization see [*Carlson 1975*].

Relations with the arithmetic, geometric and logarithmic means have been obtained by Sándor; [*Sándor 1995a,1996; Vamanamurthy & Vuorinen*].

THEOREM 10 *If* $0 < a < b$ *then*

$$\mathfrak{G}(a,b) < \mathfrak{L}(a,b) < \mathfrak{G}\otimes\mathfrak{A}(a,b) < \mathfrak{I}(a,b) < \mathfrak{A}(a,b);$$

$$(\mathfrak{G}\otimes\mathfrak{A})^2(a,b) < \mathfrak{L}(a,b)\sqrt{\mathfrak{A}(a,b)\big(\mathfrak{A}(a,b)+\mathfrak{G}(a,b)\big)/2} < \mathfrak{A}(a,b)\mathfrak{G}(a,b);$$

$$\mathfrak{I}(a,b) = \sqrt{\mathfrak{A}(a,b)\mathfrak{G}(a,b)} < \mathfrak{G}\otimes\mathfrak{A}(a,b);$$

$$\frac{2}{\pi}\Big(\frac{1}{\mathfrak{L}(a,b)} - \frac{1}{\mathfrak{A}(a,b)}\Big) < \frac{1}{\mathfrak{G}\otimes\mathfrak{A}(a,b)} - \frac{1}{\mathfrak{A}(a,b)} < \frac{12}{\pi}\Big(\frac{1}{\mathfrak{L}(a,b)} - \frac{1}{\mathfrak{A}(a,b)}\Big).$$

A complete and masterful study of the arithmetico-geometric mean can be found in [B^2] which everyone is urged to read. Various other references for this topic are: [*Gauss*], [*Allasia 1983; Almkvist & Berndt; Aumann 1935c; Barna 1934,1939; Ciorănescu 1936a; Cox 1980,1985; Dávid 1907,1909,1913,1928; Faragó; Foster & Phillips 1984a; Fricke; Frisby; Geppert 1932,1933; Gosiewski 1909a,b; Hofsommer & van de Riet; Lohnsein 1888a,b; Salamin; Stöhr; Tietze; Uspensky; van de Riet 1963; Žuravskiĭ*].

3.2.2 OTHER ITERATIONS Various modifications of the Gauss procedure have been studied by several authors. We assume in this section that $0 < a = a_0 < b_0 = b$.

(a) [*Gauss*] Gauss also defined: $a_n = (a_{n-1}+b_{n-1})/2$, $b_n = \sqrt{a_n b_{n-1}}$, $n \ge 1$, and Pfaff showed that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \dfrac{\sqrt{b^2-a^2}}{\arccos(a/b)}$.
Such algorithms go back to Archimedes as an interesting article by Miel points out; [B^2 *p.250*], [*Heath vol.II pp.50–56*], [*Kämmerer; Miel*]. This iteration is what has been called an Archimedean product, see 3.1 Remark (iii), so the above scheme gives $\mathfrak{A}\otimes_a\mathfrak{G}(a,b)$. See also [*Foster & Phillips 1984b; Toader1987a*].

REMARK (i) It is of some interest to note that $\mathfrak{H}\otimes_a\mathfrak{G}(2\sqrt{3},3) = \pi$; this is a particular case of [B^2 *(8.4.2)*].

(b) [*Beke*] Beke defined $a_n = (a_{n-1}+b_n)/2$, $b_n = \sqrt{a_{n-1}b_{n-1}}$, $n \ge 1$. The limits exist and both equal $\sqrt{a(b-a)}/\cos(\sqrt{a/b})$.

(c) [*von Bültzingslöven*] This author defines $a_n = (a_{n-1}+b_{n-1})/2$, $b_n = \sqrt{a_n b_{n-1}}$, $n \ge 1$. The limits of these sequences exist and are both equal to $(a+2b)/3$; see also [*Aczél*].

(d) [*Borchardt 1861,1876,1877,1878; Gatteschi; Hettner; Schering*] These authors were the first to see if the association of the Gauss procedure with elliptic functions could be extended to other functions, such as the hyperelliptic functions. Starting with four numbers a_0, b_0, c_0, d_0 Borchardt defined, for $n \ge 1$,

$$a_n = \mathfrak{A}_4(a_{n-1},b_{n-1},c_{n-1},d_{n-1}),\ b_n = \mathfrak{A}_2\big(\sqrt{a_{n-1}b_{n-1}},\sqrt{c_{n-1}d_{n-1}}\big),$$
$$c_n = \mathfrak{A}_2\big(\sqrt{a_{n-1}c_{n-1}},\sqrt{b_{n-1}d_{n-1}}\big),\ d_n = \mathfrak{A}_2\big(\sqrt{a_{n-1}d_{n-1}},\sqrt{c_{n-1}c_{n-1}}\big).$$

This iteration has been studied in detail in [*Veinger*]; see in addition [B^2 *p.272*], [*Kuznecov*].

(e) [*Ory*] Ory has extended the Heron method of computing square roots, II 1.3.5, that is connected with $\mathfrak{H}\otimes\mathfrak{A} = \mathfrak{G}$, to the problem of finding higher order roots. Consider the case of cube roots: if $0 < a = a_0 < b = b_0 < c = c_0$ define for $n \geq 1$, $a_n = \mathfrak{H}_3(a_{n-1}, b_{n-1}, c_{n-1})$, $c_n = \mathfrak{A}_3(a_{n-1}, b_{n-1}, c_{n-1})$, and $a_n b_n = \mathfrak{H}_3(a_{n-1}b_{n-1}, b_{n-1}c_{n-1}, c_{n-1}a_{n-1})$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \mathfrak{G}_3(a, b, c) = \sqrt[3]{abc}$.

(f) [*Sternberg; Bellman 1956*]. Sternberg and Bellman used elementary symmetric polynomial means to give a natural extension of $\mathfrak{G}\otimes\mathfrak{A}$ to n-tuples. Let $\underline{a}$ be a positive strictly decreasing n-tuple and define the sequence of n-tuples $\underline{a}^{(0)} = \underline{a} = (a_1 \ldots, a_n) = (a_1^{(0)}, \ldots, a_n^{(0)})$, $\underline{a}^{(k)} = (a_1^{(k)}, \ldots, a_n^{(k)})$, $k \geq 1$, by $a_i^{(k)} = \mathfrak{S}_n^{[i]}(\underline{a}^{(k-1)}; \underline{w})$, $1 \leq i \leq n$, $k \geq 1$. By S(r;s) $a_1^{(k)} > \cdots > a_n^{(k)} > 0$, and by simple properties of the elementary symmetric polynomial means $a_1^{(k-1)} > a_1^{(k)} > a_n^{(k)} a_n^{(k-1)}$. So we can define $A = \lim_{k\to\infty} a_1^{(k-1)}$, and $B = \lim_{k\to\infty} a_n^{(k-1)}$; further$A \geq B$. In fact $A = B$ as we now prove by showing that $A \leq B$. If $k \geq 1$ then $(a_n^{(k)})^n = \prod_{i=1}^n a_n^{(k-1)}$, and so $\prod_{j=1}^k a_n^{(j)} = \prod_{i=1}^n \prod_{j=0}^n a_i^{(j)} > \left(\prod_{j=0}^{k-1} a_i^{(j)}\right)\left(\prod_{j=0}^{k-1} a_i^{(j)}\right)^{n-1}$. On simplification this gives

$$\prod_{j=1}^{k-1} a_n^{(j)} \geq \left(\frac{a_n^{(k)}}{a_n^{(0)}}\right)^n \prod_{j=1}^{k-1} a_1^{(j)} > \left(\frac{a_n^{(k)}}{a_n^{(0)}}\right)^n \prod_{j=0}^{k-1} a_i^{(j)}.$$

Hence

$$\lim_{k\to\infty} \left(\prod_{j=1}^{k-1} a_n^{(j)}\right)^{1/k} \geq \lim_{k\to\infty} \left(\frac{a_n^{(0)}}{a_n^{(0)}}\right)^{n/k} \lim_{k\to\infty} \left(\prod_{j=0}^{k-1} a_1^{(j)}\right)^{1/k}; \text{ that is } B \geq A.$$

4 Some General Approaches to Means

In this section we consider some general approaches to means that differ from the more systematic approach in Chapter IV.

4.1 LEVEL SURFACE MEANS A method of proof of the two variable case of (GA) can be generalized to give a very simple definition of a mean; see II 2.2.1 Lemma 3 proof (*viii*), II 2.2.2 Lemma 4 proof (*iii*); [*de Finetti p.56*], [*Chisini 1929,1957; de Finetti 1930,1931; Dodd 1936a,b,1940; Jecklin 1948c,1949d; Martinotti 1931*]. Let $F : (\mathbb{R}_+^*)^n \mapsto \mathbb{R}$ then the *F-level mean of the n-tuple* $\underline{a}$ is μ where

$$F(\mu, \ldots, \mu) = F(a_1, \ldots . a_n),$$

provided F is such that μ is unique; μ is the value of t where the level of F through $\underline{a}$ meets the line $\underline{x} = t\underline{e}$.

Example (i) For $F(\underline{a}) = \sum_{i=1}^{n} w_i \mathcal{M}(a_i)$ the F-level mean is the quasi-$\mathcal{M}$-mean.

Example (ii) If $F(\underline{a}) = \sum_r ! \prod_{k=1}^{r} a_{i_k}$ then the F-level mean is the r-th elementary symmetric polynomial mean.

Example (iii) Example (i) can be generalized by taking $F(\underline{a}) = \sum_{i=1}^{n} g_i(a_i)$ when the F-level-mean is the solution of

$$\sum_{i=1}^{n} g_i(\mu) = \sum_{i=1}^{n} g_i(a_i). \tag{1}$$

A particularly interesting case of this has been studied by Landsberg, [*Landsberg 1980a*].

Let ϕ and γ_i, $i \le i \le n$, be positive functions defined for positive real numbers and for some given positive a put $g_i(x) = \int_a^x \gamma_i/\phi, 1 \le i \le n$, $x > 0$. In this case, from (1), the F-level mean $\mu = \mu(\phi) = \mu(\phi, \underline{\gamma})$ is given by

$$\sum_{i=1}^{n} \int_{a_i}^{\mu} \frac{\gamma_i}{\phi} = 0. \tag{2}$$

Putting $\Phi(x) = \sum_{i=1}^{n} \int_{a_i}^{x} \gamma_i/\phi$ it is easy to deduce that Φ is strictly increasing and, by (2), $\Phi(\mu(\phi)) = 0$. In addition we can deduce that $\Phi(x) \ge \dfrac{1}{\phi(x)} \sum_{i=1}^{n} \int_{a_i}^{x} \gamma_i$. Hence, again by (2), $\Phi(\mu(1)) \ge 0$. It follows, from the monotonicity of Φ that $\mu(1) \ge \mu(\phi)$.

By appropriate choices of ϕ and $\underline{\gamma}$ this simple result implies both (GA) and (r;s). Further details can be found in the paper by Landsberg where there are more inequalities; see also [*Landsberg & Pečarić*].

Example (iv) In the case $n = 2$ the equal weight power means are pictured as F-level means in Figure 1; where the curves nMN, nRn, nSN, nAN, nGN, nHN, nmN are $\max\{x,y\} = b$, $x^s + y^s = a^s + b^s$, $x^r + y^r = a^r + b^r, (1 < r < s)$, $x + y = a + b$, $xy = ab$, $x^{-1} + y^{-1} = a^{-1} + b^{-1}, \min\{x,y\} = a$ respectively; and the points M, R, S, A, G , H, m have both coordinates equal and equal to $\mathfrak{M}^{[\infty]}(a,b), \mathfrak{M}^{[s]}(a,b), \mathfrak{M}^{[r]}(a,b)$, $\mathfrak{A}(a,b), \mathfrak{G}(a,b), \mathfrak{H}(a,b), \mathfrak{M}^{[-\infty]}(a,b)$ respectively.

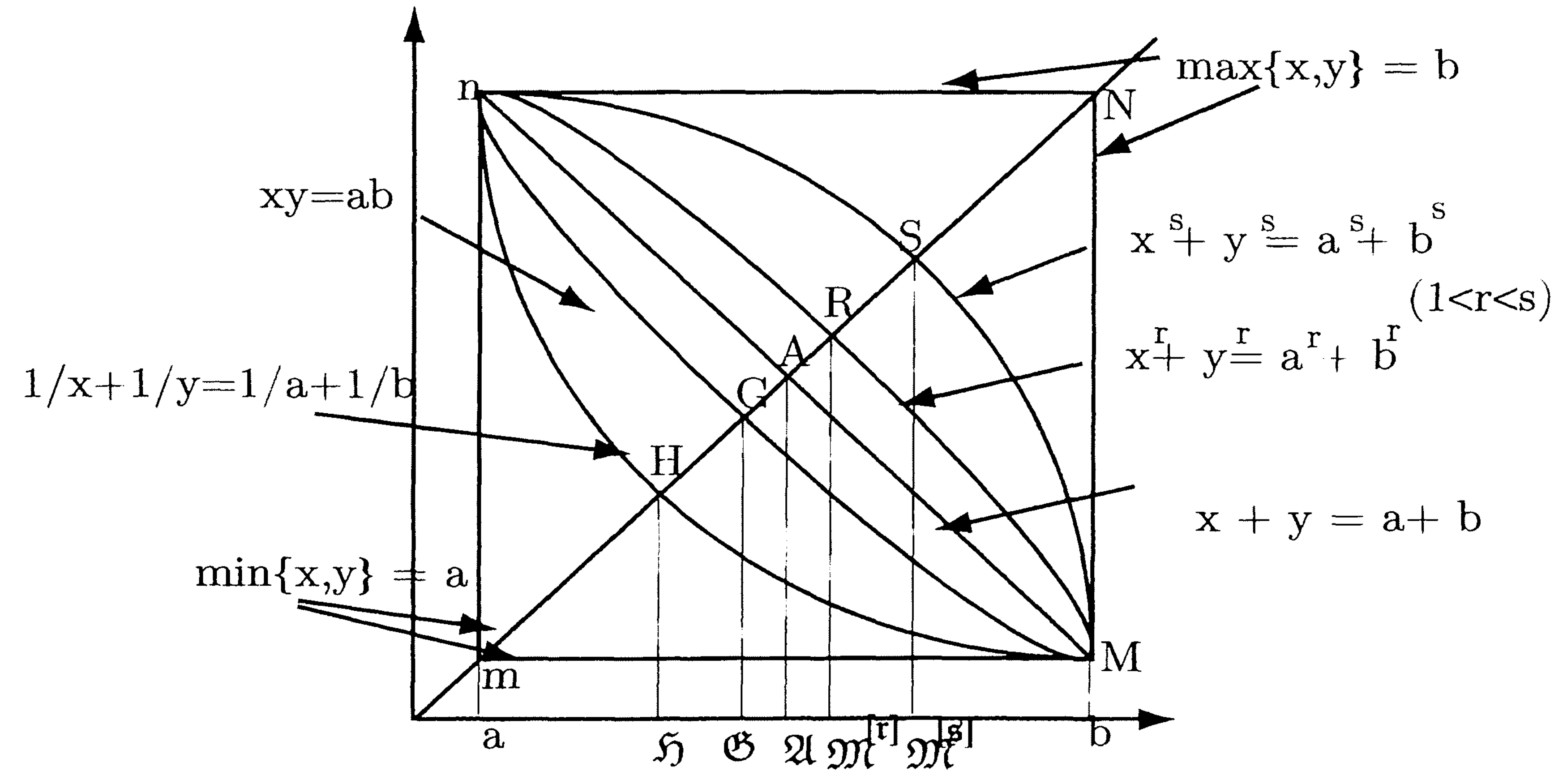

Figure 1

Consideration of Figure 1 leads naturally to the consideration of a mean, $\mathfrak{M}_{(s)}(a,b)$ that is *symmetric to the mean* $\mathfrak{M}(a,b)$ *with respect to the arithmetic mean* . This is illustrated by the level curves MNn, Mmn, associated with the means $\max\{a,b\}$, $\min\{a,b\}$, that are symmetric with respect to the line Mn. Simple calculations show that $\mathfrak{M}_{(s)} = 2\mathfrak{A} - \mathfrak{M}$. The particular means $\mathfrak{G}_{(s)} = 2\mathfrak{A} - \mathfrak{G}$ and $\mathfrak{H}_{(s)} = 2\mathfrak{A} - \mathfrak{H}$ have been studied by Tricomi; [*Tricomi 1965, 1969/70*]. The same author has also studied the question: for which numbers s and t is it true that $\mathfrak{M} = (1-s-t)\mathfrak{A} + s\mathfrak{H} + t\mathfrak{G}$ is internal?

4.2 Corresponding Means The simple relations II 1.2(7) and (8) that relate the arithmetic mean to the harmonic and geometric means respectively suggest the idea of corresponding means; [*Andreoli 1957a*]. Given a mean $\mathfrak{M}$, defined for all n-tuples, and a strictly monotonic function $f : \mathbb{R}_+^* \mapsto \mathbb{R}_+^*$ then *the mean* $\widetilde{\mathfrak{M}} = \widetilde{\mathfrak{M}}_{(f)}$ *is said to correspond to* $\mathfrak{M}$ *by* f if for all n-tuples $\underline{a}$

$$\widetilde{\mathfrak{M}}(\underline{a}) = f\Big(\mathfrak{M}\big(f^{-1}(\underline{a})\big)\Big)$$

Obviously then $\mathfrak{M}$ corresponds to $\widetilde{\mathfrak{M}}_{(f)}$ by f^{-1}; and if $\widetilde{\mathfrak{M}}$ corresponds $\mathfrak{M}$ by f and $\mathfrak{M}$ corresponds to $\mathfrak{M}^*$ by g, then $\widetilde{\mathfrak{M}}$ corresponds to $\mathfrak{M}^*$ by $f \circ g$.

Example (i) If $\mathfrak{M} = \mathfrak{A}_n$ and (*i*) $f(x) = x^{-1}$ then $\widetilde{\mathfrak{M}}_{(f)} = \mathfrak{H}_n$; (*ii*) $f(x) = (1+e^x)^{-1}$ and if $0 < \underline{a} < 1$, then $\widetilde{\mathfrak{M}}_{(f)}(\underline{a}) = \mathfrak{G}_n(\underline{a}) \Big/ \big(\mathfrak{G}_n(\underline{a}) + \mathfrak{G}'_n(\underline{a})\big)$, using the notation of IV 4.4.

Example (ii) $\widetilde{\mathfrak{M}}_{(f)} = \mathfrak{G}_n$ if either $\mathfrak{M} = \mathfrak{H}_n$ and $f(x) = e^{1/x}$, or $\mathfrak{M} = \mathfrak{M}_n^{[2]}$ and $f(x) = e^{-x^2}$.

EXAMPLE (iii) If $\mathfrak{M} = \mathfrak{S}_n^{p,q}$ and $f(x) = x^{-1}$ then $\widetilde{\mathfrak{M}}_{(f)} = \left(\dfrac{\mathfrak{A}_{n,\underline{\alpha}}}{\mathfrak{A}_{n,\underline{\alpha}'}}\right)^{1/(p-q)}$, where $\underline{\alpha} = (p-q, p, \ldots, p)$ and $\underline{\alpha}' = (0, p, \ldots, p)$.

4.3 A MEAN OF GALVANI Galvani, [*Galvani*], suggested that $\mathfrak{m} = \mathfrak{m}(\underline{a})$ be called a mean of the n-tuple $\underline{a}$ if it satisfies one of the following equivalent conditions. If $1 \le k \le n$:

(*a*)
$$\sum_k ! \Big(\prod_{j=1}^{k} (\mathfrak{m} - a_{i_j}) \Big) = 0;$$

(*b*)
$$\binom{n}{k}\mathfrak{m}^k - \binom{n-1}{k-1}\Big(\sum_{i=1}^{n} a_i\Big)\mathfrak{m}^{k-1}\binom{n-2}{k-2}\Big(\sum_{i\neq j,1}^{n} a_i\Big)\mathfrak{m}^{k-2} + \cdots\cdots\cdots (-1)^k \sum_k ! a_{1_1} \ldots a_{i_k} = 0;$$

(*c*)
$$\text{if } f(x) = \prod_{i=1}^{n} (x - a_i) \text{ then } f^{(n-k)}(\mathfrak{m}) = 0.$$

If $k = 1$ then $\mathfrak{m}(\underline{a}) = \mathfrak{A}_n(\underline{a})$, but in general $\mathfrak{m}(\underline{a})$ is not well defined; for if $a_1, \ldots, a_n$ are all distinct then $f^{(n-k)}$ has k zeros. Uniqueness can be recovered by always choosing the largest, or smallest, value. In those cases, as k decreases the mean $\mathfrak{m}(\underline{a})$ increases, respectively decreases, from the arithmetic mean to $\max \underline{a}$, respectively to $\min \underline{a}$.

A similar idea was studied in [*Chimenti*] as a result of the following problem of Jackson, [*Jackson*]: *if* $a_1 < \cdots < a_n$ *find the unique* $\mathfrak{m}$ *such that: (i)* $a_k < \mathfrak{m} < a_{k+1}$, *and (ii)* $\prod_{i=1}^{k}(\mathfrak{m} - a_i) = \prod_{i=k+1}^{n}(a_i - \mathfrak{m})$. When $n = 2$, $\mathfrak{m}$ is just the arithmetic mean.

4.4 ADMISSIBLE MEANS OF BAUER An interval $I \subseteq \mathbb{R}_+^*$ is said to be *contractive* if $x \in I$ implies that $x^n \in I$, $n = 1, 2, \ldots$. This means that: either $I = \mathbb{R}_+^*$, $I = [a, \infty[$ or $]a, \infty[$ for some $a \ge 1$, or $I =]0, a]$ or $]0, a[$ for some $a \le 1$.

If $u : I \mapsto \mathbb{R}_+^*$ is continuous and either (a) is decreasing, or (b) $u(x)/x$ is strictly increasing, see [*DI pp.237–238*], then u is said to be *admissible*; if (a) holds it is *admissible of type 1* , and if (b) holds it is *admissible of type 2* . These concepts are due to Bauer; [*Alzer 1987b; Bauer 1986a,b*].

LEMMA 1 *If u is an admissible function, $k \in \mathbb{N}^*$ and $v(x) = u(x^k)/x$, then v is strictly decreasing if u is of type 1, and strictly increasing if it is of type 2.*

□ Elementary. □

If u is an admissible function on the contractive interval I, and if $\underline{a}$ an n-tuple, $n \geq 2$, with elements in I write $\alpha_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} a_j$, $1 \leq i \leq n$, and define

$$S_u(\underline{a}) = \frac{\sum_{i=1}^n u(\alpha_i)}{\sum_{i=1}^n a_i}. \tag{3}$$

THEOREM 2 *With the above notation the equation*

$$\frac{u(x^{n-1})}{x} = S_u(\underline{a}) \tag{4}$$

has a unique solution c, with $\min \underline{a} \leq c \leq \max \underline{a}$, *and equality if and only if $\underline{a}$ is constant.*

□ The case of equality follows from Lemma 1 and (3).
Assume without loss in generality that $a_1 \leq \cdots \leq a_n$, $a_1 \neq a_n$.
(i): `u is of type 1`. For all i, $i \leq i \leq n$,

$$u(\alpha_i) \geq u(a_n^{n-1}) \geq \frac{a_i}{a_n} u(a_n^{n-1}). \tag{5}$$

Hence, since at least one of the inequalities (5) is strict, $S_u(\underline{a}) > u(a_n^{n-1})/a_n$.
A similar argument will also give that $S_u(\underline{a}) < u(a_n^{n-1})/a_1$.
The result now follows from Lemma 1 and the continuity of u.
(ii): `u is of type 2` . In this case we have that $\dfrac{u(\alpha_i)}{\alpha_i} \leq \dfrac{u(a_n^{n-1})}{a_n^{n-1}}$, $1 \leq i \leq n$.
Hence if we put $a_{n+1} = a_1$ then, $u(\alpha_i) \leq a_n^{n-2} a_{i+1} \frac{u(a_n^{n-1})}{a_n^{n-1}}$, $1 \leq i \leq n$, and again, at least one of these inequalities is strict.
So $S_u(\underline{a}) < u(a_n^{n-1})/a_n$; a similar argument gives $S_u(\underline{a}) > u(a_n^{n-1})/a_1$, and the result follows as in Case 1. □

The unique c that satisfies (4) is called the *admissible u-mean of the n-tuple* $\underline{a}$, $n \geq 2$, written $\mathfrak{B}_n^{[u]}(\underline{a})$.

EXAMPLE (i) If $I = \mathbb{R}_+^*$, $u = 1$ then $\mathfrak{B}_n^{[u]}(\underline{a}) = \mathfrak{A}_n(\underline{a})$; while if $u(x) = x^{-1}$ then $\mathfrak{B}_n^{[u]}(\underline{a}) = \mathfrak{G}_n(\underline{a})$.

EXAMPLE (ii) More generally if $I = \mathbb{R}_+^*$, $u(x) = x^p$ then u is admissible, of type 1 if $p \leq 0$, of type 2 if $p > 1$ and $\mathfrak{B}_n^{[u]}(\underline{a}) = \left(\dfrac{\mathfrak{G}_n^n(\underline{a}^p)}{\mathfrak{A}_n(\underline{a})\mathfrak{H}_n(\underline{a}^p)} \right)^{1/(np-p-1)}$; if $0 < p \leq 1$ then $u(x) = x^p$ is not admissible.

REMARK (i) No admissible mean is the harmonic mean; [*Bauer 1986a*].

THEOREM 3 *If u is a convex function that is admissible of type 1, or a concave function that is admissible of type 2, then*

$$\mathfrak{B}_n^{[u]}(\underline{a}) \le \mathfrak{A}_n(\underline{a}). \tag{6}$$

Further if u is strictly convex, respectively strictly concave, (6) is strict unless $\underline{a}$ is a constant.

□ As the two cases are similar assume that u is convex and admissible of type 1. By S(r;s) with $r = 1, s = n-1$, and the convexity of u,

$$u\big(\mathfrak{A}_n^{n-1}(\underline{a})\big) \le u\Big(\frac{1}{n}\sum_{i=1}^{n} \alpha_i\Big) \le \frac{1}{n}\sum_{i=1}^{n} u(\alpha_i).$$

Hence $\dfrac{u\big(\mathfrak{A}_n^{n-1}(\underline{a})\big)}{\mathfrak{A}_n(\underline{a})} \le S_u(\underline{a}) = \dfrac{u\big((\mathfrak{B}_n^{[u]}(\underline{a}))^{n-1}\big)}{\mathfrak{B}_n^{[u]}(\underline{a})}$. This, by Lemma 1, implies (6). The case of equality is easily considered. □

An extension to the concept of admissible means can be found in [*Dubeau 1991a*].

4.5 SEGRE FUNCTIONS In this section a general approach to mean inequalities due to Segre will be discussed; [*Segre*].

DEFINITION 4 *Let $n \ge 2$, $I =]m, M[$, $M, m \in \overline{\mathbb{R}}$. A function $f : I^n \mapsto \mathbb{R}$ that is differentiable and zero on constant $\underline{a}$, $\underline{a} \in I^n$, is called a Segre function on I.*

REMARK (i) A Segre function is not a mean, for if $\underline{a}$ is constant, a, $a \ne 0$, say, the value of a mean of $\underline{a}$ is a.

EXAMPLE (i) If $I = \mathbb{R}_+^*$ the function $f(\underline{a}) = \mathbb{D}_n^{r,s}(\underline{a};\underline{w}) = \mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) - \mathfrak{M}_n^{[r]}(\underline{a};\underline{w})$, $r, s \in \mathbb{R}$, see III 4, is a Segre function; in fact this is a homogeneous almost symmetric Segre function, symmetric in the case of equal weights.

EXAMPLE (ii) A homogeneous function $f : I^n \mapsto \mathbb{R}$ is a Segre function if and only if $f(\underline{e}) = 0$.

The following notations will be useful in what follows:

$$D = \{\underline{a};\ \underline{a} \in I^n \text{and } \underline{a} \text{ is increasing}\}; \quad D^\flat = \{\underline{a}; \underline{a} \in D \text{ and } \underline{a} \text{ is not constant}\};$$
$$\text{given } j,\ 2 \le j \le n, \quad \text{and as usual } \underline{a} = (a_1, \ldots, a_n),$$
$$D_j = \{\underline{a}; \underline{a} \in D \text{ and } a_j = a_1\}; \quad D_j^\sharp = \{\underline{a}, \ldots, a_n); \underline{a} \in D \text{ and } a_j \ge a_1\}.$$

The basic result of Segre is the following.

THEOREM 5 *A Segre function on I, f say, is non-negative on D, more precisely $f(\underline{a}) > 0$ for all $\underline{a} \in D^\flat$, if there exist functions $\rho_i : I^n \mapsto \mathbb{R}_+$, $1 \le i \le n$, such that*

$$f_1'(\underline{a}) \le \rho_1(\underline{a}) f(\underline{a}), \ \underline{a} \in D, \ \text{this inequality being strict if } \underline{a} \in D^\flat, \tag{7}$$

$$f_j'(\underline{a}) = \rho_j(\underline{a}) f_1'(\underline{a}), \ \underline{a} \in D_j, \ 2 \le j \le n. \tag{8}$$

□ If $\underline{a} \in D$ define $k = k(\underline{a}) = \max\{j;\, a_1 = \cdots = a_j\}$. If $k = n$ then $\underline{a}$ is constant and so by the definition $f(\underline{a}) = 0$.
So assume the result holds for all points with $j < k \le n$, and let $a \in D^\flat$, $k(\underline{a}) = j$. Assume further that the result is false at this point. That is assume: $\underline{a} = (a, \ldots, a, a_{j+1}, \ldots a_n)$, $a < a_{j+1}$ and $\psi(a) = f(\underline{a}) \le 0$. Since $\underline{a}$ is not constant we have, from (7) and (8), that $\psi'(a) < \rho_1(1 + \rho_2 + \cdots + \rho_n)\psi(a) \le 0$. Hence ψ is strictly decreasing on $[a, a_{j+1}]$.
So $\psi(a_{j+1}) < 0$, that is $f(a_{j+1}, \ldots, a_{j+1}, a_{j+2}, \ldots, a_n) < 0$, which contradicts the induction hypothesis. □

REMARK (ii) The above result only needs f, ρ_i, $1 \le i \le n$, defined on D.

If we now assume that f has some symmetry and homogeneity properties the same result will hold under weaker assumptions.

COROLLARY 6 *A symmetric Segre function on I is positive on $D^\flat$ if there is a $\rho_1 \ge 0$ such that (7) holds for all $\underline{a} \in D_j^\sharp$, $2 \le j \le n$.*

□ The symmetry of the function f gives (8) with $\rho_i = 1$, $2 \le i \le n$. □

COROLLARY 7 *A homogeneous symmetric Segre function on $\mathbb{R}_+^*$ is positive on $D^\flat$ if there is a $\rho_1 \ge 0$ such that (7) holds for all $\underline{a} = (1, a_2, \ldots, a_n) \in D_j^\sharp$, $2 \le j \le n$.*

Similar arguments extend these corollaries to almost symmetric Segre functions.

COROLLARY 8 *If f is an almost symmetric Segre function on I then $f(\underline{a}) > 0$ for all $\underline{a} \in D^\flat$, if there exist $\rho_i : I^n \mapsto \mathbb{R}_+$, $1 \le i \le n$, such that (7) holds for all $\underline{a} \in D_j^\sharp$, and (8) hold for all $\underline{a} \in D_j$, $2 \le j \le n$.*
If $I = \mathbb{R}_+^*$, and f is also homogeneous the same result holds if we only assume the above hypotheses for those $\underline{a}$ with $a_1 = 1$.

EXAMPLE (iii) If $f(\underline{a})$ is taken to be $\mathbb{D}_n^{0,1}(\underline{a}, \underline{w})$, see Example (i), then $f_1'(\underline{a}) = w_1\big(1 - \mathfrak{G}_{n-1}(\underline{a}_1'; \underline{w}_1')^{W_{n-1}/W_n}\big) \le 0$, with equality if and only if $a_2 = \cdots = a_n$; so (7) holds with $\rho_1 = 0$. If further $1 = a_1 = a_j$ then (8) holds with $\rho_j - w_j/w_1 > 0$. Applying Corollary 8 we get that $f(\underline{a}) \ge 0$ with equality if and only if $\underline{a}$ is constant, which is (GA).

EXAMPLE (iv) Take $I = \mathbb{R}_+^*$ and $f(\underline{a}) = W_n\big((\mathfrak{M}_n^{[s]}(\underline{a};\underline{w}))^s - W_n^{1-s/r}(\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}))^r\big)$ where $r < s$, $r, s \in \mathbb{R}_+^*$, then an argument similar to the previous one leads to (r;s).

EXAMPLE (v) Take $I =]0,1/2]$ and $f(\underline{a}) = \Big(\prod_{i=1}^n (1-a_i)/\big((\sum_{i=1}^n (1-a_i)\big)^n\Big) - \Big(\prod_{i=1}^n a_i/\big((\sum_{i=1}^n a_i\big)^n\Big)$, then f is a symmetric Segre function .
Putting $\rho_1(\underline{a}) = \sum_{i=1}^n (a_i - a_1)/(1-a_1)\sum_{i=1}^n (1-a_i)$, we get

$$f_1'(\underline{a}) - \rho_1(\underline{a})f(\underline{a}) = \frac{\sum_{i=1}^n (a_i - a_1)\prod_{i=1}^n a_i}{\big(\sum_{i=1}^n a_i\big)^n}\Big(\big((1-a_1)\sum_{i=1}^n (1-a_i)\big)^{-1} - \big(a_1\sum_{i=1}^n a_i\big)^{-1}\Big).$$

If then $0 < a_1 \le a_j$, $2 \le j \le n$, we have $\sum_{i=1}^n (a_i - a_1) > 0$, $1 - a_i > a_i > 0$, $1 \le i \le n$, and so $\rho_1 > 0$, and (5) holds. Hence by Corollary 2 $f(\underline{a}) \ge 0$ with equality if and only if $\underline{a}$ is constant. This is just Ky Fan's inequality, IV 4.4 (26).

EXAMPLE (vi) By considering $f(\underline{a}) = \mathfrak{A}_n(\underline{a}) - \mathfrak{A}_{n,\underline{\alpha}}(\underline{a})$ we can apply Corollary 7 with $\rho_1 = 0$ to get the right-hand inequality of V 6 (5). The left-hand side of this inequality is obtained using $f(\underline{a}) = \mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) - \mathfrak{G}_n(\underline{a})$, and taking $\rho_1 = |\underline{\alpha}|/n$.

Further details and results can be found in the paper of Segre.

4.6 ENTROPIC MEANS Let $\underline{a}$ and $\underline{w}$ be two n-tuples with $W_n = 1$; further let $\mathcal{F} = \{\phi;\ \phi : \mathbb{R}_+ \mapsto \mathbb{R}_+,$ strictly convex, differentiable with $\phi(1) = \phi'(1) = 0\}$. If $\phi \in \mathcal{F}$ define $d_\phi(u,v) = v\phi(u/v)$, $(u,v) \in \mathbb{R}_+^2$.

LEMMA 9 *If $\phi \in \mathcal{F}$ then, with the above notation:*

$$y > x \ge a > 0 \text{ or } 0 < y < x \le a \implies d_\phi(y,a) > d_\phi(x,a);$$
$$b > a \ge x > 0 \text{ or } 0 < b < a \le x \implies d_\phi(x,b) > d_\phi(x,a).$$

In particular if $x, a > 0$ then $d_\phi(x,a) \ge 0$ with equality if and only if $x = a$

□ These results are easy consequences of the properties of the functions in $\mathcal{F}$; for instance since ϕ' is strictly increasing $\phi'(u) > 0$ if $u > 1$ and hence $\phi(u)$ is strictly increasing if $u > 1$. Noting that if $y > x \ge a > 0$ then $y/a > x/a > 1$ gives the first result and the others are similar. □

REMARK (i) The differentiability assumption can be relaxed using the fact that a strictly convex function has strictly increasing left and right derivatives; see I 4.1 Theorem 4(b).

REMARK (ii) As a result of this lemma we see that $d_\phi(x,a)$ has some of the properties of a metric although it is not symmetric and does not satisfy (T).

The *entropic*[20] *mean of* $\underline{a}$ *with weight* $\underline{w}$, $W_n = 1$. *by* ϕ, $\phi \in \mathcal{F}$, is $x = \mathfrak{E}_n^{[\phi]}(\underline{a}, \underline{w})$ where $x > 0$ minimizes $\sum_{i=1}^n w_i d_\phi(x, a_i)$.

REMARK (iii) By Remark (ii) $d_\phi(x, a_i)$ can be regarded as a measure of the distance of x from a_i, and then the sum is an average of these distances over all the entries in $\underline{a}$; [*Ben-Tal, Charnes & Teboulle*].

THEOREM 10 *If* $\phi \in \mathcal{F}$ *then* $\mathfrak{E}_n^{[\phi]}(\underline{a}, \underline{w})$ *is unique and*

$$\min \underline{a} \le \mathfrak{E}_n^{[\phi]}(\underline{a}, \underline{w}) \le \max \underline{a}. \tag{10}$$

In particular if $\underline{a}$ *is constant, a say, then* $\mathfrak{E}_n^{[\phi]}(\underline{a}, \underline{w}) = a$. *Inequalities (10) are strict unless* $\underline{a}$ *is constant.*

□ Assume, without loss in generality, that $\underline{a}$ is increasing and consider the function $h(x) = \sum_{i=1}^n w_i d_\phi(x, a_i)$. Then $h'(x) = \sum_{i=1}^n w_i \phi'(x/a_i) < \sum_{i=1}^n w_i \phi'(1) = 0$, if $x < a_1$. Similarly if $x > a_n$ then $h'(x) > 0$. Hence by continuity there is an $x, a_1 < x < a_n$ where $h'(x) = 0$, and by the strict convexity this x is unique. This is the required unique minimum of h. The rest of the theorem is immediate. □

REMARK (iv) If we only assume ϕ to be convex then the results of Theorem 10 may fail so we then need to require that $\min \underline{a} \le x \le \max \underline{a}$.

REMARK (v) We see from the above that the mean is given by the unique x, $x > 0$, that solves the equation

$$\sum_{i=1}^n w_i \phi'(x/a_i) = 0. \tag{11}$$

REMARK (vi) The above result shows that the means defined by (9) are strictly internal and reflexive. It is easily checked that these means are also homogeneous, monotone and almost symmetric; [*Ben-Tal, Charnes & Teboulle*].

EXAMPLE (i) If $\phi(u) = u - 1 - \log u$ then $\mathfrak{E}_n^{[\phi]}(\underline{a}, \underline{w}) = \mathfrak{A}_n(\underline{a}; \underline{w})$.

EXAMPLE (ii) If $\phi(u) = 1 - u + u \log u$ then $\mathfrak{E}_n^{[\phi]}(\underline{a}, \underline{w}) = \mathfrak{G}_n(\underline{a}; \underline{w})$.

EXAMPLE (iii) If $\phi(u) = u + \dfrac{u^{1-r} - r}{r - 1}$, $r > 0$, $r \neq 1$ then $\mathfrak{E}_n^{[\phi]}(\underline{a}, \underline{w}) = \mathfrak{M}_n^{[r]}(\underline{a}; \underline{w})$.

EXAMPLE (iv) If $\phi(u) = 1 + \dfrac{u(1 - r) - u^{1-r}}{r}$, $r < 1$, $r \neq 0$ then again we have that $\mathfrak{E}_n^{[\phi]}(\underline{a}, \underline{w}) = \mathfrak{M}_n^{[r]}(\underline{a}; \underline{w})$. In this case taking $r = -1$ gives the harmonic mean.

[20] The reason for the name is not clear; see the comments by Aczél in his review, [*Zbl: 675.26007*].

THEOREM 11 *If $\phi \in \mathcal{F}$ and $\psi \in \mathcal{F}$ and if for some $K > 0$, $K\phi' \le \psi'$ then*

$$\mathfrak{E}_n^{[\phi]}(\underline{a}, \underline{w}) \ge \mathfrak{E}_n^{[\psi]}(\underline{a}, \underline{w}). \tag{12}$$

□ Let x denote the left-hand side of (12) and y the right-hand side, and suppose that $x < y$. From (11), $\sum_{i=1}^n w_i\phi'(x/a_i) = 0$ and $\sum_{i=1}^n w_i\psi'(y/a_i) = 0$.
From the hypothesis and ψ' being strictly increasing we have for $1 \le i \le n$ that $K\phi'(x/a_i) \le \psi'(x/a_i) < \psi'(y/a_i)$.
So $0 = K\sum_{i=1}^n w_i\phi(x/a_i) < \sum_{i=1}^n w_i\psi(y/a_i) = 0$. This is a contradiction and so $x \ge y$. □

REMARK (vii) By looking at the Examples (i), (ii) and (iii) we can use this theorem to deduce (GA) and (r;s).

Finally these entropic means have a means on the move property, see III 6.1 .

THEOREM 12 *Assume that $\phi \in \mathcal{F}$, and also that $\phi'''(x)$ exists and is continuous in a neighbourhood of $x = 1$ then*

$$\lim_{t\to\infty} \mathfrak{E}_n^{[\phi]}(\underline{a} + t\underline{e}, \underline{w}) = \mathfrak{A}_n(\underline{a}; \underline{w})$$

□ By Remark (vi) entropic means are homogeneous, and internal and it can be shown that $\partial\mathfrak{E}_n^{[\phi]}(\underline{a}, \underline{w})/\partial a_k = w_k$, $1 \le k \le n$; see [*Ben-Tal, Charnes & Taboulle p.550*]. The theorem then follows from the r esult of Brenner & Carlson, IV 4.5 Theorem 19. □

5 Mean Inequalities for Matrices[20]

If $A, B \in \mathcal{H}_n^+$ and $0 < t < 1$, then the *arithmetic, geometric and harmonic means of A and B with weights* $1-t$, t are defined by,

$$\mathfrak{A}(A, B; t) = (1-t)A + tB, \quad \mathfrak{G}(A, B; t) = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$
$$\mathfrak{H}(A, B; t) = \big((1-t)A^{-1} + tB^{-1}\big)^{-1}.$$

While the definitions of the arithmetic and harmonic means are obtained by an obvious analogy with the definitions in II 1.1, 1.2, the reason for definition of the geometric mean is not so clear, although if the matrices commute we have $\mathfrak{G}(A, B; t) = A^{1-t}B^t$. A lucid exposition of the reasons behind the definition can be found in [*Lawson & Lim*]; also refer to [*Ando 1979,1983; Pusz & Woronowicz*].

REMARK (i) As usual if the weights are equal, $t = 1/2$, we just write $\mathfrak{A}(A, B)$, $\mathfrak{G}(A, B)$, $\mathfrak{H}(A, B)$.

REMARK (ii) The means $\mathfrak{A}(A, B), \mathfrak{G}(A, B), \mathfrak{H}(A, B)$. are often written $A\triangle B$,[or $A\nabla B$], $A\#B$ and $A!B$ respectively.

20 The notations used in this section are defined in Notations 7.

LEMMA 1 *The arithmetic, harmonic and geometric means are also positive definite Hermitian matrices, are almost symmetric, as means, and if $A = B$ have the value A.*

□ All of the lemma is immediate except the almost symmetry of the geometric mean. The following proof is due to Furuta; see also [*Lawson & Lim p.800; Ando 1979 Corollary 2.1 (i)*].

$$\begin{aligned}\mathfrak{G}(A,B;t) &= A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}\\ &= B^{1/2}(B^{1/2}A^{-1}B^{1/2})^{t-1}B^{1/2} \text{ by [\textit{Furuta p.129 Lemma A}]}\\ &= B^{1/2}(B^{-1/2}AB^{-1/2})^{1-t}B^{1/2} = \mathfrak{G}(B,A;1-t)\end{aligned}$$

□

THEOREM 2 *(a)* [GEOMETRIC MEAN-ARITHMETIC MEAN INEQUALITY] *If $A, B \in \mathcal{H}_n^+$ and $0 < t < 1$ then*

$$\mathfrak{H}(A,B;t) \le \mathfrak{G}(A,B;t) \le \mathfrak{A}(A,B;t),$$

with equality if and only if $A = B$.
(b) [CONVERSE INEQUALITIES] *If $A, B \in \mathcal{H}_n^+$, $0 < t < 1$, $\mu = \max\{\lambda_1, \lambda_n^{-1}\}$, where $\lambda_1 \ge \cdots \ge \lambda_n$ are the eigenvalues of $B^{-1/2}AB^{-1/2}$ then,*

$$\mathfrak{A}(A,B;t) \le \frac{(\mu-1)\mu^{1/(\mu-1)}}{e\log\mu}\mathfrak{G}(A,B;t), \tag{1}$$

$$\mathfrak{G}(A,B;t) \le \frac{(\mu-1)\mu^{1/(\mu-1)}}{e\log\mu}\mathfrak{H}(A,B;t), \tag{2}$$

$$\begin{aligned}&\mathfrak{A}(A,B;t)-\mathfrak{G}(A,B;t)\\ &\quad\le \frac{t(1-t)}{2}\Big(\frac{(\mu-1)\mu^{1/(\mu-1)}}{e\log\mu}\Big)^{t-2}B^{1/2}(B^{-1/2}(A-B)B^{-1/2})^2B^{1/2}.\end{aligned} \tag{3}$$

(c [NANJUNDIAH'S INEQUALITY] *If $A, B \in \mathcal{H}_n^+$ then ,*

$$\mathfrak{G}(A,\mathfrak{A}(A,B)) \ge \mathfrak{A}(A,\mathfrak{G}(A,B)); \tag{4}$$

$$\mathfrak{G}(A,\mathfrak{H}(A,B)) \le \mathfrak{H}(A,\mathfrak{G}(A,B)). \tag{5}$$

□ (a) For any positive number x and $t \in [0,1]$ we have

$$tx + 1 - t \ge x^t \ge [tx^{-1} + 1 - t]^{-1},$$

with equality if and only if either $t = 0$, $t = 1$ or $x = 1$. The left inequality is a form of (B), see I 4.1 Example (iii); the right inequality follows from the left inequality by replacing x with x^{-1} and taking reciprocals of both sides. By the

standard operational calculus applied to these inequalities, we have for $X \in \mathcal{H}_n^+$ and $t \in [0,1]$ that

$$tX + 1 - t \geq X^t \geq [tX^{-1} + 1 - t]^{-1}.$$

Putting $X = A^{-1/2}BA^{-1/2}$ in these inequalities and multiplying by $A^{1/2}$ on both sides we get the desired result[20]; see [*Ando 1983; Furuta & Yanagida; Sagae & Tanabe*].

(b) Apply II 4.2 Theorem 4 in the case $n = 2$, $w_1 = 1-t$, $w_2 = t$, $a_1 = \lambda_i$, $a_2 = 1$ to get $(1-t)\lambda_i + t \leq \dfrac{(\mu_i - 1)\mu_i^{1/(\mu_i - 1)}}{e \log \mu_i}\lambda_i^{1-t}$, where $\mu_i = \dfrac{\max\{\lambda_i, 1\}}{\min\{\lambda_i, 1\}} = \max\{\lambda_i, \lambda_i^{-1}\}$. Using the final remark in the proof of II 4.2 Theorem 4 we get, with μ as above, that $\dfrac{(\mu_i - 1)\mu_i^{1/(\mu_i-1)}}{e \log \mu_i} \leq \dfrac{(\mu - 1)\mu^{1/(\mu-1)}}{e \log \mu}$. So

$$(1-t)\lambda_i + t \leq \frac{(\mu - 1)\mu^{1/(\mu-1)}}{e \log \mu}\lambda_i^{1-t}, \ 1 \leq i \leq n. \tag{6}$$

Now if $C \in \mathcal{H}_n^+$ then $C = UD(\underline{\lambda})U^{-1}$, where U is unitary. From (6) we have that $(1-t)D + tD \leq \dfrac{(\mu - 1)\mu^{1/(\mu-1)}}{e \log \mu}D^{1-t}$, and so $(1-t)C + tC \leq \dfrac{(\mu - 1)\mu^{1/(\mu-1)}}{e \log \mu}C^{1-t}$. Finally taking $C = B^{-1/2}AB^{-1/2}$ gives inequality (1).

Inequality (2) follows by applying (1) to A^{-1} and B^{-1}

Inequality (3) is a special case of a more general inequality and the proof is in the reference below.

(c) The square-root function is operator convex, I 4.9, and so

$$\Bigl(\frac{1}{2}(I + A^{-1/2}BA^{-1/2})\Bigr)^{1/2} \geq \frac{1}{2}\bigl(I + (A^{-1/2}BA^{-1/2})^{1/2}\bigr).$$

From this we get that

$$A^{1/2}\bigl(A^{-1/2}\frac{A+B}{2}A^{-1/2}\bigr)^{1/2}A^{1/2} \geq \frac{1}{2}\bigl(A + A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2}\bigr),$$

which is just (4).

Now, in (4), substitute A^{-1} and B^{-1} for A and B respectively; then take the inverse of both sides of the resulting inequality and we get (5). □

REMARK (iii) Inequality (1) is the analogue of Dočev's inequality II 4.2(3).

REMARK (iv) The first converse inequality for the matrix form of (GA) is in [*Mond & Pečarić 1995a*]; the results in (b) are in [*Alić, Bullen, Pečarić & Volenec 1995*]; see also [*Alić, Mond, Pečarić & Volenec 1997a,b*].

[20] The author thanks Professor Furuta for this succinct proof.

REMARK (v) The extension of Nanjundiah's inequality, II 3.4 (31), to matrices is due to Mond & Pečarić, [*Mond & Pečaric 1996d*]. See also [*Ando 1979, 1983*].

The arithmetic and harmonic means for sets of m matrices are readily defined but the extension for the geometric mean is less obvious. The following definition has been given by Sagae & Tanabe: given $A_i \in \mathcal{H}_n^+$, $1 \le i \le m$, and a positive m-tuple $\underline{w}$ with $W_m = 1$ then

$$\begin{aligned}\mathfrak{G}_m(A_1,\ldots,A_m;\underline{w}) =& A_m^{1/2}\Big(A_m^{-1/2}A_{m-1}^{1/2}\cdots\\ &\cdots\big(A_3^{-1/2}A_2^{1/2}(A_2^{-1/2}A_1A_2^{-1/2})^{v_1}A_2^{1/2}A_3^{-1/2}\big)^{v_2}\cdots\\ &\cdots A_{m-1}^{1/2}A_m^{-1/2}\Big)^{v_{m-1}}A_m^{1/2},\end{aligned}$$

where $v_i = \dfrac{W_i}{W_{i+1}}$, $1 \le i \le m-1$; if $m = 2$ this reduces to the previous definition with $A = A_1$, $B = A_2$, and $w_1 = 1-t$, $w_2 = t$[21]. Then the general form of (GA) can be proved.

THEOREM 3 *If $A_i \in \mathcal{H}_n^+$, $1 \le i \le m$, and $\underline{w}$ a positive m-tuple with $W_m = 1$ then*

$$\mathfrak{H}_m(A_1,\ldots,A_m;\underline{w}) \le \mathfrak{G}_m(A_1,\ldots,A_m;\underline{w}) \le \mathfrak{A}_m(A_1,\ldots,A_m;\underline{w}).$$

with equality if and only if $A_1 = \cdots = A_m$.

□ The case $m = 2$ is just Theorem 2(a) The case for general m is obtained by induction;. see [*Sagae & Tanabe*]. □

Other results can be generalized; for further details see [*Pečarić & Mond*] and the items in the bibliography of that paper. Using the obvious notation we can state the following analogue of (J); [*Mond & Pečarić 1993*].

THEOREM 4 *If I is a interval in $\mathbb{R}$, $f : I \mapsto \mathbb{R}$ a a convex matrix function of order n and $A_i \in \mathcal{H}_n^+$, $1 \le i \le m$, with all of the eigenvalues in I and $\underline{w}$ a positive m-tuple with $W_m = 1$ then*

$$f\big(\mathfrak{A}_m(A_1,\ldots,A_m;\underline{w})\big) \le \mathfrak{A}_m\big(f(A_1),\ldots,f(A_m);\underline{w}\big).$$

□ The case $m = 2$ is just the definition of matrix convexity, I 4.9 ((29), and the proof then follows the induction in the proof of (J), I 4.2 Theorem 12 proof(i). □

To give an extension of Nanjundiah's inequality we need the the following results due to Kedlaya and Anderson & Duffin; [*Anderson & Duffin; Kedlaya 1994,1999; Matsuda*].

[21] This definition, which is obtained by recurrence from the n=2 case, gives, if n>2, a geometric mean that is not almost symmetric. Let n=3 and A,B,C= $\begin{pmatrix}3&1\\1&1\end{pmatrix}, \begin{pmatrix}2&0\\0&1\end{pmatrix} \begin{pmatrix}1&1\\1&2\end{pmatrix}$, respectively. Then $\mathfrak{G}(A,B,C)\ne\mathfrak{G}(C,B,A)$; see [*Feng & Tonge*].

LEMMA 5 *(a)* [KEDLAYA] *If* $w_k^{(i,j)}$, $1 \le i,j \le n$, $1 \le k \le n$, *are defined by*

$$w_k^{(i,j)} = \frac{\binom{n-1}{j-1}\binom{i-1}{k-1}}{\binom{n-1}{j-1}} = \frac{(n-i)!(n-j)!(i-1)!(j-1)!}{(n-1)!(k-1)!(n-i-j+k)!(i-k)!(j-k)!},$$

they have the following properties:
(i) they are non-negative; *(ii) if* $k > \min\{i,j\}$ *then* $w_k^{(i,j)} = 0$;
(iii) for all i,j,k *we have* $w_k^{(i,j)} = w_k^{(j,i)}$; *(iv) for all* i,j *we have that* $W_n^{(i,j)} = 1$;
(v)

$$\sum_{i=1}^{n} w_k^{(i,j)} = \begin{cases} \dfrac{n}{j} & \text{if } k \le j, \\ 0 & \text{if } k > j. \end{cases}$$

(b) [ANDERSON & DUFFIN] *If* $A_{i,j} \in \mathcal{H}_n^+$, $1 \le i,j \le m$, *then*

$$\frac{1}{m}\sum_{j=1}^{m}\Big(\frac{1}{m}\sum_{i=1}^{m} A_{i,j}^{-1}\Big)^{-1} \le \Big(\frac{1}{m}\sum_{i=1}^{m}\Big(\frac{1}{m}\sum_{j=1}^{m} A_{i,j}^{-1}\Big)^{-1}\Big)^{-1}.$$

□ (a), (i), (ii) and (iii) are immediate.
Note that the numerator of $w_k^{(i,j)}$ can be considered as the number of j-element subsets of $\{1, 2, \ldots, n\}$ with k-th element i.
Summing this over k counts the subsets of $\{1, 2, \ldots, n\}$ containing i, gives $\binom{n-1}{j-1}$, which proves (iv).
Summing this over i counts all the j-element subsets of $\{1, 2, \ldots, n\}$ for $k \le j$, that is $\binom{n}{j}$ which proves (v).
(b) A proof of this matrix inequality can be found in the reference. □

THEOREM 6 *If* $A_k \in \mathcal{H}_n^+, 1 \le k \le m$, *then*

$$\mathfrak{H}_m(A_1, \ldots A_m; \underline{w}) \le \mathfrak{A}_m(A_1, \ldots A_m; \underline{w}); \tag{7}$$

$$\mathfrak{H}_m\big(A_1, (A_1 + A_2)/2, \ldots, \mathfrak{A}_m(A_1, \ldots A_m)\big) \tag{8}$$
$$\ge \mathfrak{A}_m\Big(A_1, \big((A_1^{-1} + A_2^{-1})/2)\big)^{-1}, \ldots, \mathfrak{H}_m(A_1, \ldots A_m)\Big).$$

□ (a) Inequality (7), a matrix analogue of (HA), follows from Theorem 4 just as (r;s), $r = -1, s = 1$, follows using convexity, III 3.1.1 Theorem 1 proof *(viii)*. All that is needed is to note that $f(x) = x^{-1}$ is matrix convex; I 4.9 Example (ii).

(b) Using the notation of Lemma 5 (a),

$$\begin{aligned}\mathfrak{A}_j(A_1,\ldots,A_j) &= \frac{1}{m}\sum_{k=1}^{m} A_k \sum_{i=1}^{m} w_k^{(i,j)}, \quad \text{by Lemma 5(a) (ii)}, &(9)\\ &= \frac{1}{m}\sum_{i=1}^{m}\mathfrak{A}_m(A_1,\ldots,A_m;\underline{w}^{(i,j)}), \quad \text{by Lemma 5(a) (v)},\\ &\ge \frac{1}{m}\sum_{i=1}^{m}\mathfrak{H}_m(A_1,\ldots,A_m;\underline{w}^{(i,j)}), \quad \text{by (7)}. &(10)\end{aligned}$$

Now the left-hand side of (8) is obtained by taking the harmonic means of the left-hand sides of (9), so by the above is greater than the harmonic mean of the right-hand sides of (10); that is, than $\Big(\frac{1}{m}\sum_{j=1}^{1}\big(\frac{1}{m}\sum_{i=1}^{m}\mathfrak{H}_m(A_1,\ldots,A_m;\underline{w}^{(i,j)})\big)^{-1}\Big)^{-1}$ So,

the left-handside of (8)

$$\begin{aligned}&\ge \Big(\frac{1}{m}\sum_{j=1}^{m}\big(\frac{1}{m}\sum_{i=1}^{m}\mathfrak{H}_m(A_1,\ldots,A_m;\underline{w}^{(i,j)})\big)^{-1}\Big)^{-1}\\ &\ge \frac{1}{m}\sum_{i=1}^{m}\big(\frac{1}{m}\sum_{j=1}^{m}\mathfrak{H}_m(A_1,\ldots,A_m;\underline{w}^{(i,j)})\big)^{-1}, \text{ by Lemma 4(b)},\\ &= \frac{1}{n}\sum_{i=1}^{m}\big(\frac{1}{m}\sum_{j=1}^{m}\sum_{k=1}^{m} w_k^{(i,j)}A_k^{-1}\big)^{-1} = \frac{1}{m}\sum_{i=1}^{m}\big(\frac{1}{m}\sum_{k=1}^{m}A_k^{-1}\sum_{j=1}^{m}w_k^{(i,j)}\big)^{-1}\\ &= \frac{1}{m}\sum_{i=1}^{m}\big(\frac{1}{i}\sum_{k=1}^{m}A_k^{-1}\big)^{-1}, \text{ by Lemma 5(a) (iii) and (a)(v)},\\ &= \text{ the right-hand side of (8).}\end{aligned}$$

□

Other ways of generalizing inequalities to the matrix situation have been explored. There is the following analogue of (J).

THEOREM 7 *If f is convex on an interval I and $A_k \in \mathcal{H}_n^+, 1 \le k \le m$, all having eigenvalues in I, and if $\underline{w}$ is a complex m-tuple with $\sum_{k=1}^{m}\langle w_k, w_k\rangle = 1$ then*

$$f\Big(\sum_{k=1}^{m}\langle A_k w_k, w_k\rangle\Big) \le \sum_{k=1}^{m}\langle f(A_k)w_k, w_k\rangle.$$

Using this approach the power means can be defined as follows.
Let $\underline{A} = (A_1,\ldots,A_m)$ where $A_i \in \mathcal{H}_n^+$, $1 \le i \le m$, and $\underline{w}$ be a complex m-tuple with $\sum_{i=1}^{m}\langle w_i, w_i\rangle = 1$, then:

$$\mathfrak{M}_m^{[r]}(\underline{A};\underline{w}) = \begin{cases}\big(\sum_{i=1}^{m}\langle A_i^r w_i, w_i\rangle\big)^{1/r}, & \text{if } r \in \mathbb{R}^*,\\ \exp\big(\sum_{i=1}^{m}\langle(\log A_i)w_i, w_i\rangle\big), & \text{if } r = 0.\end{cases}$$

Analogues of (r;s), (H), (M), Čebišev's inequality and of the converse inequality III 4.1 Theorem 3 have been given; see [*Mond 1965a,b,1996; Mond & Pečarić 1993, 1995a,c,d, 1996e*].

Of particular note is the inequality due to Kantorovič, see III 4.1 (11):

$$\langle Ax, x\rangle\langle A^{-1}x, x\rangle \le \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n},$$

where λ_1, λ_n denote the maximum and minimum eigenvalues of A respectively; see [*Furuta pp.188–1898*], [*Greub & Rheinboldt; Householder; Mond 1996; Mond & Pečarić 1995c*].

The methods of proof are based on the representation of matrix functions given in at the end of Notations 7.

Another approach is to use the *Hadamard product* [22] $A \circ B = (a_{ij}b_{ij})_{1\le i,j\le n}$ of two real $n\times n$ matrices $A = (a_{ij})_{1\le i,j\le n}$, $B = (b_{ij})_{1\le i,j\le n}$. The set of such matrices is then ordered by $A \ll B$, meaning $||A\circ C|| \le ||B\circ C||$ for all real $n\times n$ matrices C, where $||A|| = \big(\sum_{1\le i,j\le n} a_{ij}^2\big)^{1/2}$. Then the arithmetic, geometric, and harmonic means of a positive n-tuple $\underline{a}$ are defined as the following $n\times n$ matrices:

$$\mathfrak{A}^\circ(\underline{a}) = \Big(\frac{a_i+a_j}{2}\Big)_{1\le i,j\le n}, \mathfrak{G}^\circ(\underline{a}) = (\sqrt{a_ia_j})_{1\le i,j\le n}, \mathfrak{H}^\circ(\underline{a}) = \Big(\frac{2}{a_i^{-1}+a_j^{-1}}\Big)_{1\le i,j\le n}.$$

THEOREM 8 *With the above notation,*

$$\mathfrak{H}^\circ(\underline{a}) \ll \mathfrak{G}^\circ(\underline{a}) \ll \mathfrak{A}^\circ(\underline{a})$$

□ The reader is referred to the original paper for a proof; [*Mathias*]. □

The concept of arithmetic mean and other elementary means can be extended to more general operators than matrices, to means of ellipsoids, to semi-groups and there is even a fuzzy arithmetic mean; [*Furuta*], [*Bhagwat & Subramanian; Fujii, Kubo & Kubo; Gao L; Kubo & Ando; Mond & Pečarić 1996b; Pečarić 1991c; Ralph; Thompson*]. The idea of means on more general spaces is mentioned in the next section.

6 Axiomatization of Means

The extreme generality and variety of means leads naturally to the questions of what is a mean, what conditions on a function imply that it is a mean, or under what conditions is a function a particular mean? These questions quickly lead

[22] Sometimes called the *Schur product*.

to problems in functional equations and functional inequalities, see [*Pompeiu; Schweitzer A*]. These topics are beyond the scope of this book and are discussed fully elsewhere; [*Aczél 1966*], [*Hille*].

Means can be considered in the following way. There is an interval $I \subseteq \mathbb{R}$, and sequence of functions $\mathfrak{m}_k : I^k \mapsto \mathbb{R}$, $k \in \mathbb{N}^*$, or just $\{\mathfrak{m}_k\}$. We then say that $\{\mathfrak{m}_k\}$ has a property when every function in the sequence has that property, and we are interested in the properties introduced in I 1.1 Theorem 2: (Ad), additivity; $(\text{As})_m$, m-associativity; (Co), continuity; (Ho), homogeneity; (In), internality; (Mo), monotonicity; (Re), reflexivity; (Sy), symmetry.

CONVENTION We will always assume (Co), and even that the functions are differentiable if necessary.

The property (In) is so basic as often to be taken as the only requirement for a function to be a mean; it holds if (Mo) and (Re) hold; [*B^2 pp.230–232*].

Schiaparelli was probably the first to give a system of axioms sufficient for the arithmetic mean; his result was given another proof by Broggi, and Beetle proved the independence of the axioms; [*Beetle; Broggi; Schiaparelli 1907*]. Other authors to give such systems of axioms have been [*Aczél & Wagner; Huntington; Matsumura; Nakahara; Narumi; Schimmak; Sutô 1913,1914; Teodoriu*]. The following result of Teodoriu is particularly easy.

THEOREM 1 *If the sequence of functions $\{\mathfrak{m}_k\}$ has the properties (Ad), (Re) and (Sy) then $\mathfrak{m}_k = \mathfrak{A}_k$, $k \in \mathbb{N}^*$.*

□

$$\mathfrak{m}_k(\underline{a}) = \frac{1}{k!}\sum !\mathfrak{m}_k(\underline{a}), \text{ by (Sy)}, = \mathfrak{m}_k\big(\mathfrak{A}_k(\underline{a})\ \underline{e}\big), \text{by (Ad)}, = \mathfrak{A}_k(\underline{a}), \text{by (Re)}.$$

□

Obviously these conditions, (Ad), (Re) and (Sy), are both necessary and sufficient for $\{\mathfrak{m}_k\} = \{\mathfrak{A}_k\}$. These conditions are independent as is shown by the following examples.

EXAMPLE (i) $\mathfrak{m}_k(\underline{a}) = \dfrac{k}{k+1}\mathfrak{A}_k(\underline{a})$, $k \in \mathbb{N}^*$, satisfies (Ad), (Sy), but not (Re).

EXAMPLE (ii) $\mathfrak{m}_k(\underline{a}) = \mathfrak{A}_k(\underline{a}; \underline{w})$, $k \in \mathbb{N}^*$, satisfies (Ad), (Re), but not (Sy).

EXAMPLE (iii) $\mathfrak{m}_k(\underline{a}) = \mathfrak{M}_k^{[r]}(\underline{a})$, $r \neq 1$, $k \in \mathbb{N}^*$, satisfies (Re), (Sy), but not (Ad).

Huntington, in the above reference, gave seven sets of axioms for the geometric mean, the following result is one of these.

THEOREM 2 *If the sequence of functions* $\{\mathfrak{m}_k\}$ *has the properties (As)*$_2$*, (Re), (Sy) and if :* $\mathfrak{m}_2(a_1,a_2) = \mathfrak{G}_2(a_1,a_2)$*; then* $\mathfrak{m}_k = \mathfrak{G}_k$*,* $k \in \mathbb{N}^*$.

□ The case $k = 1$ is trivial and $k = 2$ is the hypothesis, so assume that $k \geq 3$.

$$
\begin{aligned}
\mathfrak{m}_k(\underline{a}) =&\mathfrak{m}_k\big(\mathfrak{m}_2(a_1,a_2),\mathfrak{m}_2(a_1,a_2),a_3,\ldots,a_k\big), \text{ by } (\mathrm{As})_2,\\
=&\mathfrak{m}_k\big(\sqrt{a_1a_2},\sqrt{a_1a_2},a_3,\ldots,a_k\big), \text{ by hypothesis},\\
=&\mathfrak{m}_k\big(\mathfrak{m}_2(1,a_1a_2),\mathfrak{m}_2(1,a_1a_2),a_3,\ldots,a_k\big), \text{ by hypothesis},\\
=&\mathfrak{m}_k\big(1,(a_1a_2),a_3,\ldots,a_k\big), \text{ by } (\mathrm{As})_2 = \mathfrak{m}_k\big((a_1a_2),a_3,\ldots,a_k,1\big), \text{ by (Sy)}.
\end{aligned}
$$

On repeating the argument we get that

$$
\mathfrak{m}_k(\underline{a}) = \mathfrak{m}_k\big((a_1\ldots a_k),1,1\ldots,1\big). \tag{1}
$$

Let $a_1 = \cdots = a_k = a$ then from (Re) and (1) that $a = \mathfrak{m}_k(a^k,1,1\ldots,1)$; so, again using (1), $\mathfrak{m}_k(\underline{a}) = \mathfrak{G}_k(\underline{a})$. □

Huntington also gives seven sets of axioms for the harmonic and quadratic means; the following is an example of his results.

THEOREM 3 *If the sequence of functions* $\{\mathfrak{m}_k\}$ *has the properties (As)*$_2$*, (Re), (Sy) and if : (a)* $\mathfrak{m}_2(a_1,a_2) = \mathfrak{H}_2(a_1,a_2)$*, (b)* $\mathfrak{m}_2(a_1,a_2) = \mathfrak{Q}_2(a_1,a_2)$*; then (a)* $\mathfrak{m}_k = \mathfrak{H}_k$*,* $k \in \mathbb{N}^*$*, (b)* $\mathfrak{m}_k = \mathfrak{Q}_k$*,* $k \in \mathbb{N}^*$.

Axiomatic definitions of the quasi-arithmetic means were originally given by [*Chisini 1929,1930; Fuchs 1950,1953; Kolmogorov; Nagumo 1930,1931,1933; Veress*]. For instance there is the following result of Kolmogorov.

THEOREM 4 *If the sequence of functions* $\{\mathfrak{m}_k\}$ *has the properties (As)*$_m$*,* $m \geq 1$*, (Mo), (Re) and (Sy), then for some function* $\mathcal{M} : I \mapsto \mathbb{R}$*,* $\mathfrak{m}_k(\underline{a}) = \mathfrak{M}_k(\underline{a})$*,* $k \in \mathbb{N}^*$.

□ Let $\mathfrak{m}_{k+n}(ka,nb)$ denote $\mathfrak{m}_{k+n}(\underline{a})$ for an $\underline{a}$ with k terms equal to a and n terms equal to b; by (Sy) we can assume $\underline{a} = (\underbrace{a,\ldots,a}_{k\,terms},\underbrace{b,\ldots,b}_{n\,terms})$. By (Re) and $(\mathrm{As})_m$, $m \geq 1$, $\mathfrak{m}_{p(k+n)}(pka,pnb) = \mathfrak{m}_{k+n}(ka,nb)$. Hence if $kn' = k'n$, then $\mathfrak{m}_{k+n}(ka,nb) = \mathfrak{m}_{k'+n'}(k'a,n'b)$.

Now if x is rational number, $0 \leq x = p/q \leq 1$, define $F(x) = \mathfrak{m}_q\big(pM,(q-p)m\big)$. It is easy to check that F is strictly increasing and continuous on the rationals and so can be extended to function with the same properties at all points. If then x_i be rational and $a_i = F(x_i)$, $1 \leq i \leq k$, it can be shown that $\mathfrak{m}_k(\underline{a}) = F\big(\mathfrak{A}_k(F^{-1}(\underline{a}))\big)$, so taking $\mathcal{M} = F^{-1}$, gives the result. □

A considerable amount of work has been done on axiomatizing means and quasi-arithmetic means in particular; see [*Aczél 1947,1948a,b,c,1949a; Allasia 1980; Aumann 1933a,b,1934,1935a, 1976; Bajraktarević 1951,1953; Barone & Moscatelli; Bertillon, de Finetti 1930, 1931; Dodd 1934,1936a,1937,1941a; Fenyő 1947,1948, 1949a,b; Girotto & Holzer; Horváth 1947,1948a,b; Hosszú; Howroyd; Jessen 1931a, 1933a,b; Kitagawa 1934,1935; Marichal; Mikusiński; Ostasiewicz & Ostasiewicz; Pizzetti 1950; Ryll-Nardzewski; Thielman; Toader, S; Ulam*].

Bos has considered a sequence of functions $\{\mathfrak{m}_k\}$ with the properties (Re) and (Sy), and such that each $\mathfrak{m}_{k+1}$ has the property $(\mathrm{A})_k$, and $u(x) = \mathfrak{m}_k(a_1, \ldots, a_{k-1}, x)$, $k \in \mathbb{N}^*$, is an injection defined on a fairly general topological space X. They are then said to define a *mean space structure on* X. For details the reader is referred to the interesting papers [*Bos 1971,1972a,b,c,1973*].

Extensions of the mean concept to general structures have also been given by Kubo; see [*Kubo & Ando*], and the references in the bibliography of that paper. He has defined arithmetic, geometric and harmonic means of operators on Hilbert spaces, and, in this context, proves an extension of (GA). See also [*Antoine*], [*Antoine*].

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NAME INDEX

[1] С.Б.Аблялимов; also transliterated as Abljalimov.

[2] Also written Afuwapẹ.

[3] Г.А.Бекишев;also transliterated as Bekishev.

[4] С. Т. Берколайко.

[5] I have not been able to determine if this is Jacques or Louis- Adolphe.

[6] Я. А.Боярский.

[7] В. Я. Буняковский; also transliterated as Buniakovsky.

[8] Любомир Н. Чакалов; also tranliterated as Tchakaloff.

[9] И. В. Чебевская; also transliterated Chebaevskaya.

[10] П. Л. Чебышев; also transliterated as Čebyšev, Chebyshev, Tchebicheff, Tchebishev, Tchebychef, Tchebycheff.

[11] Also transliterated as Chên Chi.

E

[12] Also occurs as Đoković.

[13] Кирил Гечов Дочев.

[14] Владислав Кирилович Дзядык.

F

[15] Istvan is variously abreviated as I., S., St.

G

[16] А. Е. Гельман.

[17] Е. К. Годунова.

H

I

J

K

[18] Also known as Hero. The date is not certain see [*Heath vol.II pp.298–302*].

[19] Леонид Витальевич Канторович; also transliterated as Kantorovich.

L

[20] А Н Колмогоров.

[21] Павел Петрович Коровкин.

[22] В М Кузнецов;also transliterated as Kuznetsov.

[23] Д. Н. Лабутин.

[24] Виктор Иосович Левин.

[25] Александр Михаилович Лиапунов; also translterated as Liapounoff, Lyapunov.

[26] Also written as Youngdo Lim.

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27 Also known as Karl Loewner.

28 А.В.Маркашин.

N

O

P—Q

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[29] А.Н.Николаев.

[30] А. Л. Перельдик.

R

[31] Р. Т. Рахмаил.

[32] Also written as Frédéric Riesz.

[33] Всеволод Иванович Романовский.

[34] Ф. В. Русйнов.

[35] А. А. Сироткина.

T

[36] П. С. Смирнов.
[37] С. А. Смолиак.
[38] В. Н .Соловиов.
[39] Also written Takahashi.
[40] Also transliterated as Têng K'ai Yü.
[41] Владимир Михайлович Тихомиров.

U—V

W

[42] Р.П.Ушаков.

[43] Н. И. Ваингер

[44] Also transliterated as Wang Chung Hsin.

X—Y

Z

45 Also transliterated asYang Kuo Shêng.

46 Also occurs as Yôita.

47 А.П. Южаков.

48 Also occurs as István.

49 А. М. Журавский; also transliterated as Zhuravskii.

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